Reactions of simple structures

Definition.: An assembly composed of a single rigid body which is connected to its neighbourhood by constraints and is able to remain in equilibrium under an arbitrary arrangement of loads is called a *simple (load-bearing) structure*. (material properties or failure are not considered here). This definition is often used in contrast to *mechanisms* that may not be in equilibrium but move under certain loads. In a more neutral context, the term *assembly* can be used without the above distinction.

A (kinematic) *constraint* may be imposed on a body to disallow a particular component of the displacement of that body. It appears in a form of *supports* or *connections*. The number of allowed displacement components is called the (kinematic) *degree of freedom* (*DOF* for short) of the body which is normally reduced by each constraint applied to the body. Different kinds of supports can restrain displacement components of different number and type but they exert forces or torques on the body in the sense of restrained displacements. These forces and torques transmitted from the support to the body are called *reactions*. (in contrast to known forces that are acting on the body, these latter ones are also called *constraint forces* or *passive forces*). Supports (connections) can be characterized by the number of constraints they represent: this equals the number of scalars required to describe reactions at a support. In the next paragraph, support types of higher relevance in engineering are introduced.

Figure B1.1 shows a straight beam supported by a *roller* (a) and another by a vertical *link* or *bar* (b) at one end. Both supports have a single constraint: the roller prevents the body from being lifted from or pushed into the surface it rolls on, while a link restrains any displacement parallel to its direction. The remaining two components of displacements are both possible: a roller does not restrain a sliding along the plane of support or a rotation about it, as well as the link does not restrain any displacement perpendicular to its axis (note that it holds only for small displacements) or a rotation about its upper end. The beam is exerted upon by a force aligned with the restrained component of displacement; that is, in a given line of action in both cases. Links are commonly denoted by numerals, other supports are marked by uppercase letters. Forces transmitted from a link to the body is ('force in a (link) member') are normally denoted as *S* with the number of the link in the subscript. Reactions at other supports are normally referred to by the same uppercase letters used for naming them. The reaction forces exerted on the beam are shown in the bottom figures. Further examples for a single-constraint support is the *simple support* (i.e., without friction) and the *cable*.

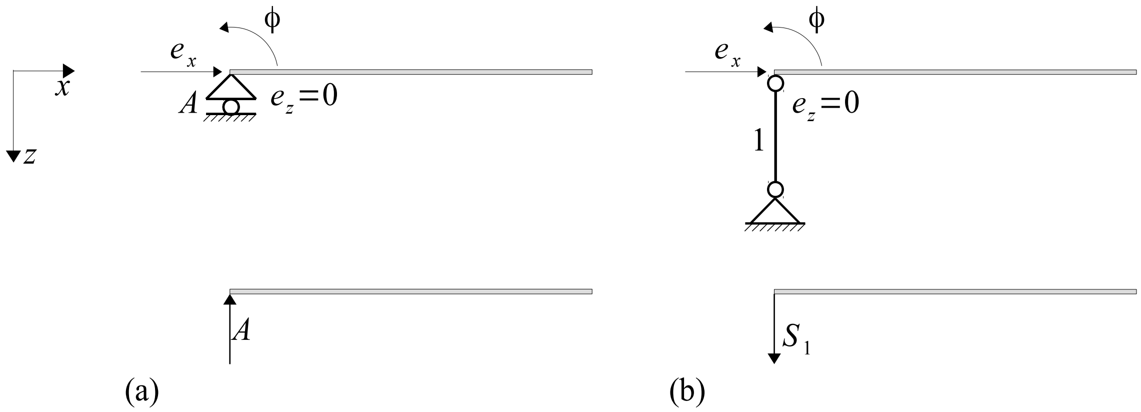


Figure B1.1: supporting by a roller (a) and a link or bar (b)

Figure B1.2 shows a beam with *hinged* or *pinned support* (a) and another with a *fixed* or *clamped support* (b). A pin-joint or hinge prevents the supported point from being translated in any direction but a rotation of the body about the pin is still allowed. As it follows immediately, the line of action of a force exerted on the body at a pin-joint always passes through the same pin-joint (which is considered as a point with no extension in our model). A pin or hinge has two constraints, the corresponding reaction can be given by two scalars (for convenience, two components). A fixed support restrains any translation and the rotation of the body and has therefore three constraints. In turn, a force of arbitrary direction as well as torque can be exerted on the supported body here. Note that there exist other types of supports but they are not considered here for their lower importance in civil engineering.

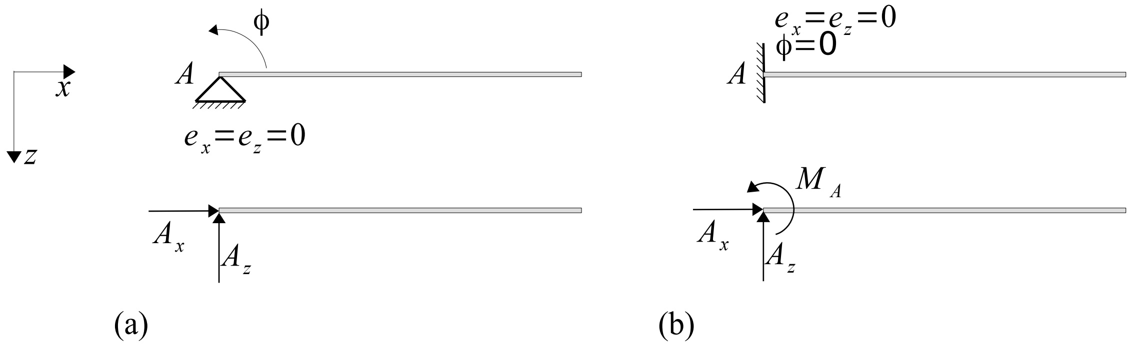


Figure B1.2: Pinned or hinged (a) and fixed or clamped (b) support

It is important to state that two-dimensional structures and problems are only dealt with here. Under this assumption, an unsupported body has exactly 3 DOFs, so at least three constraints have to be applied in order to assemble a load bearing strucure. Some examples for such simple structures are shown in Figure B1.3. All three supports in part (a) have a single constraint. If the lines of action of the three forces are neither concurrent nor parallel, the assembly behaves as a structure by being able to carry loads of arbitrary arrangement. The beam in part (b) is supported by a roller and a pin of one and two constraints, respectively. If the fixed line of action (that could also pertain to a link instead of a roller) does not pass through the pinpoint, then the beam is considered a structure again. Part (c) displays a cantilever beam which has a single fixed support by definition: it has three constraints. At the present stage, structures with more than three constraints, as well as assemblies with three constraints but with no equilibrium in some arrangements are not discussed.

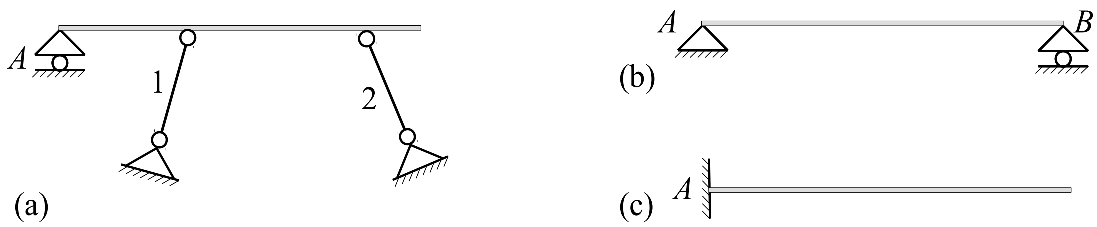


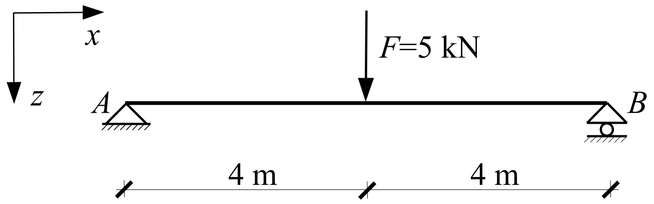
Figure B1.3: Plane structures

Structures are exerted on by several loads, which can be classified either as *live load* (e.g., loads of vehicles or furniture, environmental loads like of wind or snow) that are occasional or *dead load* (e.g., the weight of the structure) that are permanent. These loads are also qualified as *active forces* opposed to reactions transmitted to the structure at its supports. Any force system composed of active and passive forces exerted upon a structure has to remain in equilibrium. Static analysis therefore always begins with the determination of reactions. This procedure is focused on and developed step by step in the following lectures.

The first step of calculating reactions is called *isolation*. It means that the body in consideration is freed from any supports which are replaced rather by reaction forces and torques arising at them: a sketch of the body with all active and passive forces is called a *free-body diagram* (*FBD*). As soon as the body gets isolated in this way, a formal *equilibrium statement* is made by declaring an equivalence between the set of all active and passive forces exerted on the body and the zero force. Since such a statement relates a general system of plane forces, three independent *equilibrium equations* for scalars (moments or (force) resolutions) can generally be set up. The solution to this system of three equations yield values for all three components of reactions. It is important to *check* results by evaluating (a) further equation(s) of different mechanical content (those equations are still mathematically dependent on the previous ones). Computed and verified results are usually presented in a form of a *final sketch* (*FS*) which is not much different from a FBD, except that it displays all reactions with their eventual sense and magnitude. The following examples provide illustrations to the whole procedure traced above.

Example 1

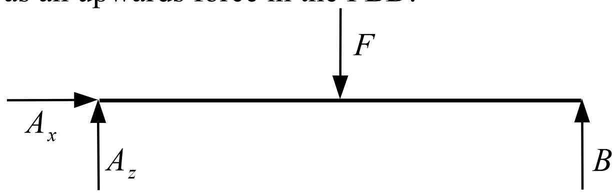
Determine support reactions of the *simply supported beam* shown below.



Solution

Draw the free-body diagram first: replace supports by the corresponding reactions. The pinned support at A is able to transmit a force of arbitrary direction which can also be given uniquely. The FBD is drawn with the assumption that horizontal and vertical components A_x and A_z are directed right- and upwards, respectively. Note that unknown reaction components can be assumed arbitrarily in a FBD (some practical exceptions will be mentioned later). The positive or negative sign of any particular result obtained from calculation will confirm or refute, respectively, the correctness of assumed senses of arrows: in the former case, an arrow in the FS is left as it appears in the FBD, in the latter, it should be reversed with respect to the sense assumed in the FBD. The roller support at B allows horizontal sliding, that is why it can transmit a vertical reaction component. Assume B as an upwards force in the FBD.

The free-body diagram of the structure:



There are three forces acting upon the body: a given active load F and passive forces A and B at the supports; they keep the structure in equilibrium. This fact can formally be expressed in an equilibrium statement as follows:

$(F, A, B) \doteq 0$.

An active load F is balanced here by a force passing through a given point (A) and another one having a given line of action (B). A reaction force acting at a given point can be specified by two scalar components, whereas the given line of action makes necessary to find a (signed) scalar magnitude only. In summary, there are three unknowns that equals the number of independent scalar equations that can be written for a general plane force system. When calculating by hand, it is always intended to get the solution through a sequence of one-variable equations: in the first equation, two out of three unknowns should therefore be eliminated. In the calculation of simply supported beams it can be taken as a thumb rule that a balance of moments about the pinned support yields the reaction at the simple support, as both components of the reaction at the pin have a zero moment arm about the same pin-joint. In the current problem there appears only the given force F and unknown B with nonzero moment arm in the balance of moments about point A; thus, force B can be obtained directly from the equation:

$$\sum M_i^{(A)}: -5 \cdot 4 + A_x \cdot 0 + A_z \cdot 0 + B \cdot 8 = 0 \quad \rightarrow \quad B = 2.5\,\mathrm{kN}(\uparrow) \cdot$$

Moments in these equations are always taken positive if they represent a counterclockwise rotation. Positive sign of the result means that force B actually has the same sense (upwards) as assumed in the FBD. This sense obtained from calculation should be displayed after the unit of the result. In order to minimize the risk of miscalculation, terms of particular forces should always be written in an equation following their order of occurrence in the equilibrium statement. Moment equations need not contain terms that vanish due to the zero moment arm but they are still displayed on the first few occurrences to demonstrate why some force components (A_x and A_z) do not appear in the equation indeed. After the reaction at the roller support having been found, each component of the reaction at the hinge *could* be obtained from single-variable resolution equations along x and z. However, this method involves a computational risk and thus it is generally not recommended: assume that B is miscalculated for any reason: would a following vertical be resolution completely free of errors, it still gives a false result because of the error of B. Alternatively, it is worth looking for an equation in the second step that includes A_z only: such an equation can again be found by eliminating both A_x and B by writing a balance of moments about the point of intersection of lines (of action) of A_x and B,which is at point B. If the structure is still equilibrium, those moments should add up to zero:

$$\sum M_i^{(B)}: -5 \cdot 4 + A_x \cdot 0 - A_z \cdot 8 + B \cdot 0 = 0 \quad \rightarrow \quad A_z = 2.5\,\mathrm{kN}(\uparrow)$$

Note the positive answer and so the upward arrow just copied from the FBD for A_z as well. With the same considerations as above, the horizontal reaction at the hinge is obtained, for convenience, from an equation containing the *only* unknown A_x. It means the elimination of both A_z and B but now there is no point of intersection of A_z and B as they are parallel. In such a case; however, there is always a *direction* perpendicular to both of them: let us write therefore the resolution in x:

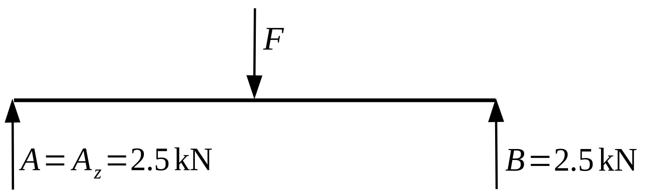
$$\sum F_{ix} \quad \rightarrow \quad A_x = 0\,\mathrm{kN}$$

Even if successive elimination of unknowns gives a certain safety against the accumulation of mistakes in calculation, it still needs a check after the components having been obtained. Any equation with different *mechanical* content from those already used can be appropriate for verification but, of course, none of these new equations are *mathematically* independent of the first three ones. In the pesent example, the vertical has not been written yet; use it for verification:

$\sum F_{iz}: 5 - 2.5 - 2.5 = 0$

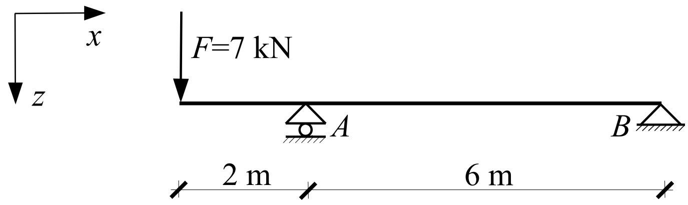
After a successful check it is only left to draw the isolated body again, now with the real senses of forces acting upon it. Reaction forces should be given now with their computed result (and unit).

Final sketch:



Exercise 1

Find support reaction of the *overhanging beam* shown.



Solution

Free-body diagram from isolation:

Equilibrium statement:

Unknowns:

Number of independent scalar equations:

Analytic solution:

...

...

...

Verification:

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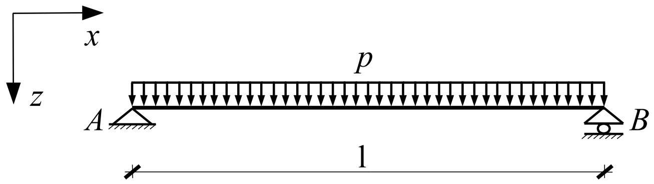
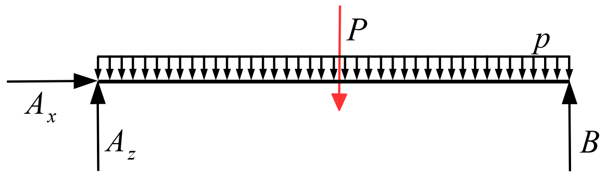
Final sketch:

Example 2
Find the support reactions as a function of given parameters for a simply supported beam.



Solution
The solution begins with the step of isolation. The resultant *P* of distributed load is also drawn.

The FBD looks as follows:



Equilibrium statement: $((p), A, B) \doteq 0$,
The distributed load is replaced by its resultant in the calculation; it is of magnitude is $P = p \cdot l$, bisecting the span of the beam. Reaction *A* can be given by two components, whereas reaction *B* in a given line of action is sufficient to be specified by its signed magnitude: the total number of scalar unknowns in the problem equals three. There exist three independent equations for any general plane force system. Write first the balance of moments for the pinned support in order to eliminate both components of *A* and to get therefore *B*. Next, eliminate A_x and *B* by writing another moment balance about *B*; finally, let both vertical reaction components be eliminated in a horizontal resolution to get . Mind the order of terms as they appear in the statement:

$$\sum M_i^{(A)} : -(p \cdot l) \cdot \frac{l}{2} + B \cdot l = 0 \quad \rightarrow \quad B = \frac{p \cdot l}{2} (\uparrow)$$

$$\sum M_i^{(B)} : (p \cdot l) \cdot \frac{l}{2} - \frac{p \cdot l}{2} \cdot l = 0 \quad \rightarrow \quad A_z = \frac{p \cdot l}{2} (\uparrow)$$

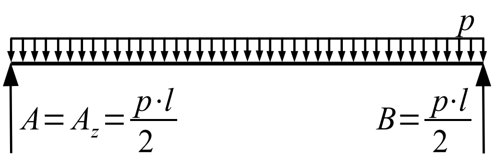
$$\sum F_{ix} : A_x = 0$$

After all reactions having been found, check the results, e.g., by writing a vertical resolution:

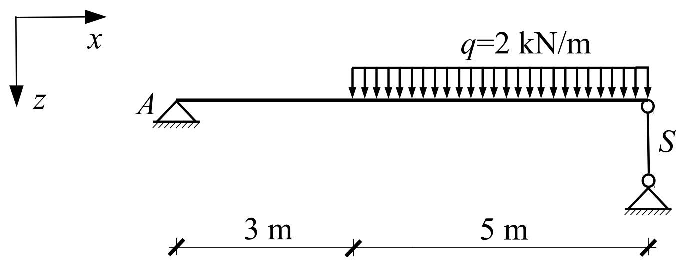
$$\sum F_{iz} : (p \cdot l) - \frac{p \cdot l}{2} - \frac{p \cdot l}{2} = 0$$

After a successful check it is only left to draw the isolated body again, now with the real senses of forces acting upon it. Reaction forces should be given now with their computed result (and unit).

Final sketch:



Exercise 2
Find support reactions of the beam shown.



Solution

FBD:

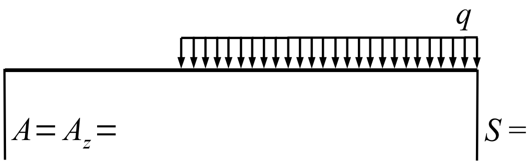
Equilibrium statement:
Number of independent scalar equations:
Unknowns:
Resultant of the distributed load, $Q =$
Analytic solution:

...

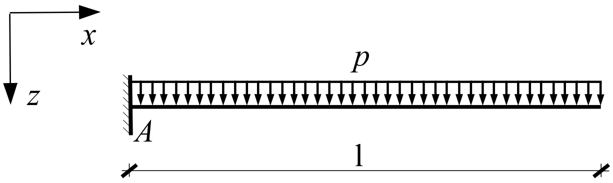
...

...
Verification:
…

Final sketch:

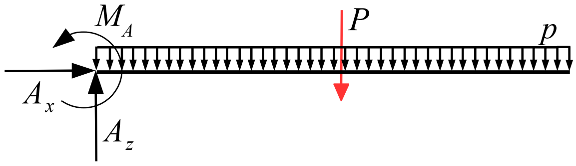


Example 3
Find support reactions of the cantilever beam as a function of given parameters.



Solution
The isolation is done first. In addition to the distributed load, the beam is acted upon by support reactions at A: a force A of unknown magnitude and direction at a given point and a support moment reaction M_A.

Free-body diagram:



Equilibrium statement: $((p), A, M_A) \doteq 0$
The body is kept in equilibrium by a general plane force system implying three independent

scalar equations to be written. The number of unknowns is also three: two components of reaction force and the moment reaction at *A*. Using the method of seeking for equations as described above, two resolution equations and a moment equation (written about the centroid of the clamped cross section) is always obtained as a system of three one-variable equations. Resolution equations have only the corresponding unknown force component, while the moment reaction stands alone in a moment equation about the support because of zero moment arms of reaction force components. (A moment can never appear in a resolution equation, since force vectors in any couple the moment is equivalent to add to the zero vector.) The equilibrium equations are:

$$\sum M_i^{(A)}: -p\cdot l\cdot\frac{l}{2}+M_A=0 \quad\rightarrow\quad M_A=\frac{p\cdot l^2}{2}\,(\curvearrowleft)$$

$$\sum F_{iz}: p\cdot l-A_z=0 \quad\rightarrow\quad A_z=p\cdot l\,(\uparrow)$$

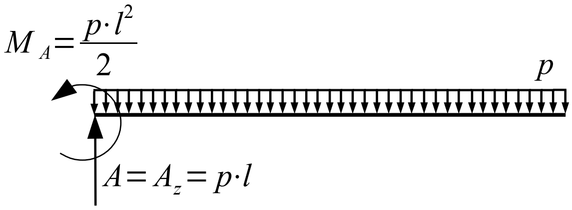
$$\sum F_{ix}: A_x=0$$

After the reaction components having been found, check the results by a moment equation written about the free end of the cantilever (to the right):

$$\sum M_i^{(r)}: p\cdot l\cdot\frac{l}{2}-p\cdot l\cdot l+\frac{p\cdot l^2}{2}=0$$

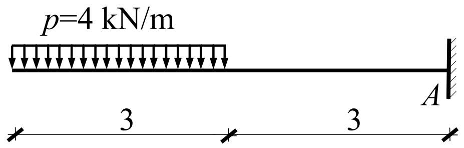
Since the check is successful, draw the final sketch:

Finalsketch:



Exercise 3

Find support reactions of the beam shown.



Solution

FBD:

Equilibrium statement:

Number of independent scalar equations:

Unknowns:

Resultant of the distributed load, P =

Analytic solution:

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Verification:

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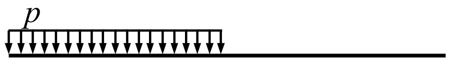
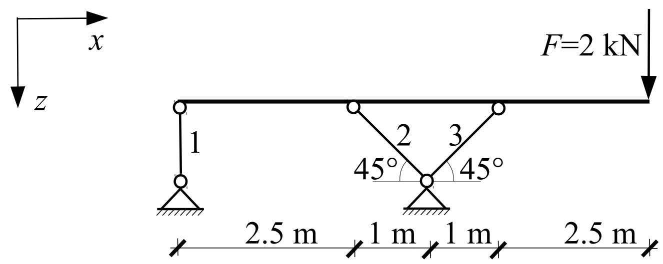
Final sketch:



Example 4

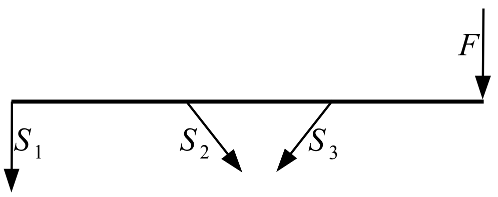
Find the reactions of the beam supported by three bars as shown in the figure.



Solution

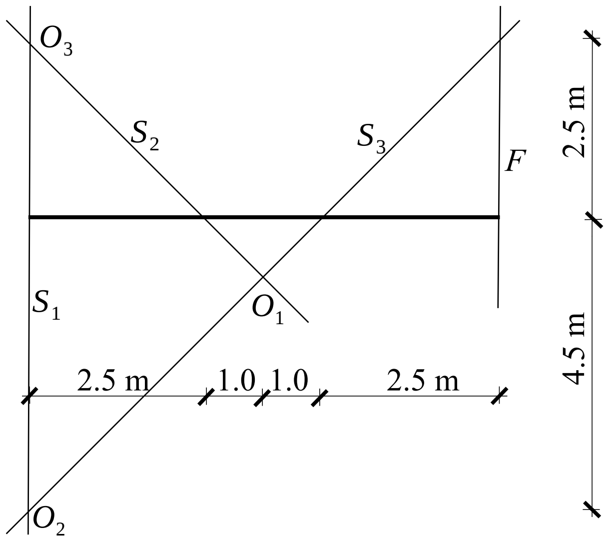
The isolation is done first. In addition to the active force, the body is kept in equilibrium by three reaction forces along the supporting links (those three forces are therefore of known line of action). Forces in bar members are always assumed to be tensile, which means that there is tension in the interior of the bar member and there is also a tensile force exerted by the bar on the beam and on the ground at the same time. The force in the *i*th bar is denoted by S_i (i=1, 2, 3).

The free-body diagram:



Equilibrium statement: $(F, S_{1,} S_{2,} S_3) \doteq 0$

All three forces in bars can be given by a signed magnitude each, these mean three unknowns of the problem. The number of independent equations is also three since the equilibrium of a general plane force system is analysed. The problem of balancing a body with three forces of given lines of action is commonly solved by *Ritter's method*: three moment equations are written; each of them is about a point of intersection of lines of unknown forces. These equations contain only one unknown member force out of the three. If a member force S_i is the only unknown force not passing through a point (i.e., having a nonzero moment arm about it), then that point is denoted by O_i and is called the principal point of the ith member. The following figure shows all three principal points as intersection of lines of action.



After the principal points having been found by some geometric arguments, let moment equations about all three principal points be written an d the equations solved:

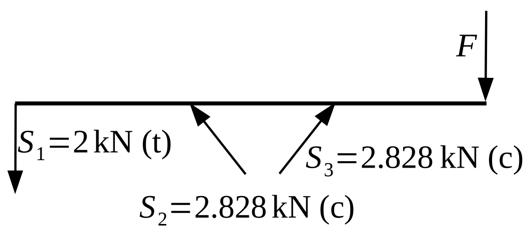
$$\sum M_i^{(O_1)}: -2\cdot 3.5 + S_1\cdot 3.5 = 0 \quad \rightarrow \quad S_1 = 2\,\mathrm{kN}\,(\text{tension})$$
$$\sum M_i^{(O_2)}: -2\cdot 7 - S_2\cdot\cos(45°)\cdot 7 = 0 \quad \rightarrow \quad S_2 = -2.828\,\mathrm{kN}\,(\text{compression})$$
$$\sum M_i^{(O_3)}: -2\cdot 7 - S_3\cdot\cos(45°)\cdot 7 = 0 \quad \rightarrow \quad S_3 = -2.828\,\mathrm{kN}\,(\text{compression})$$

In the second (third) equation, force S_2 (S_3) is resolved into components at point O_3 (O_2), making sufficient to account only for horizontal force components in evaluation of moments. A positive result indicates tensile behaviour (as it was originally assumed), while negative answers refer to bar members rather in compression. The tensile or compressive property of a bar is of extreme importance in design, that is why it is always referred to within brackets together with the calculated result.
Since moment balances were only used in the solution, both 'remaining' resolutions can be used for verification:

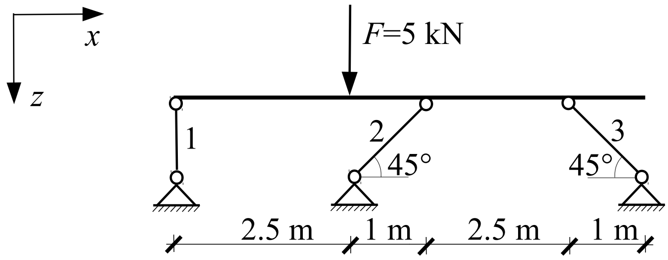
$$\sum F_{ix}: -2.828\cdot\cos(45°) + 2.828\cdot\cos(45°) = 0$$
$$\sum F_{iz}: 2 + 2 - 2.828\cdot\sin(45°) - 2.828\cdot\sin(45°) = 0$$

After a successful check having been completed, a final sketch is made with true senses of arrows of bar member forces.

Final sketch:

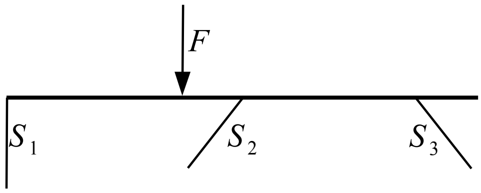


Exercise 4
Find the reactions of the beam supported by three bars as shown in the figure.



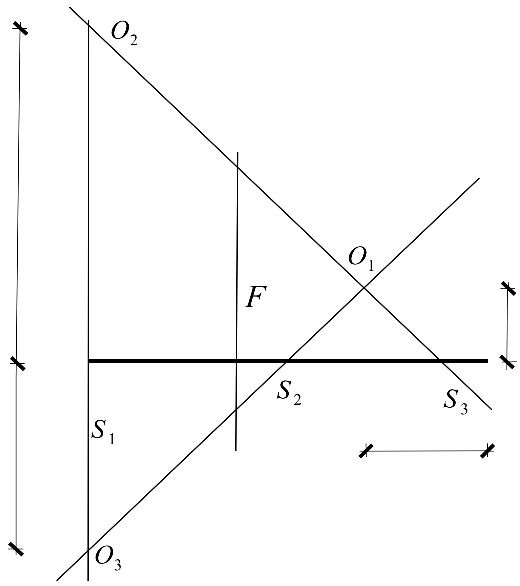
Solution

Free-body diagram:



Equilibrium statement:

Locating principal points:



Number of independent scalar equations:

Unknowns:

Analytic solution:

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Equation(s) for verification:

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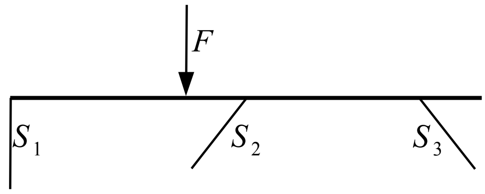
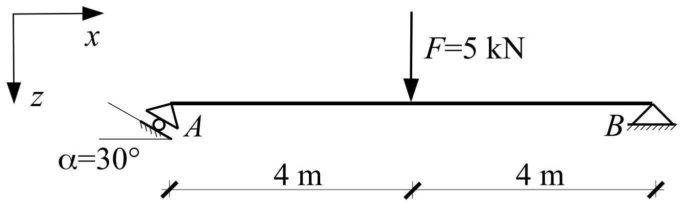
...

Final sketch:



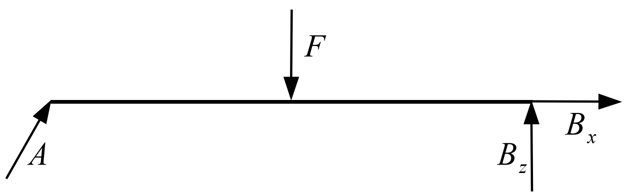
Example 5
Find the reactions at *A* and *B* of the simply supported beam shown.



Solution

Isolate the structure first. A reaction perpendicular to the oblique surface arises at the roller: assume that it points towards right and up. Pinned support exerts a force of unknown magnitude and direction on the beam, assume that horizontal and vertical components of that reaction force point to the right and upwards, respectively.

The free-body diagram :



The body is kept in equilibrium by active force *F* and support reactions *A* and *B*:
$(F, A, B) \doteq 0$.

The active force is balanced here by a force with given line of action (*A*) and another one with known point of application (*B*): this latter one can be specified by two components (signed scalars), while the former one can be given by a single scalar only. The system of forces is still a general one in plane, so three independent equations can be set up which equals the number of unknowns.

If the reaction component A is sought, a moment about the point of intersection of other unknowns (i.e., the pinned support) should be written that yields reaction A directly:

$$\sum M_i^{(B)}: 5\cdot 4 - A\cdot\cos(30°)\cdot 8 = 0 \rightarrow A = 2.887\,\text{kN}\ (\nearrow)$$

Positive sign of the result confirms the orientation of reaction A as assumed in the FBD. In a quite similar manner, write now a balance of moments about point A in order to find the vertical component of B:

$$\sum M_i^{(A)}: -5\cdot 4 + B_z\cdot 8 = 0 \rightarrow B_z = 2.5\,\text{kN}(\uparrow)$$

Following the scheme of finding reactions by numerically independent equations, now an equation not including A or B_z would follow. Such an equation could be obtained by writing moments about the point of intersection of an inclined and a vertical line but that point might sometimes be difficult to find: it must be considered whether such preliminary geometric calculations involve more risk than the use of recently obtained numeric values in further equilibrium equations. Here an example is given for the latter approach by writing a simple horizontal resolution that is also based on the value of force A:

$$\sum F_{ix}: 2.887\cdot\sin(30°) + B_x = 0 \rightarrow B_x = -1.444\,\text{kN}(\leftarrow)$$

After all unknown reaction components having been found, let a vertical resolution equation is written for checking our results:

$$\sum F_{iz}: 5 - 2.887\cdot\cos(30°) - 2.5 = -0.0002 \approx 0$$

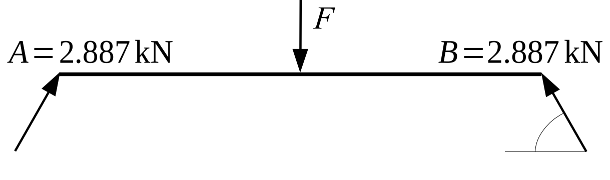
The 'small' error obtained on the right hand side is due to roundoff. It is never evaluated on its own but should be compared to the order of magnitude of forces appearing in the same equation. Since there is a difference still of at least three orders of magnitude, the results can be accepted (note that the error itself is not evaluated for four significant figures because only its order of magnitude is relevant).

Since the check is successful, find the magnitude and direction of the force at B:

$$B = \sqrt{(1.444)^2 + (2.5)^2} = 2.887\,\text{kN}$$

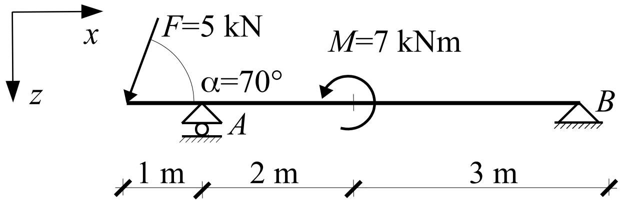
$$\alpha_B = \arctan\frac{2.5}{1.444} = 59.99°$$

FInally, make a final sketch:



Exercise 5

Find the reactions at A and B of the simply supported beam shown.



Solution

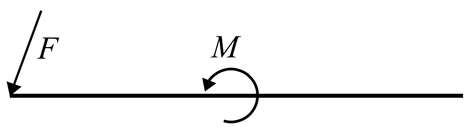
The free-body diagram:

Equilibrium statement:

Analytic solution:

Verification:

Final sketch:



Statical determinacy

The behaviour of structures essentially depends on the arrangement of supports and their number of constraints. All structures considered in the preceding lectures were simple ones with exactly three constraints arranged in a way that an equilibrium statement could be satisfied for any load. Support reactions for such structures can always be obtained from three independent equilibrium equations. The conditions above can be generalized for assemblies composed of more than one rigid bodies to set up a formal definition for statical determinacy.

Definition: An assembly is said to be ***statically determinate*** or ***isostatic*** if it is able to remain in equilibrium under arbitrary arrangement of loads and its reactions can uniquely be found from equilibrium equations.

Statical determinacy requires therefore two conditions to hold. If any of them is not met, the assembly is qualified differently.

Definition: An assembly is said to be ***statically overdeterminate*** or ***hypostatic*** if there exists an arrangement of loads for which the assembly cannot be balanced (i.e., there is no solution to the system of equilibrium equations because of a contradiction among them).

Definition: An assembly is said to be ***statically indeterminate*** or ***hyperstatic*** if there exists an arrangement of loads for which the assembly can be balanced in many different ways (i.e., there are several solutions to the system of equilibrium equations, or alternatively, reactions of the assembly cannot be determined uniquely just from equilibrium equations). Such structures frequently appear in engineering because of some its advantageous properties (numerical methods for their solutions will be discussed later in the subject 'Structural Analysis I').

Let us consider some examples for the cases above from the set of assemblies composed of a single body. Figure B2.1 lists statically determinate structures: assemblies shown to the left are all supported at three points with one constraint each. The arrangement of constraints prevents the body from translations or rotation: all three components of displacement of the body are restrained. The assembly in part (c) also has a total of three constraints and is still prevented from any displacement since the hinge at A disallows any translation but the roller at B also restrains the rotation about A. Part (d) illustrates the determinacy of a cantilever beam: the support disallows any translations or rotation; a single body with one fixed support is always statically determinate.

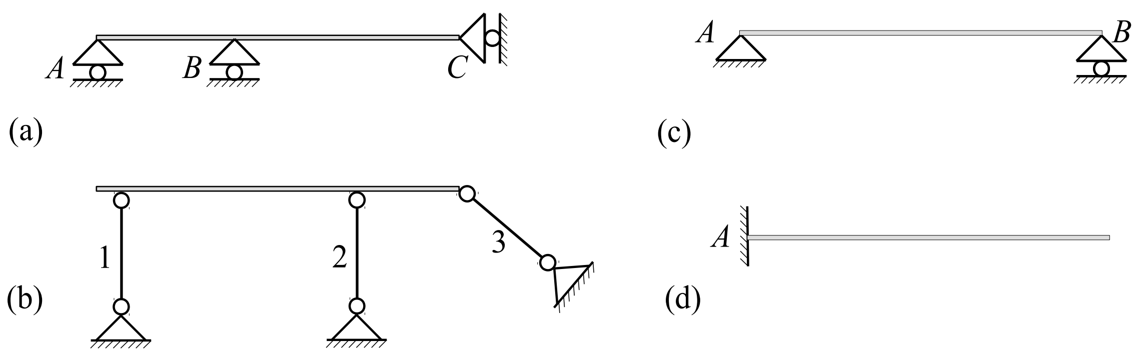


Figure B2.1: Statically determinate (isostatic) simple structures

Statically overdeterminate assemblies are shown in Figure B2.2: all of them are variants of structures (a), (b) and (c) of Figure B2.1 that are obtained by a special rearrangement of some of their supports. Obviously, this does not affect the number of constraints (it remains still three); nevertheless, there can be found a load with no possibility of equilibrium for each assembly. In the case (a), all three reactions are concurrent, which means that the beam can rotate about that point of intersection B (force F in the figure would generate a counterclockwise rotation). Viewing the same from the aspect of statical determinacy, the equilibrium of moments can never be restored if the moment of active loads about point B is nonzero. The assembly shown in part (b) is supported by three parallel links; thus, the beam can be translated in a direction perpendicular to them (force F generates a leftwards translation). In terms of Statics, the sum of components of reactions in a direction perpendicular to the links is always zero which makes horizontal equilibrium impossible under an active load with nonzero horizontal force resultant. Finally, the beam in part (c) is supported by a pin and a roller in a way that the line of action of the force in the roller passes through the pinned support, causing a degeneracy of supports similar to that found in part (a). Force F would result in a clockwise rotation about A; or in terms of equilibrium, active forces with a resultant not passing through the pin cannot be balanced. IMPORTANT: matching numbers of unknown reaction components and independent equilibrium equations ($u = e$) DO NOT imply statical determinacy of an assembly: that relationship constitutes only a *necessary but not sufficient* condition for the structure to be statically determinate (isostatic).

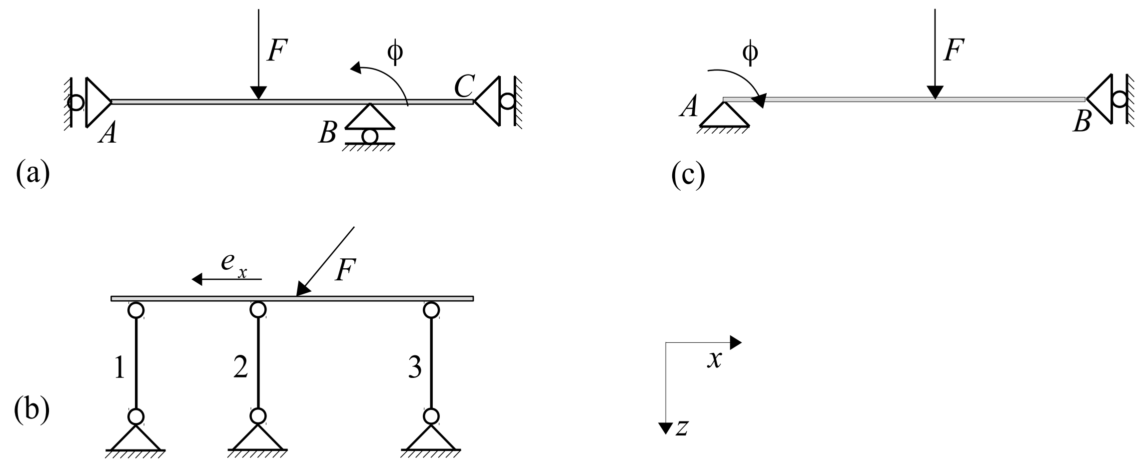


Figure B2.2: Statically overdeterminate (hypostatic) assemblies ($u = 3$)

Any simple assembly (i.e., composed of a single body) is statically overdeterminate if the number of its constraints is less than three. This is a *sufficient but not necessary* condition for the statical overdeterminacy. It was shown earlier that three components of displacement of a rigid body in a plane can only be blocked by at least three constraints (see the definition of constraint). In some kind of a dual approach it can also be seen that equilibrium conditions for a general plane force system (with arbitrary active loads) cannot be satisfied with less than three reaction components: an unrestrained displacement is just generated according to Newton's second law by unbalanced forces in the same sense. Figure B2.3 exemplifies assemblies with less than three constraints. All of them are drawn with components of allowed displacements and violated equilibrium conditions. It should be emphasized that a *single arrangement of loads* with no possibility of equilibrium is *already a proof* for statical overdeterminacy (it is often realized by finding an equation with no unknowns involved).

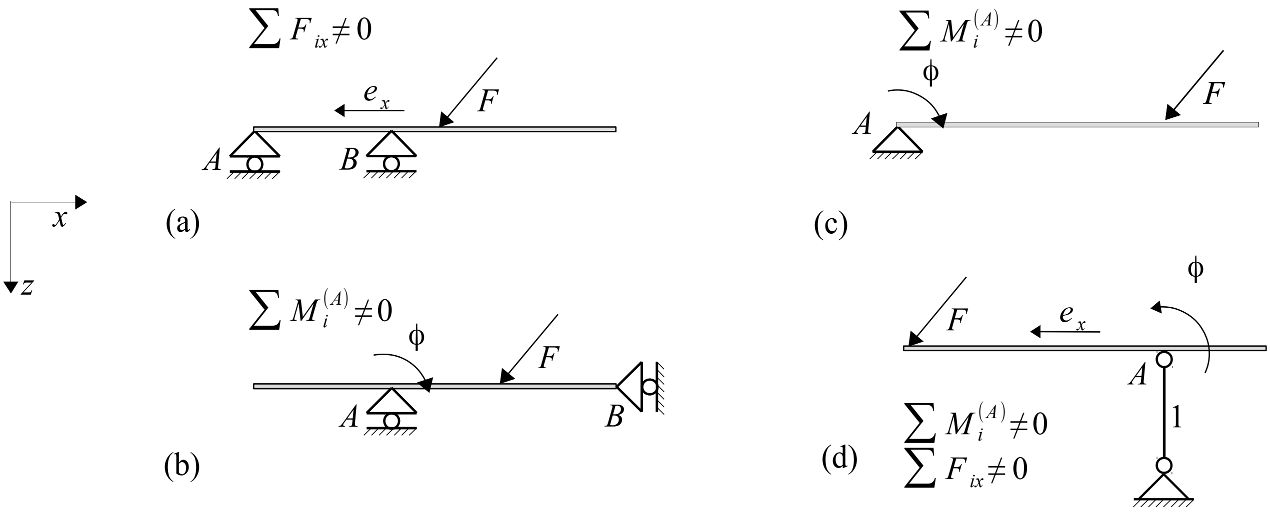


Figure B2.3: Statically overdeterminate (hypostatic) structures ($u < 3$)

Statically indeterminate structures are shown on the left hand side of Figure B2.4. Total number of constraints of supports amounts to four in examples (a), (b) and five in example (c). These assemblies are able to remain in equilibrium for arbitrary loading (thus, they can indeed be called 'structures') because they can be transformed into statically determinate structures simply by removing some of its constraints. On the right hand side of the figure one can see such determinate structures obtained by removal of constraints: any removed constraint is replaced by a corresponding reaction force component. Note here again that a roller is interpreted to be able to exert a reaction force also against lifting from the surface it is lying on. As we put no restrictions on the magnitude of active forces, these forces at removed constraints are also of arbitrary magnitude (if such a force can be balanced by the unknown ones, its n-tuple can as well). This means that the original structure could be in equilibrium with an arbitrary scaling of the force system; consequently, reactions of an indeterminate assembly cannot be found uniquely even in the lack of active forces. If the number of constraints exceeds that of independent equations, the assembly is sure to be statically indeterminate; however, as it will be seen later, it is not a necessary condition for statical indeterminacy.

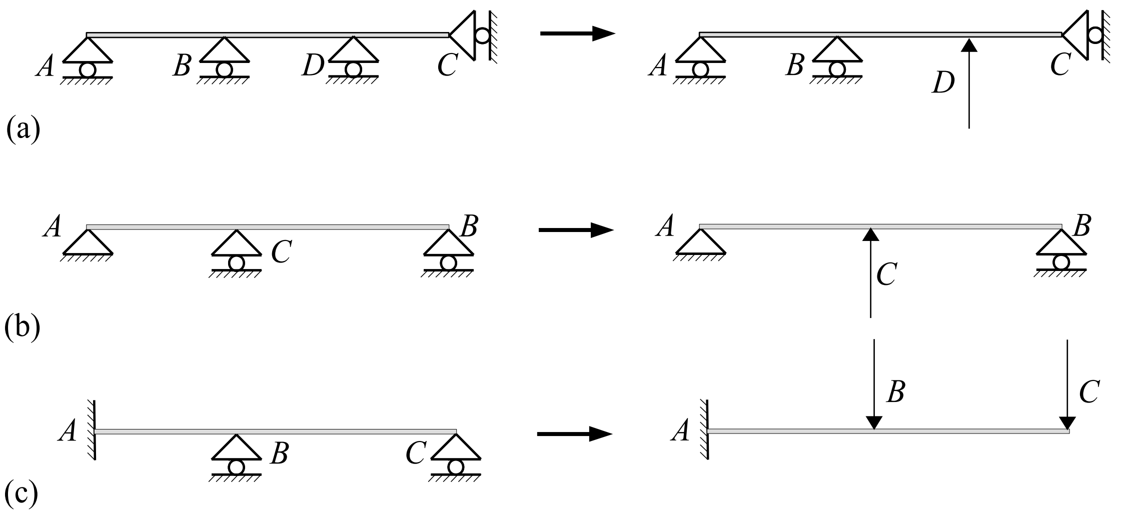


Figure B2.4: Statically indeterminate (hyperstatic) structures ($u > 3$)

On the left hand side of Figure B2.5 a beam supported by three parallel links is drawn. The replacement of one of the links by a vertical force S_2 yields a structure and a load that can be balanced on it. If the equilibrium exists for some vertical load S_2 then it still exists for an arbitrary n-tuple of it: the reactions cannot be found from equilibrium equations even if no active loads were present on the structure.

This assembly has already been qualified as statically overdeterminate: it means that an assembly can be statically in- and overdetereminate at the same time. It can be shown that if the number of constraints (reaction components) and that of independent equations are equal and the assembly is statically overdeterminate then it is statically indeterminate at the same time and vice versa (indeterminacy implies overdeterminacy as well).

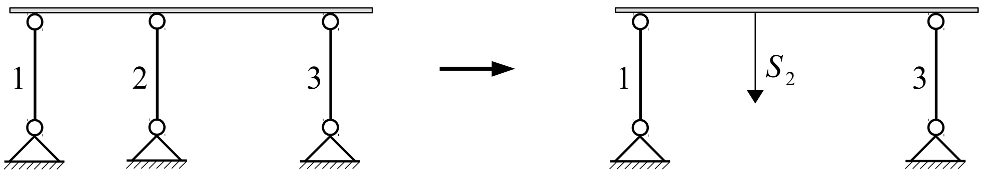
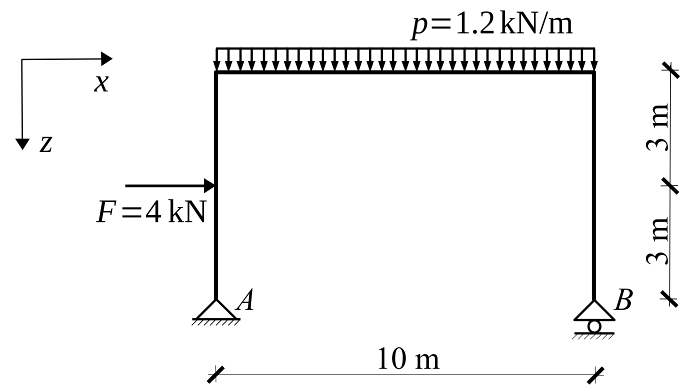


Figure B2.5: A statically indeterminate (hyperstatic) structure (u = 3)

Reactions of simple structures II.

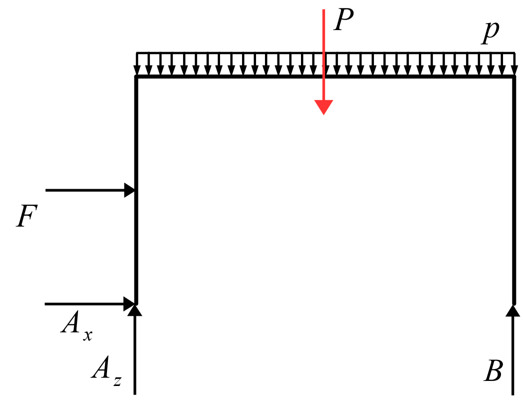
Example 1

Qualify the given frame with respect to statical determinacy. Find all support reactions.



Solution

Prepare the free-body diagram first:



The equilibrium statement must contain all active and passive forces exerted on the frame:
$(F,(p),A,B) \doteq 0$.

Let the statical determinacy of the assembly be analyzed before starting calculations. All active forces are balanced by a reaction A passing through a given point and another one (B) lying in a given line of action. Thus, forces exerted at point A and B can be given by two and one scalar variable, respectively: the total number of unknowns is three ($u = 3$). There are three independent equilibrium equations for a general force system in a plane ($e = 3$). The number of scalar unknowns and that of independent scalar equations are equal ($e = u$), making the necessary but not sufficient condition of statical determinacy be satisfied. Sufficiency can be tested by tracing the procedure of solution: its uniqueness can now be seen from the property that any unknowns can be found by systematic elimination of the others: vertical reaction components are obtained from moments written about points where remaining force components intersect; the horizontal component can be found from a resolution independent of any vertical forces. This means that the assembly is statically determinate indeed: those three equilibrium equations (with one variable each) can always be solved for unknown components with arbitrary active forces.

For the purposes of further calculation, let the distributed force be replaced by its resultant: its

line of action is vertical and bisects the span; its magnitude is $P=1.2\cdot 10=12\,\mathrm{kN}\,(\downarrow)$.
Following the method described above, component B is obtained from moments written about point A as follows:

$$\sum M_i^{(A)}:-4\cdot 3-12\cdot 5+B\cdot 10=0 \quad\rightarrow\quad B=7.2\,\mathrm{kN}\,(\uparrow)$$

The positive sign means here that the assumption for the sense of B was correct, it can then be confirmed by an upwards arrow. Likewise, vertical component of the reaction at hinge A is found from the balance of moments about B:, hiszen a B erőt korábban már meghatároztuk (B előjele az egyenletben azért negatív, mert felfelé, a z tengely negatív irányába mutat):

$$\sum M_i^{(B)}:-4\cdot 3+12\cdot 5-A_z\cdot 10=0 \quad\rightarrow\quad A_z=4.8\,\mathrm{kN}\,(\uparrow)$$

The result is positive again, showing that vertical component at A is directed upwards as assumed.
The horizontal reaction component at the same point is obtained from a resolution equation in the direction of x, not influenced by any unknown vertical components:

$$\sum F_{ix}:4+A_x=0 \quad\rightarrow\quad A_x=-4\,\mathrm{kN}\,(\leftarrow)$$

Here the result is found to be negative: it simply means that the arrow of original assumption must be reversed in the final sketch (it points to the left rather than to the right). Now an equation is set up and evaluated in order to check the recently obtained results. For that purpose one can use, e.g., a vertical resolution equation:

$$\sum F_{iz}:12-4.8-7.2=0$$

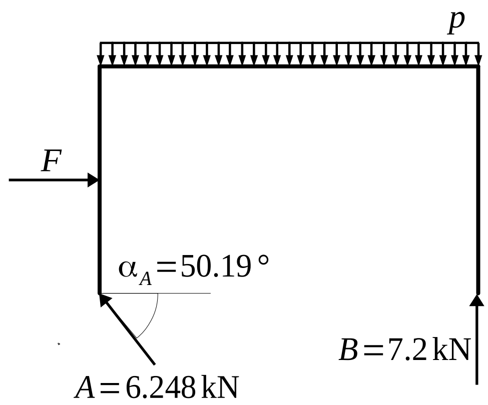
After the reactions having been checked, let the reaction force at A be given by magnitude and direction as follows:

$$A=\sqrt{(4^2+4.8^2)}=6.248\,\mathrm{kN}$$

$$\alpha_A=\arctan\frac{4.8}{4}=50.19\,°$$

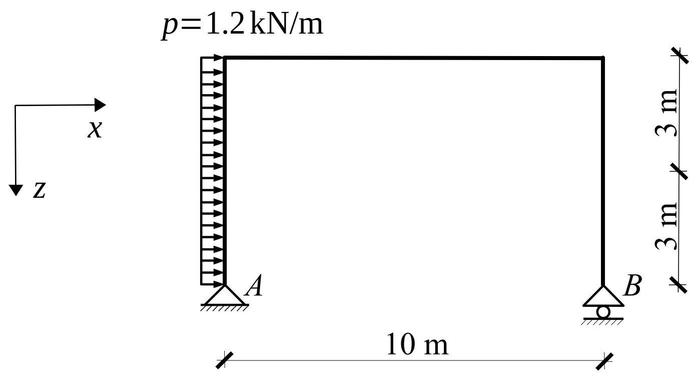
Finally, draw the isolated body with all forces exerted on it. Calculated scalar magnitudes wuth their units should be written out in details.

Final sketch::



Exercise 1

Determine the support reactions of the frame.



Solution

The free-body diagram:

Equilibrium statement:

Unknowns: $u =$ Number of independent scalar equations: $e =$

Before the analytic solution, determine lines of action for passive forces based on inspection.

Analytic solution:

.

.

.

Verification:

.

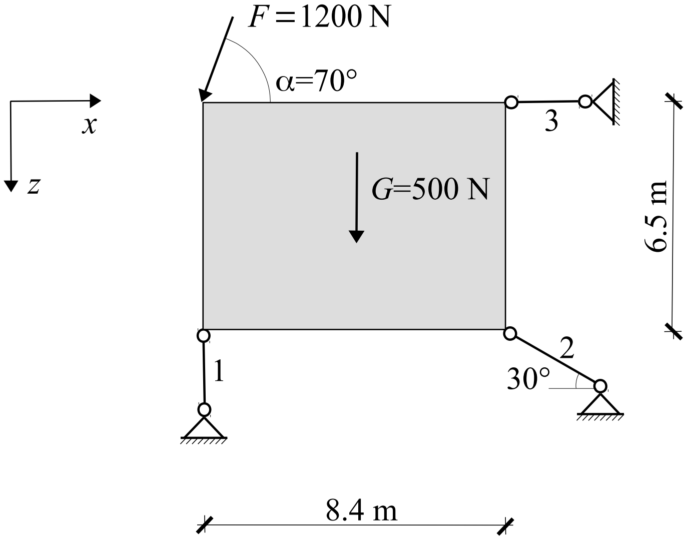
$A =$

$\alpha_A =$

Final sketch::

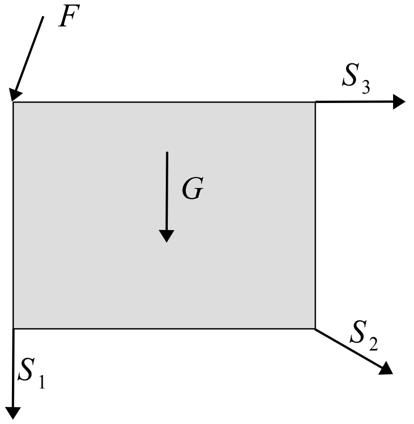
Example 2

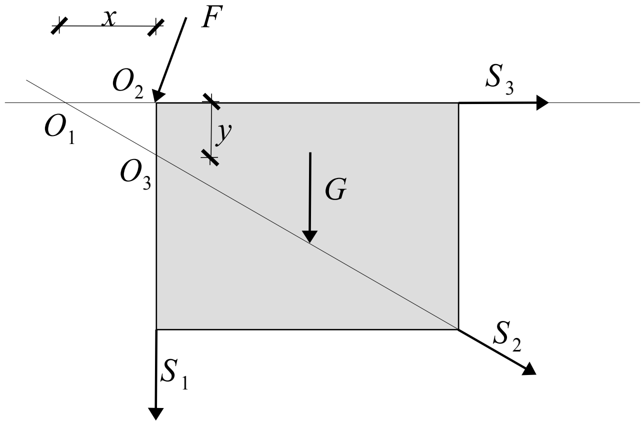
Determine the support reactions of the body shown.



Solution

Isolation is done first (see the figure to the left). Forces in links are always assumed to be tensile.





Equilibrium is maintained by two active and three passive forces (in links), so the statement reads $(F, G, S_1, S_2, S_3) \doteq 0$.

Each force in a link can be given by a signed scalar variable, so the number of unknowns is three ($u = 3$). It also equals the number of independent scalar equations since a general plane force system is dealt with ($e = 3$). In a problem of balancing with three forces in given lines of action, Ritter's method is applied: balances of moments about principal points (points of intersection of lines of remaining unknowns) are evaluated. Sometimes the calculation must be preceded by the determination of positions of principal points. distances x and y can be obtained from considerations on elementary geometry:

$$(x+8.4)\cdot\tan 30° = 6.5 \quad \rightarrow \quad x = 2.858\,\text{m}$$
$$y = x\cdot\tan 30° \quad \rightarrow \quad y = 1.650\,\text{m}$$

Once the principal points are already located, equations for moments are set up and solved:

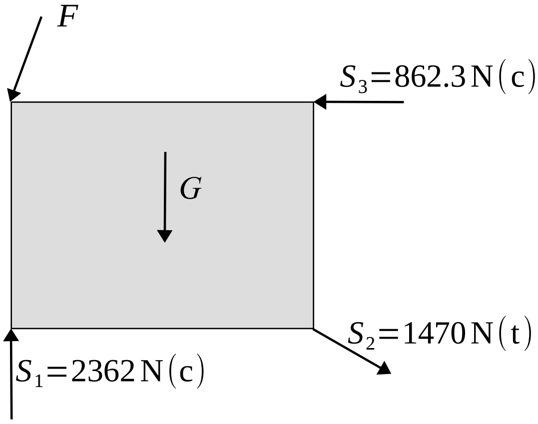
$\sum M_i^{(O_1)}: -1200 \cdot \sin(70°) \cdot 2.858 - 500 \cdot 7.058 - S_1 \cdot 2.858 = 0 \quad \rightarrow \quad S_1 = -2362\,\text{N}\,(\text{compression})$
$\sum M_i^{(O_2)}: -500 \cdot 4.2 + S_2 \cdot \cos(30°) \cdot 1.650 = 0 \quad \rightarrow \quad S_2 = 1470\,\text{N}\,(\text{tension})$
$\sum M_i^{(O_3)}: 1200 \cdot \cos(70°) \cdot 1.650 - 500 \cdot 4.2 - S_3 \cdot 1.65 = 0 \quad \rightarrow \quad S_3 = -862.3\,\text{N}\,(\text{compression})$

The second equation involves only a horizontal component of force S_2 since it was resolved into components at point O_3. The complete solution is based exclusively on moments, so both resolution is left for checking:

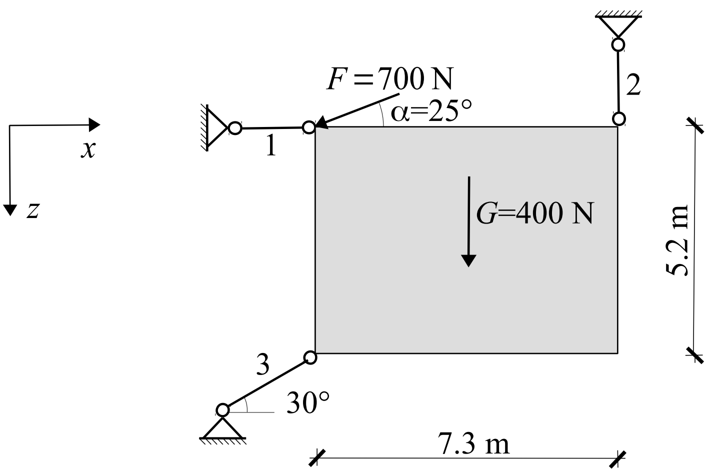
$\sum F_{ix}: -1200 \cdot \cos(70°) + 1470 \cdot \cos(30°) - 862.3 = 0.3 \approx 0$
$\sum F_{iz}: 1200 \cdot \sin(70°) + 500 - 2362 + 1470 \cdot \sin(30°) = 0.6 \approx 0$

Note that the obtained error in both cases is smaller by about three orders of magnitude than active or passive forces in the equations; the results are successfully checked. In the last step, the final sketch is drawn with all forces exerted on the body (in their computed sense), completed by the corresponding numeric results.

Final sketch::



Exercise 2
Determine the support reactions of the body shown.



Solution
The free-body diagram:

Equilibrium statement:
Number of independent scalar equations:
Unknowns:
Finding positions of principal points:

Analytic solution:

.

.

.

Equations for checking the results:

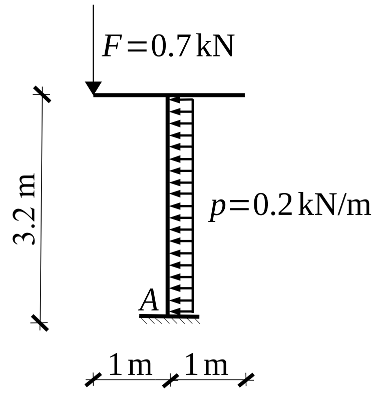
.

.

Final sketch::

Example 3

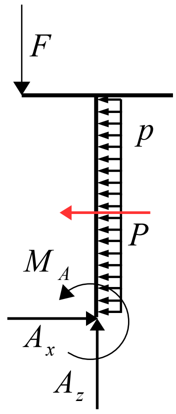
Find support reactions at point A. The reaction force may be given by components in the final sketch.



Solution

Let the structure be isolated first, then an equilibrium statement is formulated. In addition to active forces, the frame is exerted on by a reaction force A of unknown magnitude and direction and a torque M_A at the clamping.

The free-body diagram:



Equilibrium is stated as $(F, (p), A, M_A) \doteq 0$

The body is in equilibrium under the effect of a general plane force system, making possible to write three independent scalar equations. This matches the number of unknown scalar components (two from the reaction force, one from the torque). A cantilever structure can always be uniquely solved by three one-variable equations (two resolution equations and a balance of moments written about the support). Write and solve these equations:

$$\sum M_i^{(A)}: 0.7\cdot 1 + 0.2\cdot 3.2\cdot 1.6 + M_A = 0 \quad \rightarrow \quad M_A = -1.724\,\mathrm{kNm}\,(\curvearrowright)$$

$$\sum F_{iz}: 0.7 - A_z = 0 \quad \rightarrow \quad A_z = 0.7\,\mathrm{kN}\,(\uparrow)$$

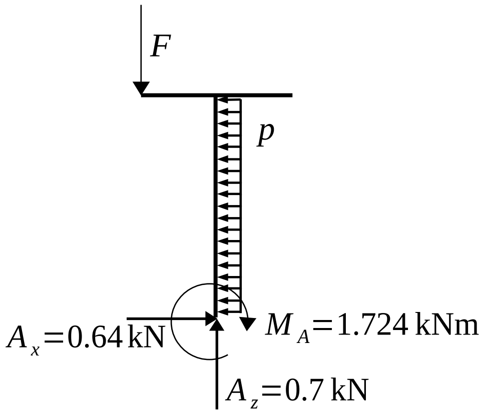
$$\sum F_{ix}: -0.2\cdot 3.2 + A_x = 0 \quad \rightarrow \quad A_x = 0.64\,\mathrm{kN}\,(\rightarrow)$$

The determination of components is always followed by a check, let it now be written as a balance of moments about the point of application of force F:

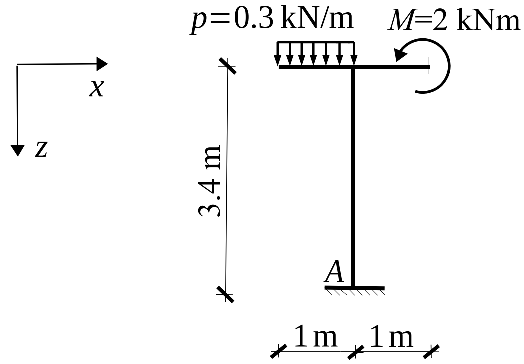
$$\sum M_i^{(F)}: -0.2\cdot 3.2\cdot 1.6 - 1.724 + 0.7\cdot 1 + 0.64\cdot 3.2 = 0$$

A successful check is then followed by presenting the final sketch with computed results:

Final sketch::

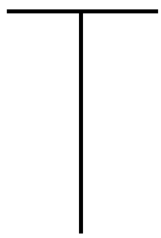


Exercise 3
Find support reactions at point *A*. The reaction force may be given by components in the final sketch.



Solution

The free-body diagram:

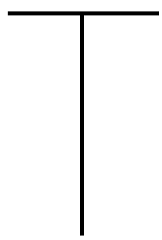


Equilibrium statement:
Analytic solution:

.

Verification:
.

Final sketch::



Compound structures I

Assemblies that are composed of more than one rigid bodies and are able to remain equilibrium under an arbitrary arrangement of loads are called *compound structures*. Bodies in such a compound are connected to each other or to the fixed neighbourhood by constraints. *Constraints* at connections to the neighbourhood of the compound are called *external* (these are the same kinds of supports learnt in simple assemblies), whereas constraints between bodies within the compound are called *internal*. Figures 13.1 – 13.4 illustrate statical models of most frequent structural types appearing in building and bridge construction.

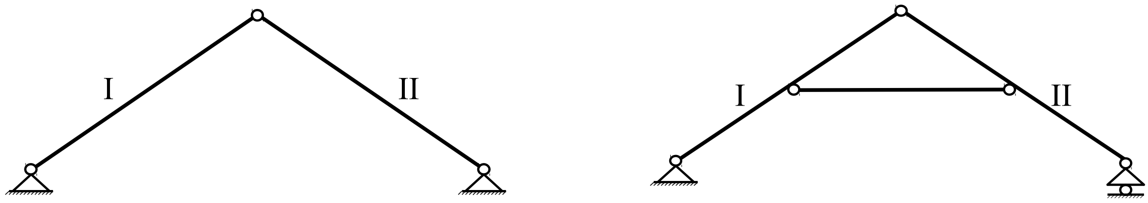


Figure B3.1 Statical models of roofs: single roof (to the left) and collar roof (to the right)

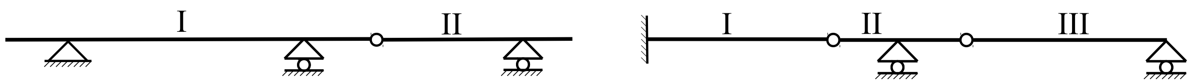


Figure B3.2 Compound beams (Gerber beams)

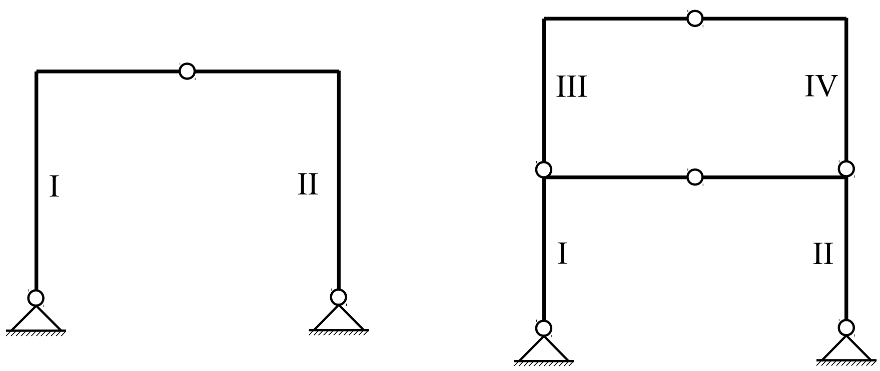


Figure B3.3 Compound frames

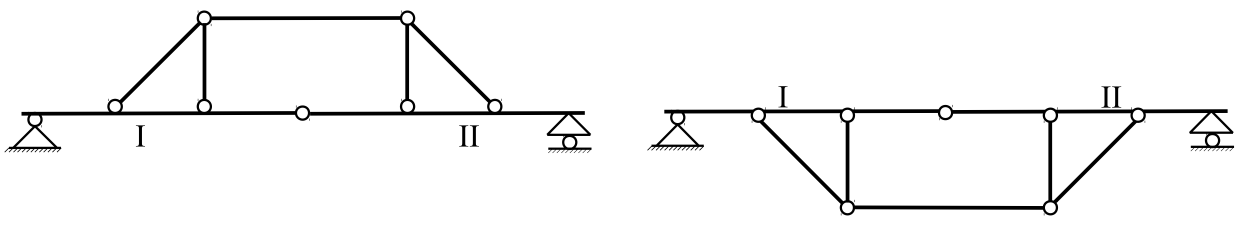


Figure B3.4 Queen post structure (to the left) and inverted queen post structure (to the right)

Calculation of reactions of compound structures

A compound structure can only be in equilibrium if all of its components are also in equilibrium. Steps of finding reactions of compound structures are analogous to that discussed in simple ones.

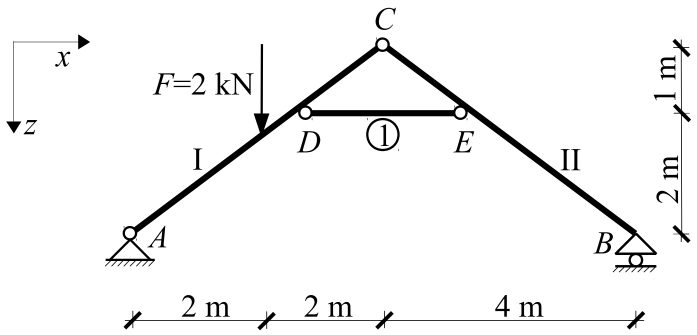
Firstly, rigid members are *isolated* and their individual *free-body diagrams* are drawn one by one, following the main rule of isolation of compounds: *multi-force members* (i.e., members acted upon by more than two forces or torques) *are only isolated*. It must be noted that forces are counted here by physical contacts and not by components: an external force, a two-component reaction at a hinge or a single-component reaction at a roller are all counted once. The final number of FBDs must be equal to the number of isolated bodies (those involved in the analysis of equilibrium).

Secondly, *equilibrium statements* are made for each FBD: an individual statement contains all active and passive forces and torques acting upon the body in case. It means that if the equilibrium of the entire structure is also stated, it can contain external active and passive forces and torques only, that is, no internal reactions can appear in it. Equilibrium statements of bodies that have a nonzero extension ('finite bodies' for short) are related to a general plane force system, so they imply three independent scalar equations each. Isolation of joints results in a statement of equilibrium of a concurrent plane force system that makes possible to write two independent scalar equations only. If equations are written based on the statement of equilibrium of the complete structure, there can also be found three independent ones but those are not independent of other equations derived from the individual statements. Analytic solution of the problem means the setup and solution of the system of equilibrium equations (in contrast to some graphical and grapho-analytical solution methods that are not dealt with in details here). In calculations by hand, it is still aimed at writing and solving one-variable equations one after each other. Unlike in simple structures, however, equations that are completely free of all but one unknown reactions are not always possible to be found in compounds: one can be forced either to use recently obtained values in finding further unknowns or even to give up looking for one-variable equations and solve rather a system in two variables. Since in compounds it is not obvious which is the body the calculation should be started at, first a comparison between the numbers of equations and unknowns is made for each body: equations are normally written first for a statement involving not more unknowns than the number of independent equations that can be written for the same. It will be seen later that not only individual statements but a statement of equilibrium of an arbitrary set of connected components (e.g., the complete structure) can serve as a point of departure.

The following task is to *set up and solve equations* sucessively for internal and external reaction components. As in earlier problems, checking of results is made afterwards by (an) equation(s) not yet used (in compound structures it needs special care to find equations that perform a real check ratheer than a formal one, details will be given later). The last event in the preocedure of solution is still the preparation of a *final sketch* which follows the structure of FBDs as earlier: each isolated member has to be drawn separately with active and passive (internal and external) forces or torques exerted on it.

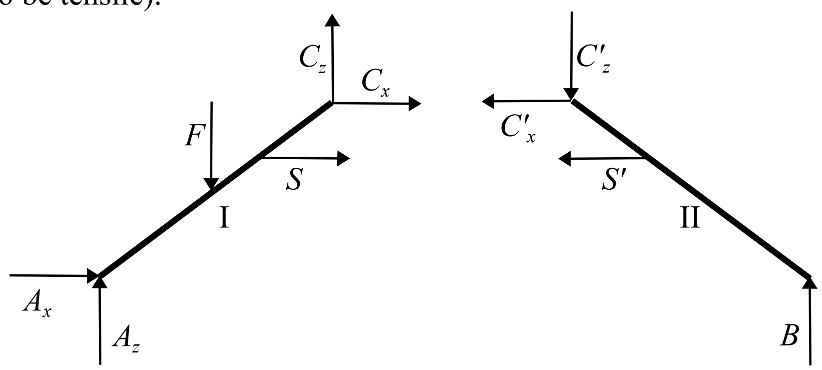
Statical models of roof structures are shown in Figure B3.1. The main difference between their behaviour is that vertical loads result in horizontal components of reaction in the case shown on the left hand side only. Examples for both structures will be given below.

Example 1
Determine external and internal reactions (calculated forces can be given by components).

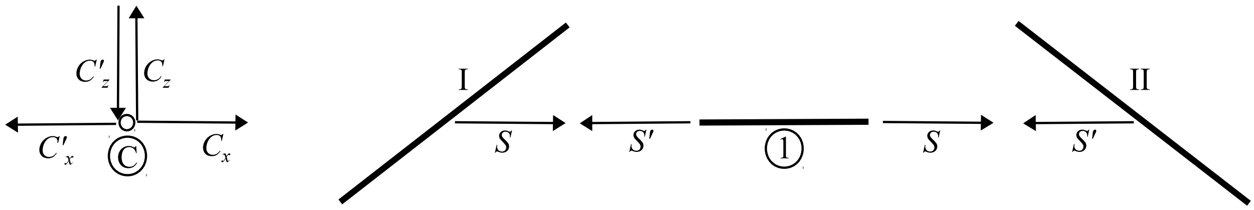


Solution

The solution is started by the isolation: multi-force members are isolated only. Body I is a four-force member since it has theree physical neighbours in addition to the active force F: it needs therefore isolation. Body II is a three-force member even without active load on it, it should also be isolated. Any of the internal pin-joints (A, C, D, E) are two-force members only and the same holds for member 1 (identified as an internal link or bar member): none of them will be isolated. It is only left to decide on the number and type of external and internal reaction components to be assumed in the (two) FBDs.
Support A and B is a hinge and roller, respectively, so reaction components are assumed there as earlier (arrowheads can be drawn in either sense). Point C represents a hinged connection where bodies I and II exert forces of unknown magnitude and direction upon each other: it implies two force components to be drawn in *both* FBDs according to Newton's third law. The principle of action and reaction has an immediate consequence that internal reactions appear always in pairs of opposite orientation: a force exerted on body I by body II has the same magnitude and opposite sense compared to the force exerted on body II by body I. Although there is absolutely no algebraic difference between those forces, one of them is normally distinguished by a prime in FBDs, equilibrium statements and equations in order to prevent confusion in assumed senses of arrows. Pin-joints (hinges) D, E, as well as member 1 are not part of the FBDs but they still represent a series of physical connection. With reference to an earlier experience of dealing with links, they are simply replaced by a pair of forces S and S' following the direction of the link (still assumed to be tensile).



Now it will be shown why two-force members are not isolated. If an object is kept in equilibrium by two forces (say, P and Q), then equilibrium equations prove that they must have equal magnitude, opposite sense and must lie in the *same line of action*. If that object is connected to another one, e.g., at a point where P is applied, then the adjacent object is acted upon by the force P' which has (this time in the sense of Newton's third law) the same magnitude, opposite sense and the same line of action as P. It means therefore that any information about Q directly applies to P', so it is simply unnecessary to assume them independently in an isolation.
A two-force member can either have a finite extension or not. A pin-joint, idealized as a point, can be in equilibrium under the action of a pair of forces of arbitrary direction: it needs an assumption of two force components. A link is, on the contrary, a finite body where the point of application of the two forces specifies a unique line of action for both forces: in a FBD it appeas therefore as a force unknown up to one scalar only.



After the isolation is complete, write equilibrium statements for each separate body as well as for the entire structure.

	e	u	new u
I $:(F,A,C,S) \doteq 0$	3	5	5
II $:(B,C',S') \doteq 0$	3	4	1
Str$:(F,A,B) \doteq 0$	(3)	3	

Next to the statements, numbers of independent equations (e), unknown reaction components (u) are shown. In addition to them, unknowns that have not appeared before (with or without a prime) are also counted as new unknowns. Number three is bracketed next to the statement of the complete structure because those equations are not independent of previous ones.
In order to find an appropriate sequence of one-variable equations, the number of independent equations and unknowns should be compared; calculation can normally be started with a member where these numbers match. In the current problem, equilibrium statements of bodies I and II both involve more unknowns than the number of equations but those are equal for the complete structure. Let equations be written therefore based on the overall equilibrium of the roof. If the structure is dealt with as a whole, it can be modelled as an assembly isolated from its external supports only: its equilibrium statement must not contain any internal reactions (this statement can also be obtained as a kind of addition of statements of isolated bodies where an internal component and its primed pair automatically cancel each other).
Equilibrium equations in a compound structure can only be written with an explicit reference to the structural part they are written about (e.g., any isolated members: 'I', 'II' or the complete structure: 'Str'). Now one can start by finding external reactions exactly as in simple structures:

$\text{Str} \sum M_i^{(A)}: -2 \cdot 2 + B \cdot 8 = 0 \quad \rightarrow \quad B = 0.5 \text{ kN} (\uparrow)$
$\text{Str} \sum M_i^{(B)}: 2 \cdot 6 - A_z \cdot 8 = 0 \quad \rightarrow \quad A_z = 1.5 \text{ kN} (\uparrow)$
$\text{Str} \sum F_{ix}: A_x = 0$

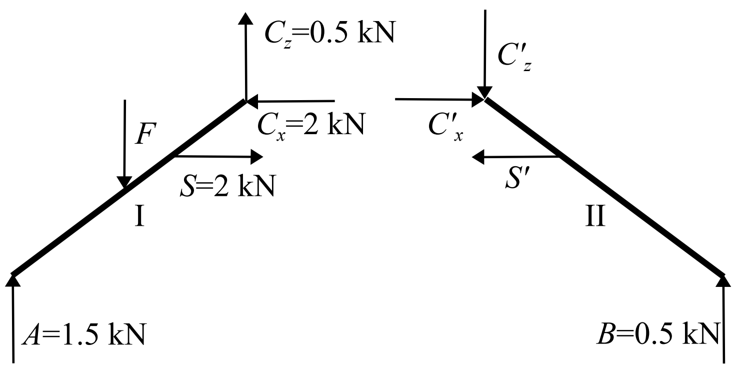
As soon as the external reactions are all found, the number of unknowns in the statement of member I is reduced to 3 from 5; the three scalar equations for member I looks sufficient for their determination (note that some results of former calculations must be used here):

$\text{I} \sum M_i^{(C)}: 2 \cdot 2 + S \cdot 1 - 1.5 \cdot 4 = 0 \quad \rightarrow \quad S = 2 \text{ kN (t)}$
$\text{I} \sum F_{ix}: C_x + 2 = 0 \quad \rightarrow \quad C_x = -2 \text{ kN } (\leftarrow) \quad \rightarrow \quad C_x' = 2 \text{ kN } (\rightarrow)$
$\text{I} \sum F_{iz}: 2 - 1.5 - C_z = 0 \quad \rightarrow \quad C_z = 0.5 \text{ kN} (\uparrow) \quad \rightarrow \quad C_z' = 0.5 \text{ kN} (\downarrow)$

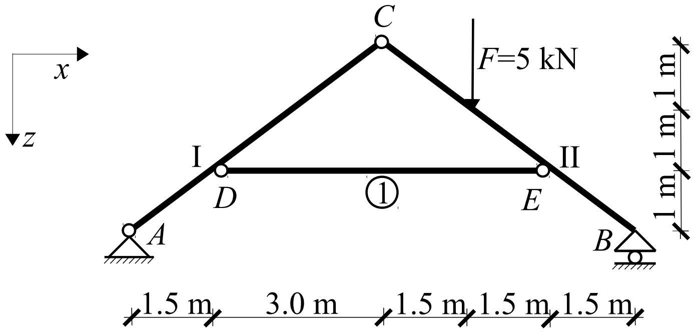
The horizontal component of internal reaction C is found to be negative, showing that its assumed sense must be reversed later in the final sketch. Magnitudes (and signs) of internal components C' are exactly the same as of C but their arrows are of different sense (if they were assumed correctly, they are both confirmed as a pair of opposite arrows, otherwise they are both reversed). Because of this algebraic equivalence, internal components are sufficient to be displayed with a single value but two arrows.
At this point, all external and internal reactions became known. The results can be checked by equations assembled on the basis of equilibrium statement of member II yet not used at all:

$\text{II} \sum F_{ix}: 2 - 2 = 0$
$\text{II} \sum F_{iz}: -0.5 + 0.5 = 0$

After checking the results, it is left only to prepare a final sketch: each isolated member should be drawn with all active and passive forces acting on it with their true senses. The figure should also contain the values of reactions; values of internal reaction components are sufficient to be given only once.



Exercise 1
Determine external and internal reactions (calculated forces can be given by components).



Solution

Free-body diagrams:

Equilibrium statements:

e u new u

I :

II :

Str:

Analytic solution:

$\text{Str} \sum M_i^{(A)}$:

$\text{Str} \sum M_i^{(B)}$:

$\text{Str} \sum F_{ix}$:

$$\mathrm{I}\ \sum M_i^{(C)}:$$

$$\mathrm{I}\ \sum F_{ix}:$$

$$\mathrm{I}\ \sum F_{iz}:$$

Check:

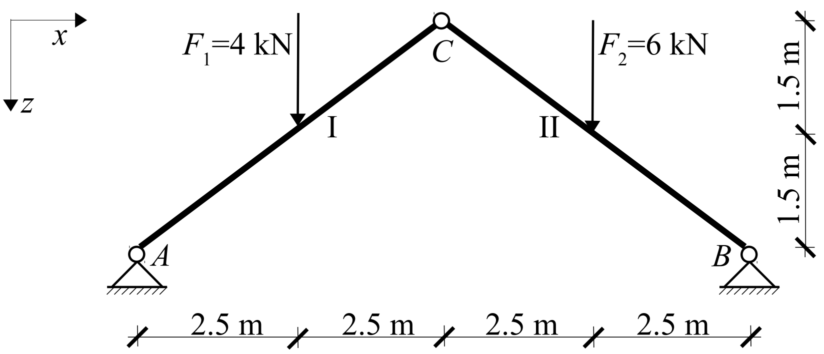
$$\mathrm{II}\ \sum F_{ix}:$$

$$\mathrm{II}\ \sum F_{iz}:$$

Final sketch:

Example 2

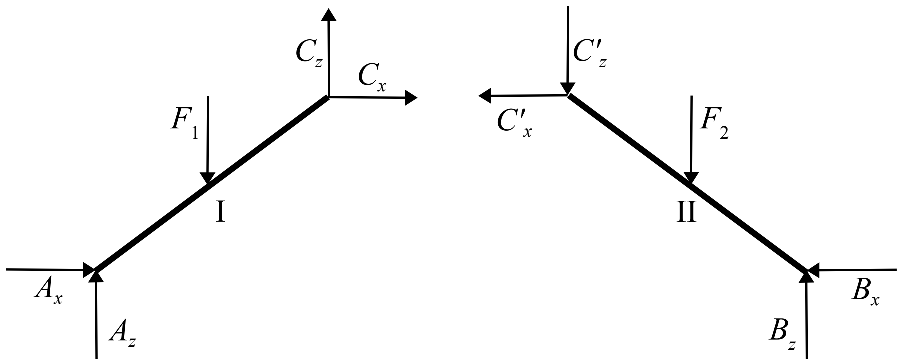
Determine external and internal reactions (calculated forces can be given by components).



Solution

The solution starts with isolation. All *multi-force members* (i.e., those acted upon by *more than*

two forces) should be displayed separately (members with *less than three* forces are *not* isolated). Body I is kept in equilibrium by the active force *F*, external reaction *A* and internal reaction *C*. Since there are only two forces exerted on internal hinge *C*, they should exactly be negatives of each other. Body II is acted upon by the negative of internal reaction *C*, the external reaction *B* at the pinned support to the right.
The free body diagams are as follows:



After the structure having been isolated, equilibrium of each separate member as well as of the entire structure should be stated. Next to the equilibrium statements, the number of independent equations and of unknown reaction components pertaining to the given force system should be written. In addition, the number of new unknowns (that is, not appearing in any previous statements of equilibrium) are also displayed:

	e	u	new u
I $:(F_1, A, C) \doteq 0$	3	4	4
II $:(F_2, B, C') \doteq 0$	3	4	2
Str$:(F_1, F_2, A, B) \doteq 0$	(3)	4	

Independent statements of equilibrium imply six scalar equations which matches the number of unknowns in the problem: the necessary condition of statical determinacy is satisfied again. Unfortunately; however, the number of scalar equations is always less than the number of reaction components in each statement. This property would normally require a two-variable system of equations to be written and solved but special gemometry of the problem still allows the solution of a one-variable equation. Based on the equilibrium of the entire structure, moments can be written about one of the external supports: it will only contain one of the four unknown components:

$$\text{Str} \sum M_i^{(A)}: -4\cdot 2.5 - 6\cdot 7.5 + B_z\cdot 10 = 0 \quad \rightarrow \quad B_z = 5.5 \text{ kN}(\uparrow)$$
$$\text{Str} \sum M_i^{(B)}: +4\cdot 7.5 + 6\cdot 2.5 - A_z\cdot 10 = 0 \quad \rightarrow \quad A_z = 4.5 \text{ kN}(\uparrow)$$

As soon as vertical components of the external reactions became known, number of unknowns is reduced to three in both bodies I and II. Solving equations for, e.g., body I, the horizontal component of external reaction at *A*, as well as both components at *C* are obtained as follows:

$$\text{I} \sum M_i^{(C)}: 4\cdot 2.5 - 4.5\cdot 5 + A_x\cdot 3 = 0 \quad \rightarrow \quad A_x = 4.167 \text{ kN}(\rightarrow)$$
$$\text{I} \sum F_{ix}: 4.167 + C_x = 0 \quad \rightarrow \quad C_x = -4.167 \text{ kN } (\leftarrow) \quad \rightarrow \quad C'_x = 4.167 \text{ kN } (\rightarrow)$$
$$\text{I} \sum F_{iz}: 4 - 4.5 - C_z = 0 \quad \rightarrow \quad C_z = -0.5 \text{ kN}(\downarrow) \quad \rightarrow \quad C'_z = 0.5 \text{ kN}(\uparrow)$$

The last unknown, B_z can then be found, e.g., from moments about hinge *C* based on the equilibrium of body II.

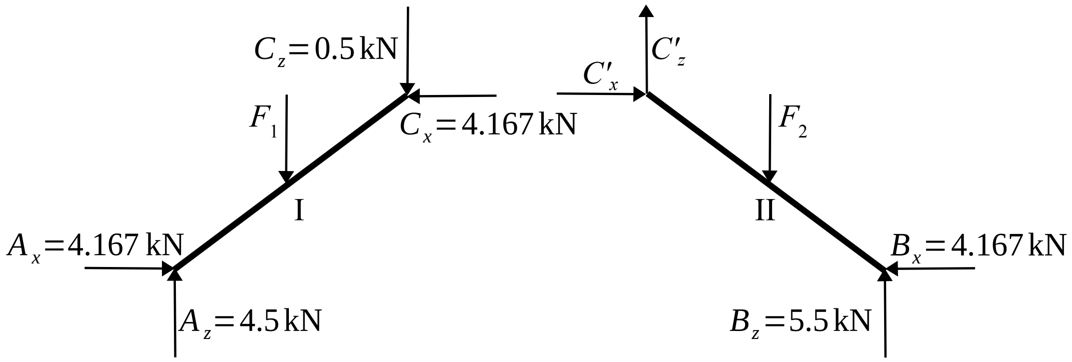
II $\sum M_i^{(C)}: -6\cdot 2.5+5.5\cdot 5-B_x\cdot 3=0 \quad \rightarrow \quad B_x=4.167$ kN ($\leftarrow$)

Check the results based on the equilibrium of body II using two resolution equations:

II $\sum F_{ix}: 4.167-4.167=0$

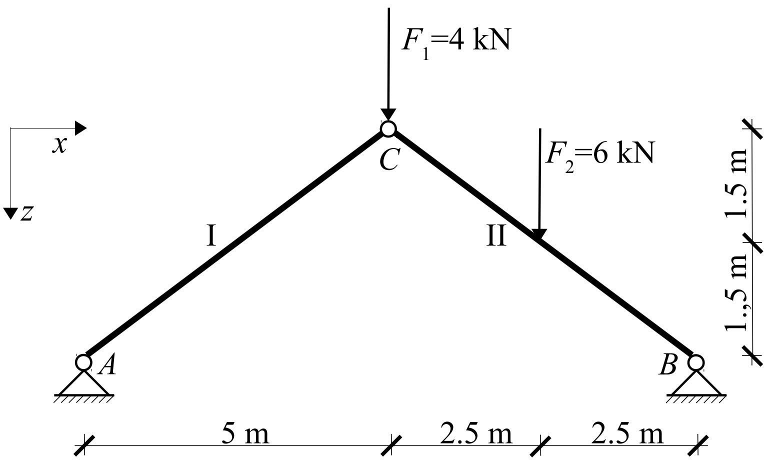
II $\sum F_{iz}: 6-5.5-0.5=0$

Warning: in order to avoid confusion of arrowheads, all equations are written first, for convenience, still with the original assumptions of unknowns; calculated results should be substituted into the equation together with their signs afterwards. Arrows are only modified when a successful check has been made. In our case, both components of the internal reaction must be reversed in the final sketch because of their negative signs; including arrows of force C' exerted on body II. Note that negative signs are not necessary to be shown in the final sketch, since arrows give the senses of forces uniquely. The final sketch



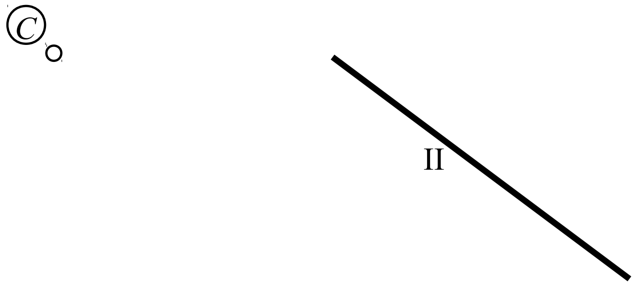
Exercise 2

Determine external and internal reactions (calculated forces can be given by components).



Solution

Free-body diagrams:



Equilibrium statements:

e u new u

II :

C :

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

$\sum$

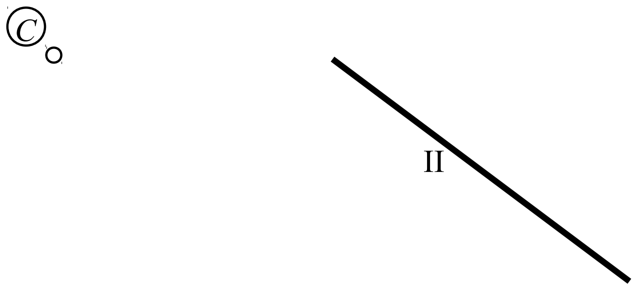
$\sum$

Check:

$\sum$

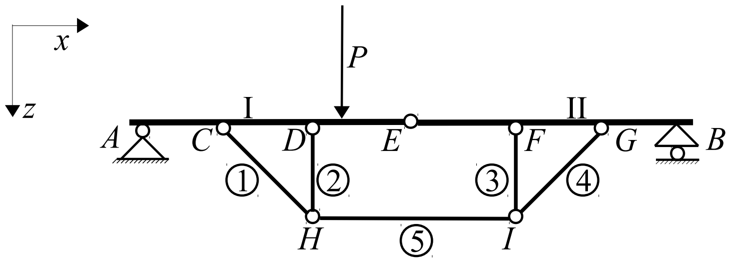
$\sum$

Final sketch:



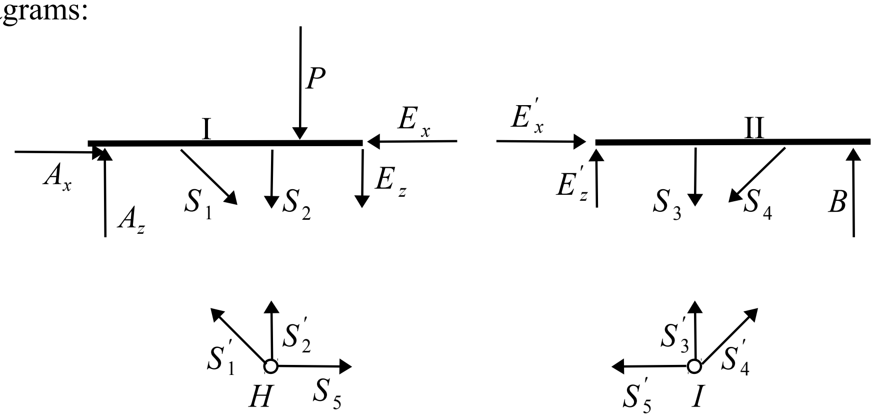
Example 3

Isolate the structure (draw the minimum necessary number of FBDs). Write the corresponding equilibrium statements and count unknown components as well as independent scalar equations.



Solution

The solution starts with isolation. All *multi-force members* (i.e., those acted upon by *more than two* forces) should be displayed separately. Beam I is acted upon by five forces: active force P, external reaction at A and three internal reactions: member forces S_1, S_2 and a force E at the central hinge. Beam II is kept in equailibrium by four forces: external reaction at B and three internal reactions: member forces S_3, S_4 and a force E' at the central hinge (the hinge itself is a two-force member and is not isolated, that is why forces E, E' on beams I and II are considered to be pairs in a single contact. Hinges H and I; however, need isolation because they are both three-force members.
Free-body diagrams:



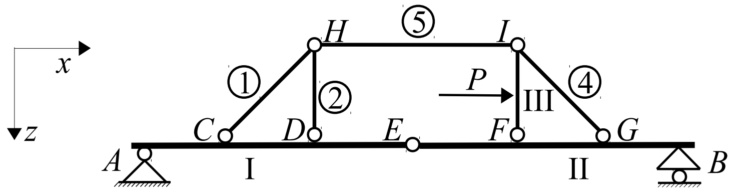
When the isolation is complete, write equilibrium statements for each separate body as well as for the entire structure. Count independent scalar equations and scalar unknowns.

	e	u	new u
I: $(P, S_1, S_2, E, A) \doteq 0$	3	6	6
II: $(S_3, S_4, E', B) \doteq 0$	3	5	3
H: $(S'_1, S'_2, S_5) \doteq 0$	2	3	1
I: $(S'_3, S'_4, S'_5) \doteq 0$	2	3	0
Str: $(P, A, B) \doteq 0$	(3)	3	

The number of independent scalar equations as well as of unknowns is both equal to ten in the problem ($e = u$): the necessary condition of statical determinacy is satisfied.

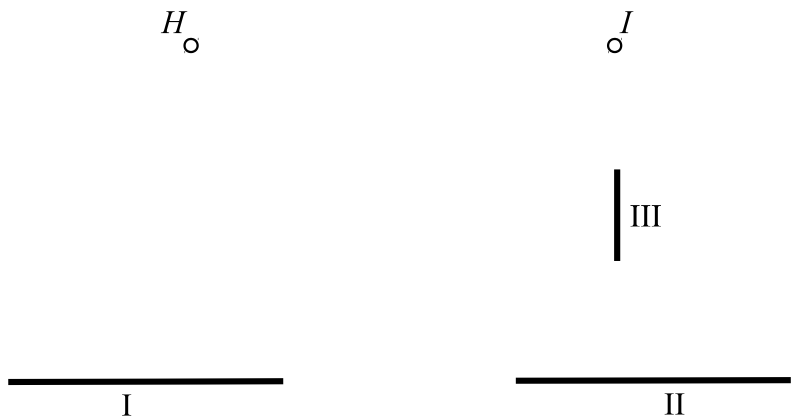
Exercise 3

Isolate the structure (draw the minimum necessary number of FBDs). Write the corresponding equilibrium statements and count unknown components as well as independent scalar equations.



Solution

Free-body diagrams:



Equilibrium statements:

Compound structures II

A compound assembled of at least one *fixed* and one *suspended* part is called a *Gerber beam* (*Gerber's beam*) after a patent of a german engineer Heinrich Gerber (1866). There is some discrepancy among terms used worldwide by engineers: fixed part is often referred to as simply supported beam or cantilever beam (depending on its structural design), while suspended parts are commonly called *drop-in* beams. Likewise, the whole compound is commonly called also *multispan hinged beam*. Among several parts of a compound, a structural part (body) is called fixed if it can be balanced merely by its external supports (i.e., without any contribution of other members) in a statically determinate (isostatic) fashion. Despite that, a suspended part itself of a structure would be statically overdeterminate (hypostatic): it is able to carry loads when supported to other parts of the structure. Notice that a Gerber beam can have more than one suspended parts. Numeric solution of a Gerber beam is reduced to a series of solutions of simple beams. Internal reaction components obtained on the suspended part are applied to the fixed part as external loads, making therefore possible to calculate external reactions of the suspended part as has been done in simple structures.

The structure shown in Figure B4.1 is a Gerber beam with part I and II being the fixed and suspended part, respectively. Body I is supported by three constraints altogether: it corresponds to the usual supporting system of a simple statically determinate beam. Body II has a single external constraint only which would result in a statical overdeterminacy. However, this structural unit is also connected to the fixed part by two internal constraints, making body II able to carry loads as well.

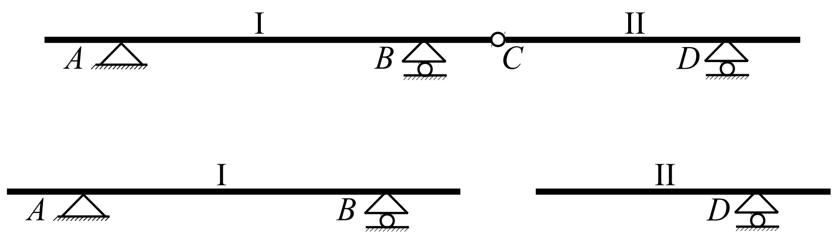


Figure B4.1 A Gerber beam (above) with its fixed (I) and suspended (II) parts (below); only external supports are displayed.

The structure shown in Figure B4.2 is also a Gerber beam, where bodies II and I are the fixed and suspended parts of the structure. Body II with its three external constraints correspond to a completely supported cantilever beam. Body I is supported externally by only one constraint, hence it would be statically overdeterminate in itself. This suspended part, however, is attached to the fixed part through a pin-joint: together with the support *A*, a system of three constraints provides the structure with a load-bearing capacity.

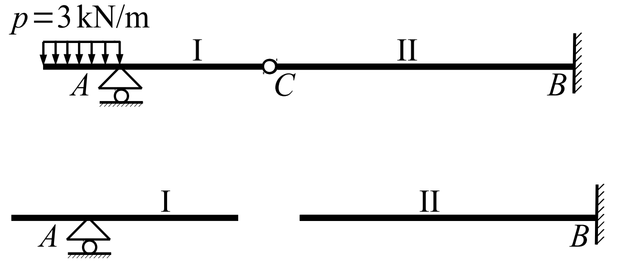
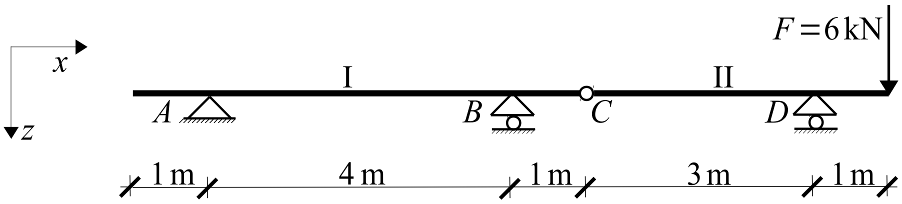
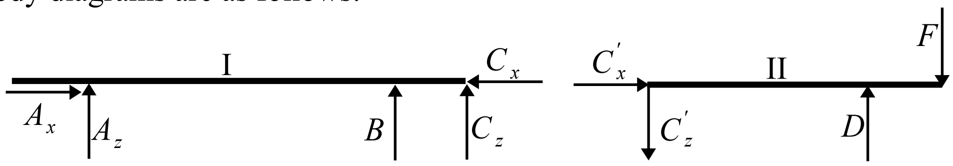


Figure B4.2 A Gerber beam (above) with its suspended (I) and fixed (II) parts (below); only external supports are displayed.

Example 1
Determine external and internal reactions of the Gerber beam (reactions can be given by components).



Solution
The solution starts with isolation. All *multi-force members* (i.e., those acted upon by *more than two* forces) should be displayed separately (members with *less than three* forces are *not* isolated). The fixed part (I) of the Gerber beam is kept in equilibrium by external reactions A, B and internal reaction C. There are only two forces exerted on hinge C, therefore it need not be considered separately. Finally, the suspended part (II) is acted upon by active force F, internal reaction C' and external reaction D.
The free body diagrams are as follows:



After the structure having been isolated, equilibrium of each separate member as well as of the entire structure should be stated. Next to the equilibrium statements, the number of independent equations and of unknown reaction components provided by the given force system should be written. In addition, the number of new unknowns (that is, not appearing in any previous statements of equilibrium) are also displayed:

	e	u	new u
I: $(A, B, C) \doteq 0$	3	5	5
II: $(F, C', D) \doteq 0$	3	3	1
Str: $(F, A, B, D) \doteq 0$	(3)	4	

Independent statements of equilibrium imply six scalar equations which matches the number of unknowns in the problem: the necessary condition of statical determinacy is therefore met. In order to find an appropriate order (and type) of equilibrium equations to be written down, numbers of independent scalar equations and unknown reaction components are compared for each statement. As a general rule, calculation is worth being started at the member where these numbers are equal. The solution of Gerber-type structures is always started at the suspended part. (if there are more than one suspended parts, calculation starts at the member having equal number of scalar equations and unknowns).
Let the solution be started at the suspended part:

$$\mathrm{II}\sum M_i^{(C)}: -6\cdot 4 + D\cdot 3 = 0 \quad \rightarrow \quad D = 8\ \mathrm{kN}(\uparrow)$$

$$\mathrm{II}\sum F_{ix}: C'_x = 0 \quad \rightarrow \quad C_x = 0$$

$$\mathrm{II}\sum F_{iz}: 6 + C'_z - (8) = 0 \quad \rightarrow \quad C'_z = 2\,\mathrm{kN}(\downarrow) \quad \rightarrow \quad C_z = 2\,\mathrm{kN}(\uparrow)$$

By determining both components of internal reaction at C, the original number five of unknowns at body I reduces to three, making three other scalar equations to be sufficient for obtaining a unique solution.

$$\mathrm{I}\sum M_i^{(A)}: B\cdot 4 + (2)\cdot 5 = 0 \quad \rightarrow \quad B = -2.5\ \mathrm{kN}(\downarrow)$$

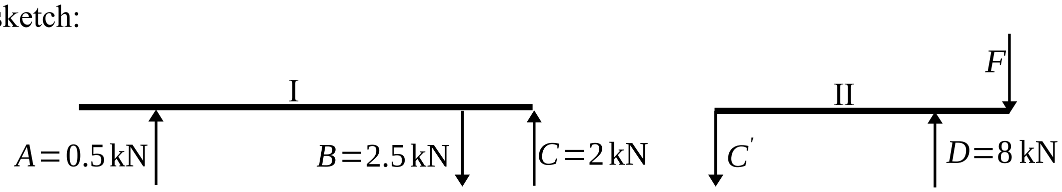
$$\mathrm{I}\sum F_{ix}: A_x = 0$$

$$\mathrm{I}\sum M_i^{B}: -A_z\cdot 4 + (2)\cdot 1 = 0 \quad \rightarrow \quad A_z = 0.5\,\mathrm{kN}(\uparrow)$$

Now all external and internal reactions are determined; let (for instance) a vertical resolution for the complete structure be written down in order to check the equilibrium:

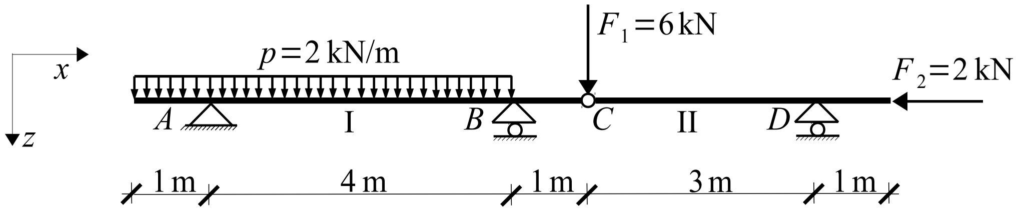
$$\mathrm{Str}\sum F_{iz}: 6 - 0.5 - (-2.5) - 8 = 0$$

After the correctness of the results having been verified, a final sketch of results should be made. Precisely it means a repetition of free body diagrams with adjusting all arrows of internal and external reaction components according to their calculated senses. The final sketch must also be provided with all calculated scalar values.
Final sketch:



Exercise 1

Determine external and internal reactions of the Gerber beam (reactions can be given by components).



Solution

Isolation:

Equilibrium statements:

e u new u

I:

C:

II:

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

$\sum$

$$\sum$$
$$\sum$$
$$\sum$$
$$\sum$$

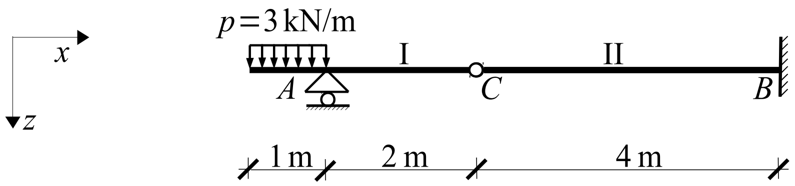
Check:

$$\sum$$

Final sketch:

Example 2

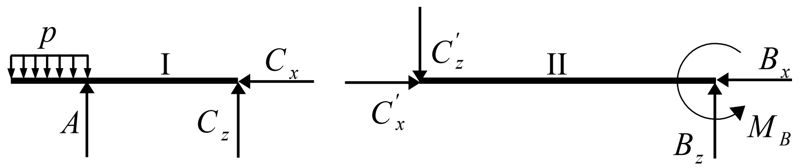
Determine external and internal reactions of the Gerber beam (reactions can be given by components).



Solution

Let the solution be started again by the isolation. All structural units acted upon by more than two forces (torques) should be considered separately. Body I is in equilibrium under the action of active distributed load p, external reaction A and internal reaction C. Since there are only two forces exerted on internal hinge C, they should exactly be negatives to each other. Body II is acted upon by the negative of internal reaction C, the external reaction B and torque M_B at the fixed support.

The free body diagams are as follows:



After the structure having been isolated, equilibrium of each separate member as well as of the entire structure should be stated. Next to the equilibrium statements, the number of independent equations and of unknown reaction components provided by the given force system should be written. In addition, the number of new unknowns (that is, not appearing in any previous statements of equilibrium) are also displayed:

	e	u	new u
I: $((p), A, C) \doteq 0$	3	3	3
II: $(C', B, M_B) \doteq 0$	3	5	3
Str: $((p), A, B, M_B) \doteq 0$	(3)	4	

Independent statements of equilibrium imply six scalar equations which matches the number of unknowns in the problem: the necessary condition of statical determinacy is satisfied again. Comparing number of scalar equations against the number of reaction components it is experimented that these numbers match only for the suspended part. Consequently, the calculation starts by writing equilibrium equations based on the equilibrium staement of the suspended part itself. It is still aimed at finding equations with only one unknown if possible:

$$\mathrm{I} \sum M_i^{(C)}: 3\cdot 1\cdot 2.5 - A\cdot 2 = 0 \quad \rightarrow \quad A = 3.75\ \mathrm{kN}\,(\uparrow)$$

$$\mathrm{I} \sum F_{ix}: C_x = 0 \quad \rightarrow \quad C'_x = 0$$

$$\mathrm{I} \sum F_{iz}: 3\cdot 1 - (3.75) - C_z = 0 \quad \rightarrow \quad C_z = -0.75\ \mathrm{kN}\,(\downarrow) \quad \rightarrow \quad C'_z = -0.75\ \mathrm{kN}\,(\uparrow)$$

As soon as components of reaction C' are obtained, there are only three unknowns left in the equilibrium statement written for the fixed part and they can be therefore determined as follows:

$$\mathrm{II} \sum F_{ix}: B_x = 0$$

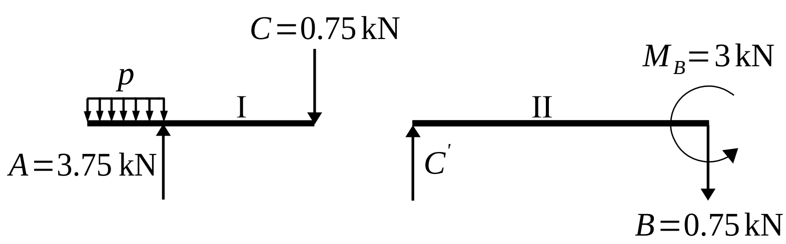
$$\mathrm{II} \sum F_{iz}: (-0.75) - B_z = 0 \quad \rightarrow \quad B_z = -0.75\ \mathrm{kN}\,(\downarrow)$$

$$\mathrm{II} \sum M_i^{(B)}: (-0.75)\cdot 4 + M_B = 0 \quad \rightarrow \quad M_B = 3\,\mathrm{kNm}\,(\curvearrowleft)$$

Now all external and internal reactions are determined; let a moment about point A for the complete structure be written down in order to check the equilibrium:

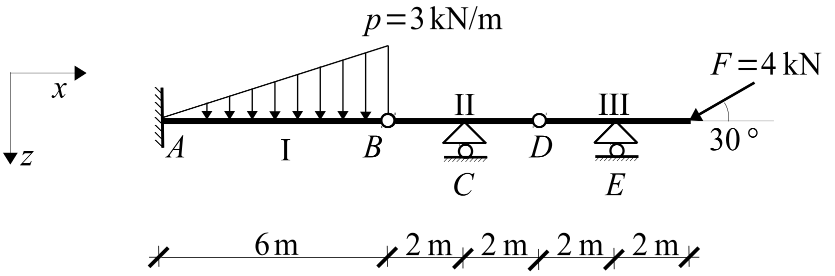
$$\mathrm{Str} \sum M_i^{(A)}: 3\cdot 1\cdot 0.5 + (-0.75)\cdot 6 + 3 = 0$$

After the results having been determined, make the final sketch:



Exercise 2

Determine external and internal reactions of the Gerber beam (reactions can be given by components).



Solution

FBDs:

Equilibrium statements:

e u new u

I:

II:

III:

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

$\sum$

$$\sum$$

$$\sum$$

$$\sum$$

$$\sum$$

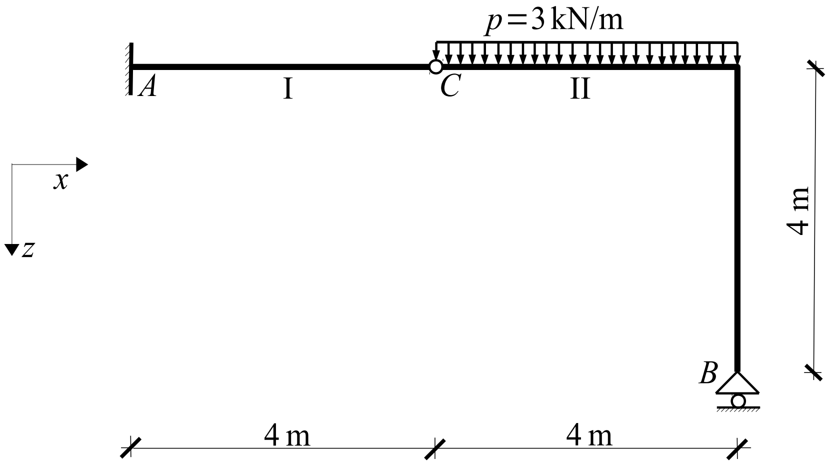
$$\sum$$

Check:

$$\sum$$

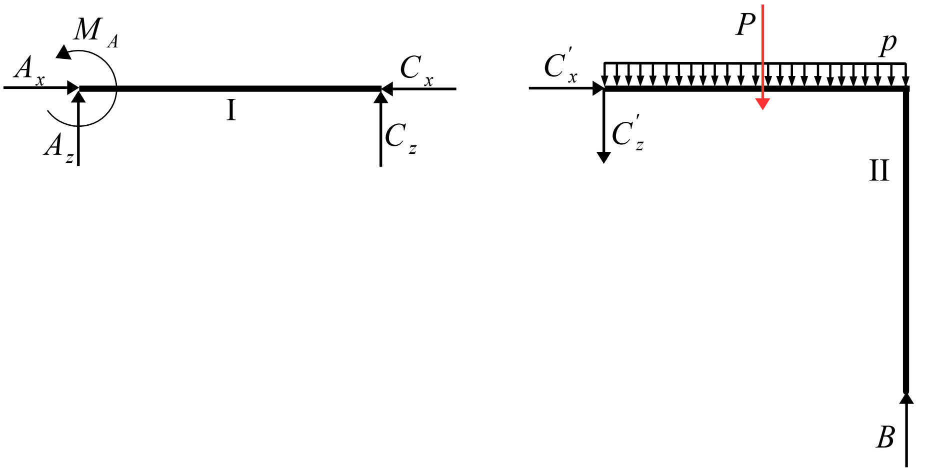
Final sketch:

Example 3
Determine external and internal reactions (calculated forces can be given by components).



Solution

The solution is started by the isolation. Body I is acted upon by external reactions A, M_A and internal reaction C. Since there are only two forces sxerted on internal hinge C, they should exactly be negatives to each other. Body II is acted upon by active distributed load p, internal reaction C' and external reaction B.
The free body diagams are as follows:



When calculating reactions, a distributed load can be replaced by its resultant of magnitude $P = 3 \cdot 4 = 12\,\text{kN}$.
After the isolation is complete, write equilibrium statements for each separate body as well as for the entire structure. Count independent scalar equations and scalar unknowns.

		e	u	new u
I:	$(A, M_A, C) \doteq 0$	3	5	5
II:	$((p), C', B) \doteq 0$	3	3	1
Str:	$((p), A, M_A, B) \doteq 0$	(3)	4	

Independent statements of equilibrium imply six scalar equations. The number of unknowns in the problem is also six: the necessary condition of statical determinacy is still satisfied. The number of scalar equations and also the number of reaction components for body II is three, so the solution process starts here:

$$\text{II} \sum M_i^{(C)} : -12 \cdot 2 + B \cdot 4 = 0 \quad \rightarrow \quad B = 6 \text{ kN} (\uparrow)$$

$$\text{II} \sum F_{ix} : C'_x = 0 \quad \rightarrow \quad C_x = 0$$

$$\text{II} \sum F_{iz} : 12 + C'_z - 6 = 0 \quad \rightarrow \quad C'_z = -6 \text{ kN} (\uparrow) \quad \rightarrow \quad C_z = -6 \text{ kN} (\downarrow)$$

After components of reaction C having been obtained, the equilibrium statement of body I remains with three unknowns only. They can be calculated, e.g., as:

$$\text{I} \sum F_{ix} : A_x = 0$$

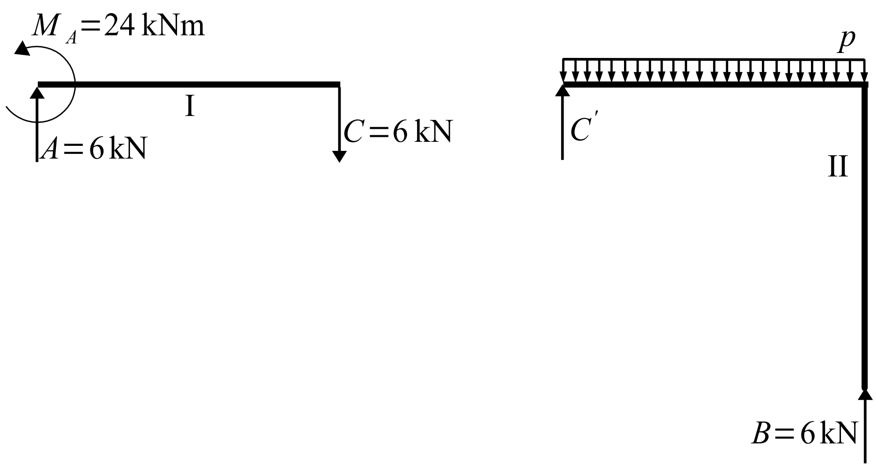
$$\text{I} \sum F_{iz} : -A_z - (-6) = 0 \quad \rightarrow \quad A_z = 6 \text{ kN} (\uparrow)$$

$$\text{I} \sum M_i^{(A)} : M_A + (-6) \cdot 4 = 0 \quad \rightarrow \quad M_A = 24 \text{ kNm} (\curvearrowleft)$$

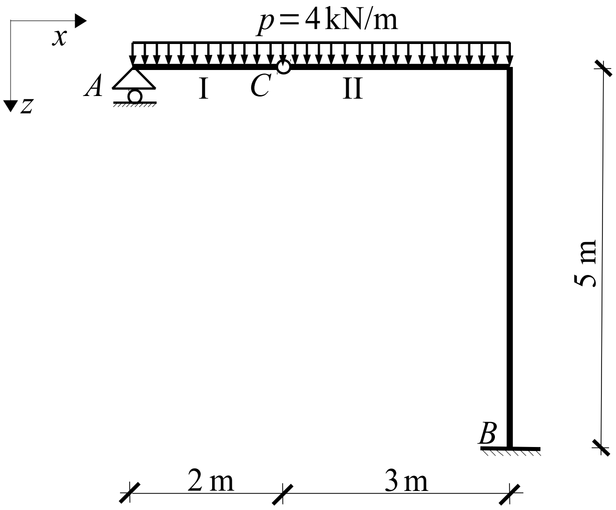
Now all external and internal reactions are determined; let moments about point B for the complete structure be summed up in order to check the equilibrium:

$$\text{Str} \sum M_i^{(B)} : 12 \cdot 2 - (6) \cdot 8 + 24 = 0$$

After checking the results, it is left only to prepare a final sketch:



Exercise 3
Determine external and internal reactions of the structure shown below.



Solution
FBDs:

Equilibrium statements:

e u new u

I:

II:

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

$\sum$

$\sum$

$\sum$

Check:

$\sum$

Final sketch:

Compound structures III

One of the most common compounds is the *three hinged structure* (frame, arch), composed of two pin-jointed members, both supported to the ground by a further pin-joint. Figure B5.1 shows different (typical) geomeries for three hinged structures.

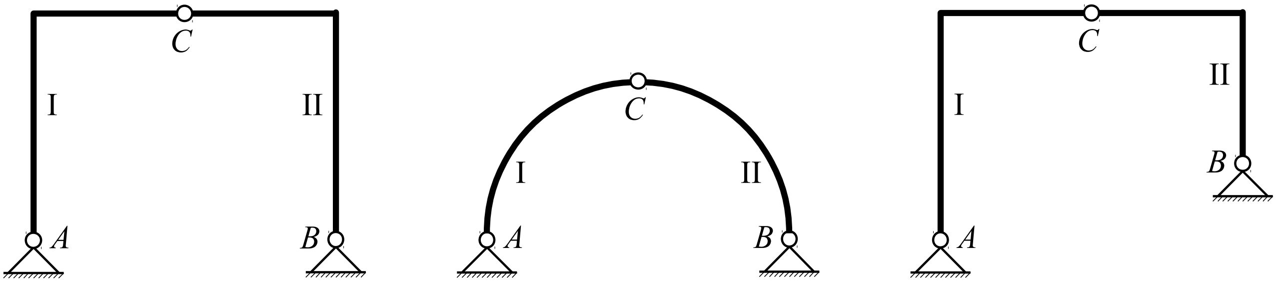
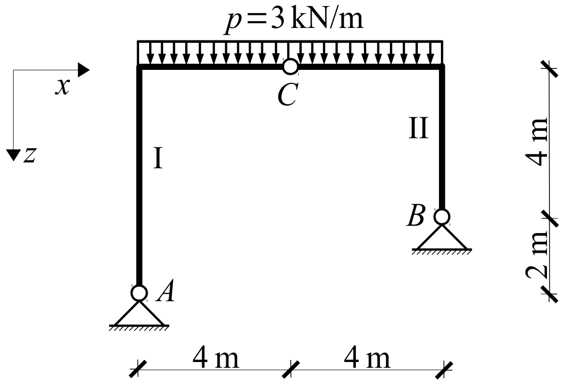


Figure B5.1 Three hinged arch and frames

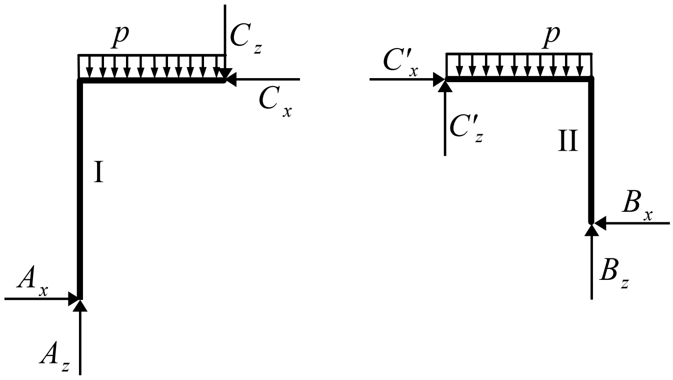
A typical problem of calculation of these three hinged structures is a smaller number of independent equations than that of unknowns implied by an equailibrium statement of any isolated bodies or even the entire compound. Nevertheless, generally there is no need of solving a system with multiple equations: a system with two unknowns and equations is always suitable to start with, and all other unknowns can be obtained afterwards by single equations each. Moreover, in such a special (but not extraordinary) case when two external supports are at the same height, all reaction components can be obtained using single-variable equations only.

Example 1
Determine all external and internal reactions of the three hinged frame shown.



Solution
The solution is started by the isolation. Both members of the structure are acted upon by more than two forces. Body I carries the left half of distributed load p and is balanced by external reaction A and internal reaction C at the middle hinge, both of unknown magnitude and direction. Hinge C does not need isolation because only two forces from adjacent bodies are exerted on it. Note that a hinge is assumed to have infinitely small extension and thus the

resultant of the distributed load over it is also infinitely small. Body II is exerted upon by the right half of load p, as well as external reaction B and internal reaction C' at the middle hinge, both are of unknown magnitude and direction.
Free body diagrams are as follows:



After the structure having been isolated, equilibrium of each separate member as well as of the entire structure should be stated. Next to the equilibrium statements, the number of independent equations and of unknown reaction components provided by the given force system should be written. In addition, the number of new unknowns (that is, not appearing in any previous statements of equilibrium) are also displayed:

	e	u	new u
I: $((p), A, C) \doteq 0$	3	4	4
II: $((p), C', B) \doteq 0$	3	4	2
Str: $((p), A, B) \doteq 0$	(3)	4	

Independent statements of equilibrium imply six scalar equations which equals the number of unknowns in the problem: the necessary condition of statical determinacy is therefore met. It is common to three hinged structures, however, that any statement of equilibrium implies less independent scalar equations than unknowns but, a system of at most two equations and two variables is sufficient to start the solution. Focusing on two unknown components at a given hinge, e.g., at B, makes possible to extract two equations from two different statements, both having only the components B_x, B_z. Let therefore the sum of moment about point A be written for the equilibrium of the entire structure and, simultaneously, another sum of moments about point C for the equailibrium of body II only:

$$\text{Str} \sum M_i^{(A)}: -3 \cdot 8 \cdot 4 + B_z \cdot 8 + B_x \cdot 2 = 0$$

$$\text{II} \sum M_i^{(C)}: -3 \cdot 4 \cdot 2 + B_z \cdot 4 - B_x \cdot 4 = 0$$

The component B_x is expressed from the first equation as follows:

$$B_x = 48 - 4 \cdot B_z$$

and writing it into the second one yields:

$$-24 + 4 \cdot B_z - 192 + 16 \cdot B_z = 0 \cdot$$

From this expression, one obtains B_z as

$$B_z = 10.8\,\text{kN}\,(\uparrow)$$

and then B_x is

$B_x = 4.8\,\text{kN}(\leftarrow)\cdot$

After both components of B having been determined, any components of A can be obtained from resolution equations written for the entire structure:

$\text{Str}\sum F_{ix}: A_x - (4{,}8) = 0 \quad \rightarrow \quad A_x = 4{,}8\,\text{kN}(\rightarrow)$

$\text{Str}\sum F_{iz}: 3\cdot 8 - A_z - (10.8) = 0 \quad \rightarrow \quad A_z = 13.2\,\text{kN}(\uparrow)$

Now all external reactions are determined, let the equilibrium be checked by writing the sum of moments about point C, based on the equailibrium of body I:

$\text{I}\sum M_i^{(C)}: 3\cdot 4\cdot 2 - (13.2)\cdot 4 + (4.8)\cdot 6 = 0$

Having the correctness of external reaction components been verified, determine components of internal reaction C based on the equilibrium of body I:

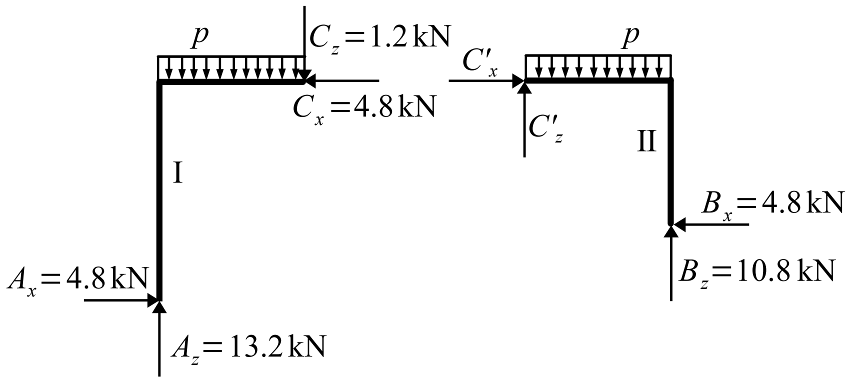
$\text{I}\sum F_{ix}: (4.8) - C_x = 0 \quad \rightarrow \quad C_x = 4.8\,\text{kN}(\leftarrow) \quad \rightarrow \quad C'_x = 4.8\,\text{kN}(\rightarrow)$

$\text{I}\sum F_{iz}: 3\cdot 4 - (13.2) + C_z = 0 \quad \rightarrow \quad C_z = 1.2\,\text{kN}(\downarrow) \quad \rightarrow \quad C'_z = 1.2\,\text{kN}(\uparrow)$

At this point, all internal reaction components became known. The results can be checked through summing up moments about point B with respect to the statement of body II:

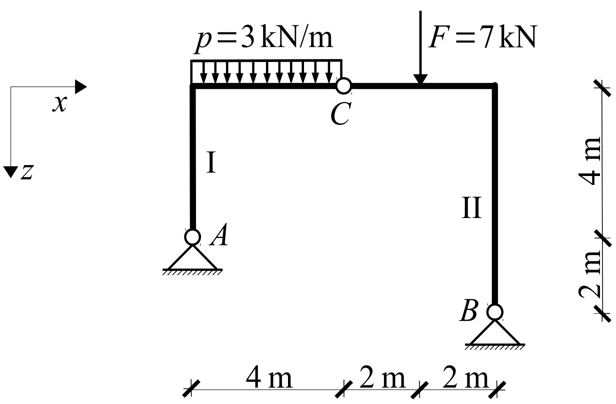
$\text{II}\sum M_i^{(B)}: 3\cdot 4\cdot 2 - (1.2)\cdot 4 - (4.8)\cdot 4 = 0$

Since all checks were successful, make a final sketch:



Exercise 1

Determine all external and internal reactions of the three hinged frame shown.



Solution

Free body diagrams:

Equilibrium statements:

e u new u

I:

II:

Str:

Analytic solution:

Str $\sum M_i^{(B)}$:

I $\sum M_i^{(C)}$:

Express component A_x from the first equation:

$A_x =$

and plug it into the second one:

...

This yields:

$A_z =$

$A_x =$

All further equations are of a single variable:

$\sum$

$\sum$

Checking external reactions:

$\sum$

Determine now the components of reaction *C*:

$\sum$

$\sum$

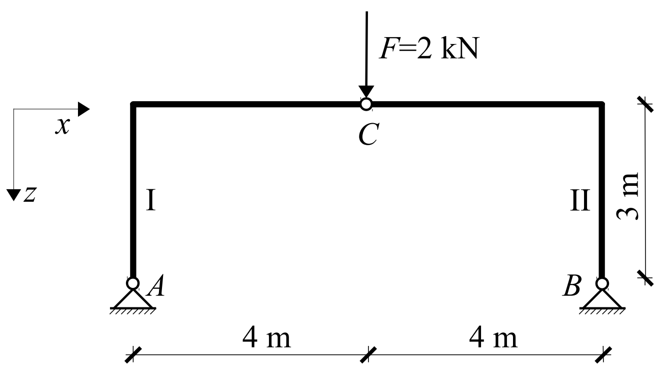
Check:

$\sum$

Final sketch:

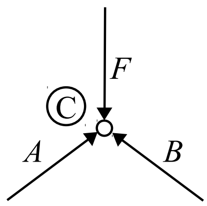
Example 2

Determine external and internal reactions of the three hinged frame shown.



Solution

The first step is the isolation. Member I is acted upon by external reaction *A* and an internal reaction at the middle hinge *C*, that is, they must be the negatives of each other. Likewise, member II is kept in equilibrium by an exernal reaction *B* and an internal one at hinge *C*, that is only possible again if they are also negatives of each other. As an immediate consequence of these conditions, both internal reactions of the hinge *C* also pass through an external hinge (*A* or *B*), or conversely, both external reactions pass also through point *C*. Hinge *C* should now be isolated because of the three forces acting on it; and its free body diagram will be the only one after isolation:



State the equilibrium of joint *C* :

	e	u	new u
$C{:}(F, A, B) \doteq 0$	2	2	2

Note that the equilibrium statement of the entire structure involves the same three forces, making it unnecessary to be written separately. The statement implies two independent scalar equations and there are also two unknowns in it, since the line of action of both external reaction forces are known. Let moment equilibria be written about both external supports:

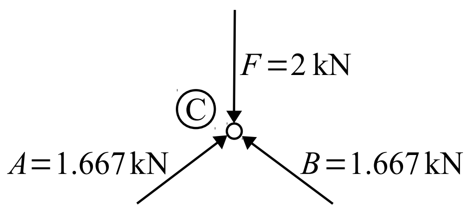
$$C\sum M_i^{(B)}: 2\cdot 4 - A\cdot\frac{3}{5}\cdot 8 = 0 \quad \rightarrow \quad A = 1.667 \text{ kN } (\square)$$

$$C\sum M_i^{(A)}: -2\cdot 4 + B\cdot\frac{3}{5}\cdot 8 = 0 \quad \rightarrow \quad B = 1.667 \text{ kN } (\square)$$

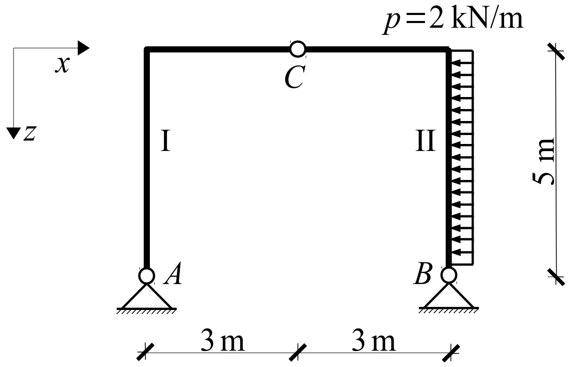
(Support reactions *A* and *B* are resolved into components in points *A* and *B*, respectively.)
Now, check the obtained results by a resolution equation written for joint *C*:

$$C\sum F_{ix}: (1.667)\cdot\frac{4}{5} - (1.667)\cdot\frac{4}{5} = 0$$

At the end, make a final sketch:

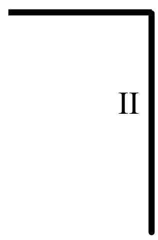


Exercise 2
Determine external and internal reactions of the three hinged frame shown.

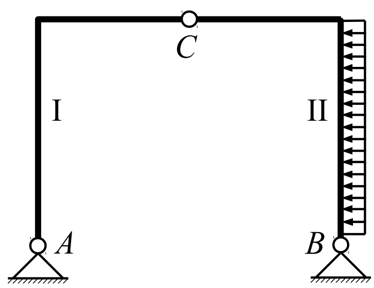


Solution

Free body diagram:



Equilibrium statement(s):



Draw lines of both support reactions into the figure.

Analytic solution:

$\sum$

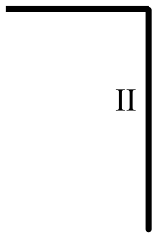
$\sum$

$\sum$

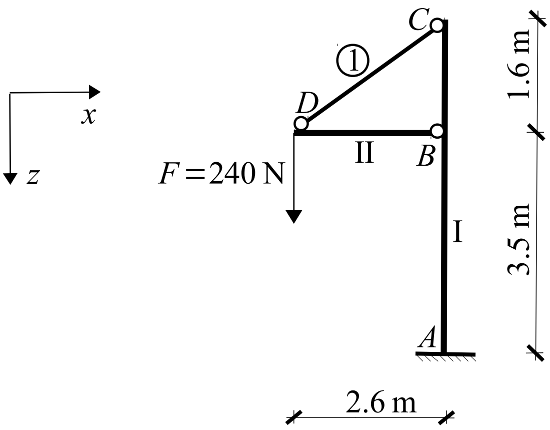
Equation for checking:

$\sum$

Final sketch:

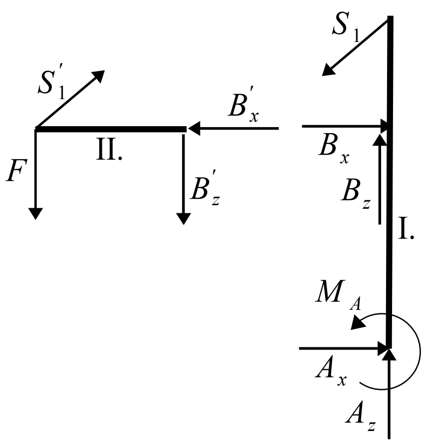


Example 3
Determine external and internal reactions of the structure given below.



Solution

The first step is the isolation. Member I is acted upon by external reactions A, M_A and internal reactions at hinges C and D. All the hinges themselves and even member CD are two-force members only, so they will not be isolated. Member II, however, carries an active load F and balanced by internal reactions at D (force S'_1 with given line of action) and B (force B' of unknown magnitude and direction). The FBDs are then as follows:



After the isolation is done, write equilibrium statements for separate bodies and the entire structure. Count independent scalar equations as well as unknown reaction components.

	e	u	new u
I: $(A, M_A, B, S_1) \doteq 0$	3	6	6
II: $(F, S'_{1,B}{}') \doteq 0$	3	3	0
Str: $(F, A, M_A) \doteq 0$	(3)	3	

Equilibrium statements imply six independent scalar equations for six unknown components, hence the necessary condition of statical determinacy is satisfied. Equilibrium of member II implies three equations and its statement involves three unknowns as well; it is convenient to start the calculations here:

$$\text{II} \sum M_i^{(B)}: 240 \cdot 2.6 - S'_1 \cdot \frac{1.6}{\sqrt{1.6^2+2.6^2}} \cdot 2.6 = 0 \quad \rightarrow \quad S_1 = 458.0\ \text{N}(\text{t})$$

$$\text{II} \sum M_i^{(C)}: 240 \cdot 2.6 - B'_x \cdot 1.6 = 0 \quad \rightarrow \quad B'_x = 390\,\text{N}(\leftarrow) \quad \rightarrow \quad B_x = 390\,\text{N}(\rightarrow)$$

$$\text{II} \sum M_{iz}^{(D)}: B'_z = 0 \quad \rightarrow \quad B_z = 0$$

Equilibrium statement of the entire structure allows for writing three independent scalar equations as well for three unknowns which are the external reactions of the compound:

$$\text{Str} \sum M_i^{(A)}: 240 \cdot 2{,}6 + M_A = 0 \quad \rightarrow \quad M_A = -624\ \text{Nm}\ (\curvearrowright)$$

$$\text{Str} \sum F_{ix}: A_x = 0$$

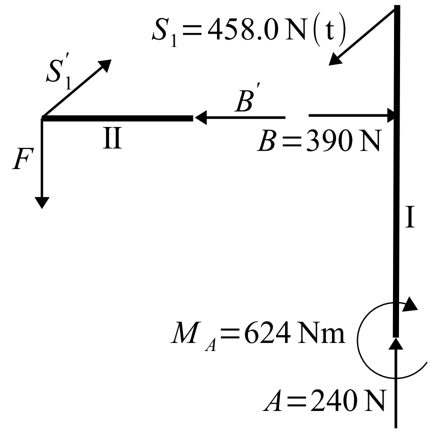
$$\text{Str} \sum F_{iz}: 240 - A_z = 0 \quad \rightarrow \quad A_z = 240\ \text{N}(\uparrow)$$

At this point, all internal and external reactions are known. Check the equailibrium by writing two resolution equations for member I as follows:

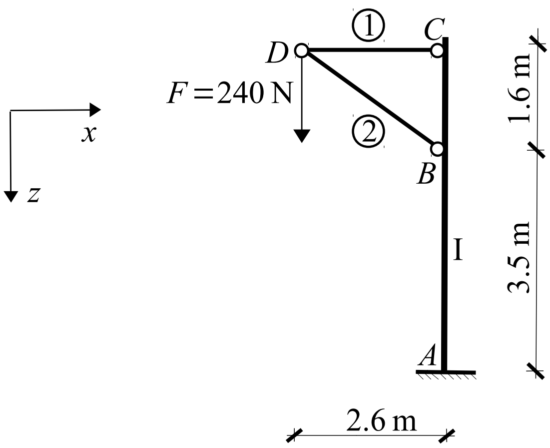
$$\text{Str} \sum F_{ix}: 390 - (458.0) \cdot \frac{2.6}{\sqrt{1.6^2+2.6^2}} = 0.04 \approx 0$$

$$\text{Str} \sum F_{iz}: -240 + (458{,}0) \cdot \frac{1.6}{\sqrt{1.6^2+2.6^2}} = 0.03 \approx 0$$

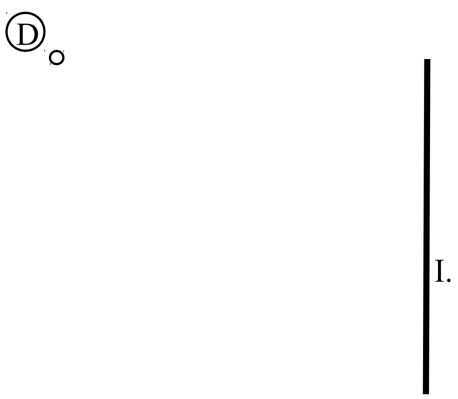
Make a final sketch after the results having been checked:



Exercise 3
Determine external and internal reactions of the structure given below.



Solution
Free body diagrams:



Equilibrium statements:

e u new u

I:

D:

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

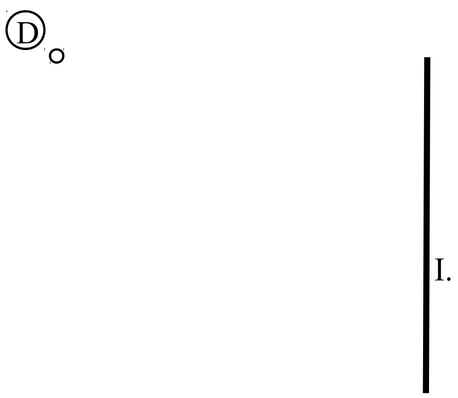
$\sum$

$\sum$

Check:

$\sum$

Final sketch:



Statical determinacy

Definitions with respect to statical determinacy were given at the analysis of simple structures in a way that they also generalize to compounds. In the present lecture only determinate structures were dealt with so far, but now here follow some examples of statically indeterminate and overdeterminate compound structures. Let former definitions be revisited as follows:

Definition: A structure is said to be ***statically determinate*** (***isostatic***) if it remains in equilibrium under arbitrary loads and all its reactions can uniquely be determined from equilibrium equations.

Statical determinacy requires therefore two conditions to be satisfied simultaneously. If any of here conditions is not met, the structure has a different classification.

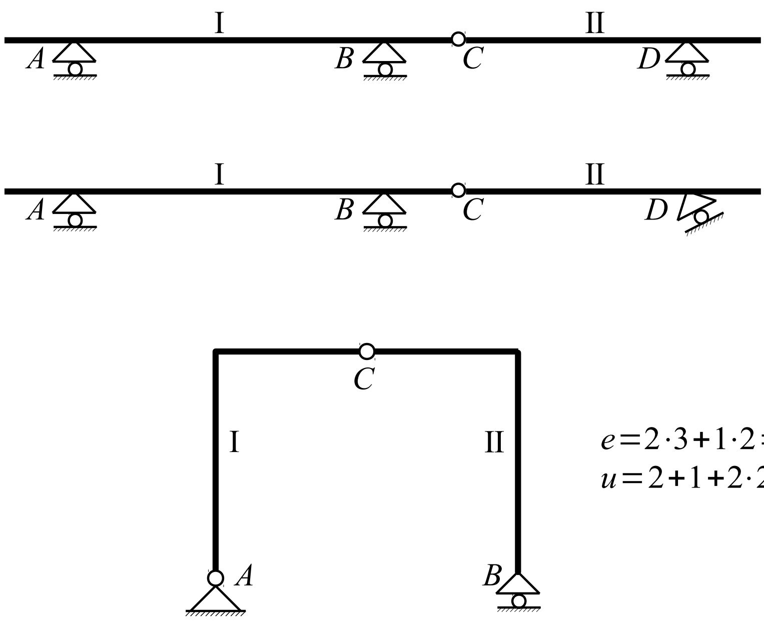
Definition: A structure is said to be ***statically overdeterminate*** (***hypostatic***) if there exists a load under which the structure does not remain in equilibrium (i.e., the system of equilibrium equations has no solution).

Definition: A structure is said to be ***statically indeterminate*** (***hyperstatic***) if there exists a load under which the system of equilibrium equations can be solved (i.e., the structure remains in equilibrium) but this solution is not unique. In other terms, reactions cannot be determined uniquely just from equilibrium equations.

Analysis of statical determinacy of a structure is commonly started by the comparison of the number e of scalar equations implied by the different equilibrium statements and the number u of scalar unknowns involved in them. If the number of equations exceeds that of unknowns, the structure is sure to be statically overdeterminate: relation $e > u$ is a sufficient (but not necessary) condition of statical overdeterminacy. It can also be said that such structures need further supports to be able to carry loads of arbitrary arrangement. Figure B5.2 shows statically overdeterminate structures obeying the condition $e > u$. When analysing an entire structure from the aspect of statical determinacy, numbers of equations and unknowns are determined on the basis of the *complete isolation* of the structure. In the lack of any specified load, all members in a compound should be treated as possibly exposed to some active load, and should therefore be isolated one by one. It is an important note that statical determinacy, in- or overdeterminacy of an assembly is a property independent from any load acting on it; it is strictly inherent to the structure itself.

If the number of scalar equations is less than that of scalar unknowns, there can surely be found a load under which the equilibrium equations can be solved but not uniquely. The relation $e < u$ is a sufficient (but not necessary) condition of statical indeterminacy. structure is sure to be statically overdeterminate: Abban esetben, ha az ismeretlenek száma nagyobb, mint a felírható skaláregyenletek száma, akkor biztosan van olyan teher, ami esetén nem tudjuk a reakcióerőket csupán az egyensúlyi egyenletek alapján egyértelműen meghatározni. Figure B5.3 shows statically indeterminate structures obeying the condition $e < u$. It can also be said that such structures have too many supports (more generally: too many internal or external connections) to get all reaction components uniquely from the equilibrium equations only.

As has already been seen in simple structures, fulfilment of e=i does not automatically imply statical determinacy of the structure (relation $e = u$ is only a necessary (but not sufficient) condition of statical determinacy). All examples shown in Figure B5.4 are statically indeterminate and overdeterminate three hinged assemblies at the same time. Both structures have their three hinges incident to a straight line: if the middle hinge C is loaded only, both adjacent members are balanced by just two forces, making their lines of action to pass hinge C. With his special geometry, a resolution equation perpendicular to this common line of internal reactions is not satisfied whenever the load on C has any component in the equation. This proves the existence of a load under which the structure is unable to remain in equilibrium: the structure is statically overdeterminate. If the structure is not acted upon by any external load, any pair of reactions of equal magnitude and opposite direction through *A* and *C* is able to maintain equilibrium. It is a direct proof again for the system of equilibrium equations to have more than one solution, making thus the condition of statical indeterminacy satisfied: there exists a load under which the structure is in equilibrium but the solution of equilibrium equations is not unique.



$e=2\cdot3+1\cdot2=8$
$u=3\cdot1+2\cdot2=7$

$e=2\cdot3+1\cdot2=8$
$u=3\cdot1+2\cdot2=7$

$e=2\cdot3+1\cdot2=8$
$u=2+1+2\cdot2=7$

Figure B5.2 Statically overdeterminate compound structures

A *B* *C* *D* *E*

$e=2\cdot3+1\cdot2=8$
$u=2+3\cdot1+2\cdot2=9$

A *B* *C* *D*

$e=2\cdot3+1\cdot2=8$
$u=2\cdot2+1+2\cdot2=9$

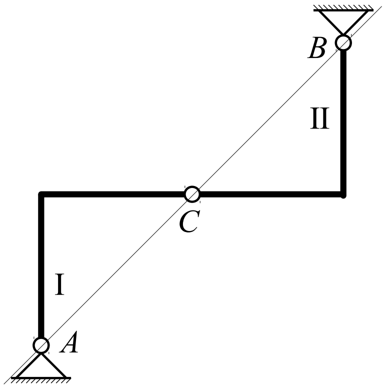
A *C*

$e=2\cdot3+1\cdot2=8$
$u=3+2+2\cdot2=9$

B

Figure B5.3 Statically indeterminate compound structures

$e = 2\cdot 3 + 1\cdot 2 = 8$
$u = 2\cdot 2 + 2\cdot 2 = 8$



$e = 2\cdot 3 + 1\cdot 2 = 8$
$u = 2\cdot 2 + 2\cdot 2 = 8$

Figure B5.4 Statically in- and overdeterminate compound structures

Trusses

Structures composed of several bars with pin-jointed connections at each ends are called trusses, see Figure B6.1. Bars in a truss are normally straight; a truss can only be supported by links (rollers) and pin-joints. Loads on a truss are commonly admitted on joints only. Note that trusses in the engineering practice are not precisely built with pin-jointed connections; however, a bar-and-joint statical model considerably simplifies computing tasks, while it provides a reliable approximation even in the case of rigid (e.g., welded) connections.

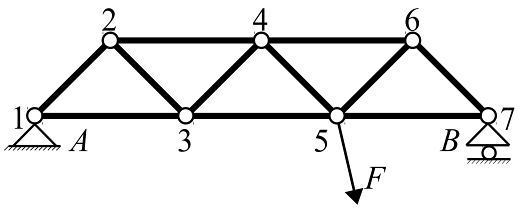


Figure B6.1 Truss loaded at its joints

Trusses are a special kind of compound structures, therefore all its internal and external reactions can be determined using techniques discussed so far. A special care about this family of structures is justified either by their frequent application or by some special procedures that have been developed and spread over the world for the calculation of truss members. In order to verify the necessary condition of statical determinacy of a truss, let a complete isolation of a truss be prepared first (see Figure B6.2). Forces in members connecting joints i and j ($i<j$) will be denoted by their signed magnitude $S_{i,j}$ henceforth. We adopt a convention that first subscript in a member force refers always to the node labeled by the smaller number. In FBDs, any member force exerted on a joint is denoted by an arrow and provided with a scalar magntude of that force keeping the assumption that tensile bars are said to have positive member forces. Two forces exerted by a member on two connected joints are obviously negative to each other which can also be seen from opposite arrows drawn to joints. For the sake of simplicity, prime notation of opposite member forces (i.e., in the sense of Newton's third law) are omitted.

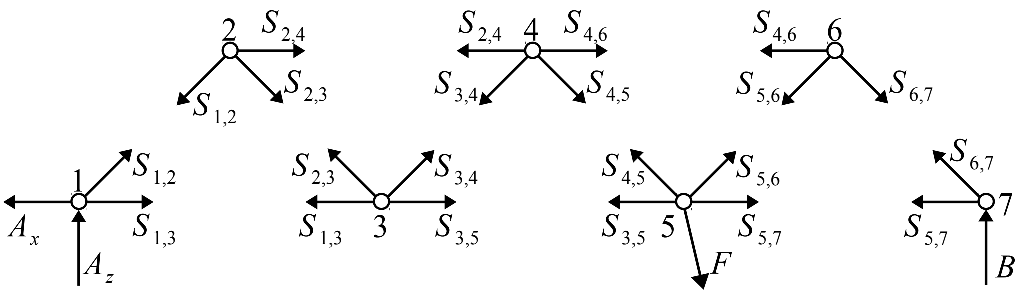


Figure B6.2 Isolated joints of a truss loaded at its joints only

The procedure of isolation means a separate analysis of joints only, since all bars are acted upon exactly by two forces. Those forces must share a line of action and should therefore pass through both joints adjacent to the truss member. Each separate joint represents a concurrent force system in 2D and implies therefore two independent scalar equations. This leads to a total number $2c$ of equations where c stands for the number of joints inside a truss. Unknowns in the system of equilibrium equations are member forces and external reaction components; their total number adds up to $r + k$ with r being the number of truss members and k is the total degree of constraints. The necessary but not sufficient condition of statical determinancy is that the number of independent scalar equations should be equal to the number of scalar unknowns ($e = u$) that can be translated for trusses as $2c = r + k$.

Members of special position within a truss have their own names. Truss members forming top and bottom boundaries of the structure are called *top chord members* and *bottom chord members*, respectively; their complete series is referred to as *top chord* and *bottom chord* of the truss. If top and bottom chord members are all parallel to each other, the truss is named *parallel chord truss* or *flat truss*. Trusses with triangular side view (mostly used in roof structures) are called *pitched trusses*. Members running between chords are called *web members* and can further be divided into *columns/struts* or *ties*, depending on their typical (compressive or tensile) loads. Figure B6.3 shows examples for members in different position. Not that trusses are commonly drawn without displaying hinges at nodes, nevertheless, calculation is made assuming pinned connections.

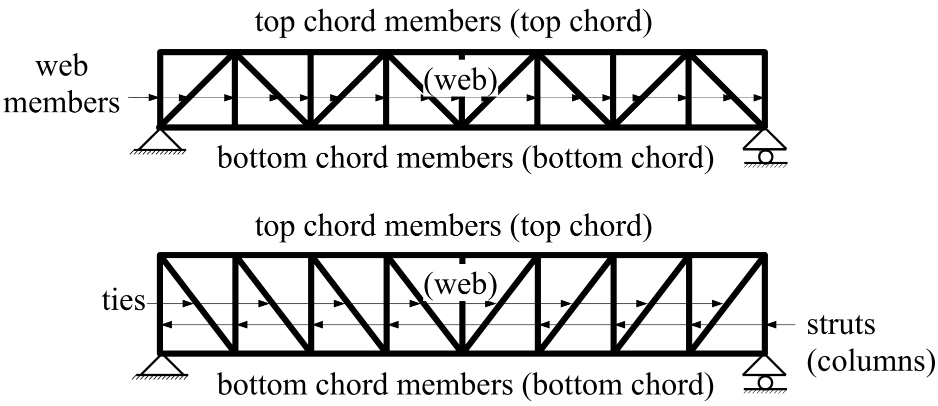


Figure B6.3 Truss member terminology

Figs. 16.4 -16.8 show different truss types which are (except for X- or Brown trusses of Figure B6.7) alll statically determinate. Note that statical indeterminacy of Brown trusses are not influenced by whether or not there are also pin-joints at crossings (the example at the bottom and top is with and without pin-joints, respectively), since insertion of a new pin-joint comes along with two new web members, leaving the difference between numbers of equations and unknowns unchanged.

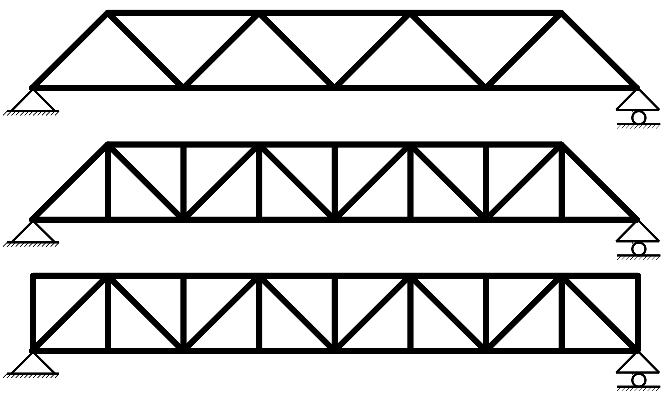


Figure B6.4 Warren truss

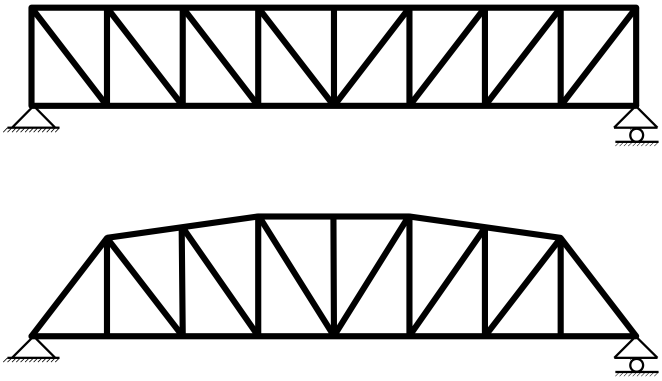


Figure B6.5 Pratt truss.

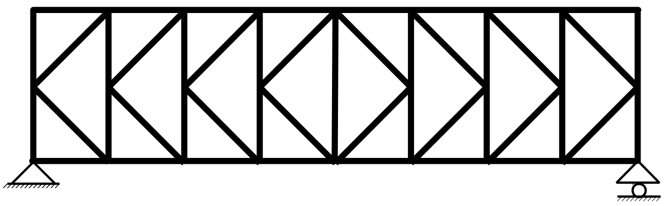


Figure B6.6 K truss

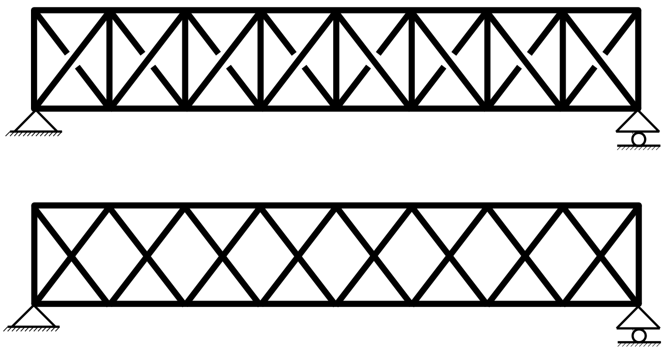


Figure B6.7 Brown truss (X-truss)

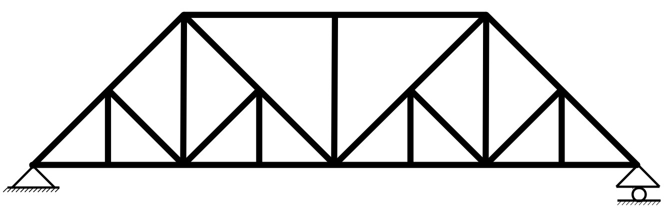


Figure B6.8 Baltimore truss (with secondary bracing)

In the following sections, two methods (*method of joints* and *method of sections*) will be presented for calculating truss member forces in the case when active loads are assumed to act exclusively at nodes.

The method of joints is based on the analysis of equilibrium of an individual joint and is applicable in two cases: 1) If a joint is exerted on by not more than two unknown member forces, then two equations (either resolution or moment equations) yield the member forces. 2) If all but one member adjacent to a joint (either of known or unknown forces) are aligned with each other, the remaining member force can be determined from a resolution equation perpendicular to all other members.

The method of sections is mainly applied for so-called triple sections: three members of the truss are replaced by their member forces such that the truss is split into two disjoint substructures. Note that it is a special kind of partial isolation that can also be recovered from the complete one by the unification of all joints on either side of that section. If the three member forces act on both substructures, they both should remain in equilibrium. Equilibrium equations written for either side of the section provides a solution to three scalars (assume that external reactions of the entire truss are determined beforehand) In a slightly generalized context, both methods could be unified under the name 'method of sections', since the method of joints is also derived from a special section where all members adjacent to a node are cut (replaced by forces).

Members that are neither tensile nor compressive under a given load are called *zero force members* and denoted by a small circle drawn to their axis. In some special cases, zero force members can easily be detected. If a joint with no active load or external reaction component exerted on it (call it 'unloaded' for brevity henceforth) is adjacent to two non-parallel members only, both members are zero force members (Figure B6.9a). This statement can easily be justified by resolution equations written in a direction perpendicular to each member. Similarly, if an unloaded joint is adjacent to three members from which two are collinear, a resolution perpendicular to them yields immediately for the third one to be a zero force member (Figure B6.9b). It is still not very different if there are only two members adjacent to a joint which is loaded either by an active or passive force whose line of action is incident to one of the two member axes. A resolution perpendicular to parallel lines proves again the remaining member to be a zero force member (Figure B6.9c).

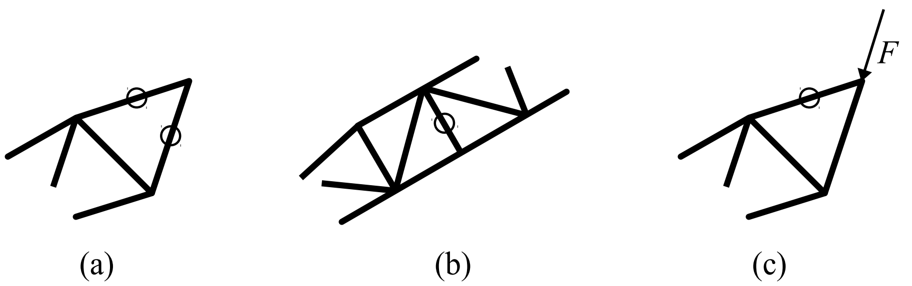
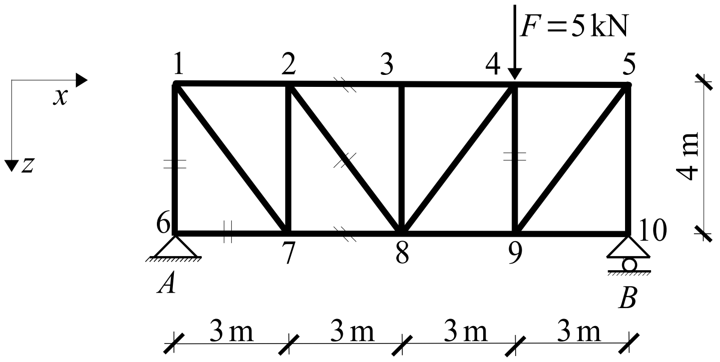


Figure B6.9 Special joints with zero force members

Calculated member forces are easier to be presented rather in a table format than in graphical sketches (being possibly different because of the different sections applied). The first column of a member force table lists the names of members, while columns 2 and 3 contain values of tensile and compressive member forces, respectively. (This strict separation of tensile and compressive members is justified by the essential difference in their mechanical behaviour: unlike tensile members, compressive members can buckle even at a relatively small level of load: difference in signs DOES mind.)

Example 1
Determine forces in all marked members of the truss shown.



Solution

Let the solution be started by determining external reactions: this task is done as if the truss was a single rigid member: reactions are calculated as in simply supported beams (even if no separate FBD is provided at this stage). Equilibrium of the truss is influenced by the only active load F and two support reactions:

Str: $(F, A, B) \doteq 0$

All reaction components are obtained from moment equilibria about supports and a horizontal resolution:

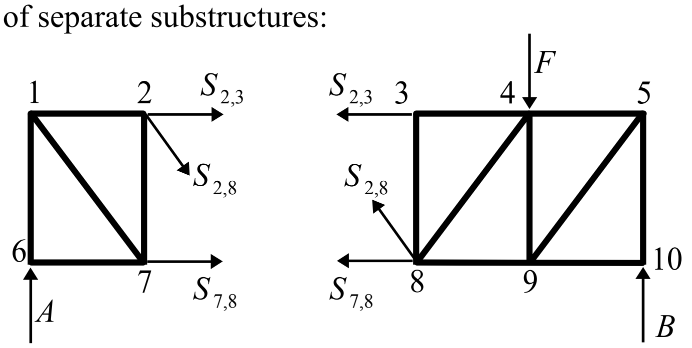
Str $\sum M_i^{(A)}: -5 \cdot 9 + B \cdot 12 = 0 \quad \rightarrow \quad B = 3.75\ \mathrm{kN}(\uparrow)$
Str $\sum M_i^{(B)}: 5 \cdot 3 - A_z \cdot 12 = 0 \quad \rightarrow \quad A_z = 1.25\ \mathrm{kN}(\uparrow)$
Str $\sum F_{ix}: A_x = 0$

In order to check them, write a resolution equation along z:

Str $\sum F_{iz}: 5 - 1.25 - 3.75 = 0$

In view of those reactions, member forces $S_{2,3}$, $S_{2,8}$, $S_{7,8}$ can already be determined from a triple section. Let all these members be removed and replaced by their member forces; both parts to the left and right of the section continues to be in equilbrium. Draw free body diagrams of both substructures and state their equilibrium one by one:

Triple section, FBDs of separate substructures:



Equilibrium statements are as follows:

left: $(S_{2,3}, S_{2,8}, S_{7,8}, A) \doteq 0$

right: $(F, S_{2,3}, S_{2,8}, S_{7,8}, B) \doteq 0$

Both statements contain the same three unknown forces and both statements imply three scalar equilibrium equations. These triples of equations are not independent of each other: equilibrium of the entire assembly has already been ensured by finding external reactions, and equilibrium of either substructure implies then directly the equilibrium of the other one. Eventually there are three independent scalar equations; for the sake of simplicity, they are mainly written based on that equilibrium statement which requires the less forces to be dealt with (for the same reason, FBD of the other side is not even drawn usually). Continue now calculating based on the equilibrium of the left hand side: reaction A is balanced by three forces in known lines of action. It is always possible to set up three single-variable equations to calculate unknown forces involved in a triple section. Member force $S_{2,3}$ is worth calculating from a moment equilibrium about the point of intersection (8) of two remaining forces as follows:

left $\sum M_i^{(8)}: -S_{2.3} \cdot 4 - 1.25 \cdot 6 = 0 \quad \rightarrow \quad S_{2.3} = -1.875 \text{ kN(c)}$

Force $S_{2,8}$ is obtained from a resolution equation perpendicular to both remaining forces:

left $\sum F_{iz}: S_{2.8} \cdot \frac{4}{5} - 1.25 = 0 \quad \rightarrow \quad S_{2.8} = 1.563 \text{ kN(t)}$

Force $S_{7,8}$ is still calculated preferably from an equation not involving any recently obtained member forces. Write a moment equation about the point of intersection (2) of two remaining forces as follows:

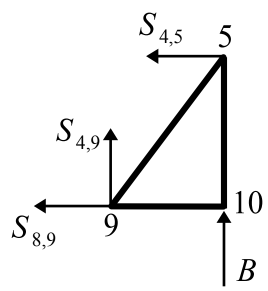
left $\sum M_i^{(2)}: S_{7,8} \cdot 4 - 1.25 \cdot 3 = 0 \quad \rightarrow \quad S_{7,8} = 0.9375 \text{ kN(t)}$

In order to check all forces in the same section, let a horizontal resolution be written down:

left $\sum F_{ix}: (-1.875) + (1.563) \cdot \frac{3}{5} + (0.9375) = 0.0003 \approx 0$

Force $S_{4,9}$ can be obtained from a section driven through members (4,5), (4,9) and (8,9). Use the equilibrium of the right hand side of the section.

FBD:



Equilibrium statement:

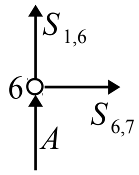
right: $(S_{4,5}, S_{4,9}, S_{8,9}, A) \doteq 0$

Since both chords are parallel, this member force is obtaind from a resolution perpendicular to both chords:

right $\sum F_{iz}: -S_{4,9} - 3.75 = 0 \quad \rightarrow \quad S_{4,9} = -3.75 \text{ kN(c)}$

Forces in members (1,6) and (6,7) are determined using the method of joints applied at joint 6.

FBD:



Equilibrium statement:

$6:(S_{1,6}, S_{6,7}, A) \doteq 0$

From a horizontal resolution it follows that member (6,7) is a zero force member ($S_{6,7}$=0).

Force $S_{1,6}$ can finally be determined from a vertical resolution equation as follows:

$6 \sum F_{iz}: -1.25 - S_{1,6} = 0 \quad \rightarrow \quad S_{1,6} = -1.25 \text{ kN(c)}$

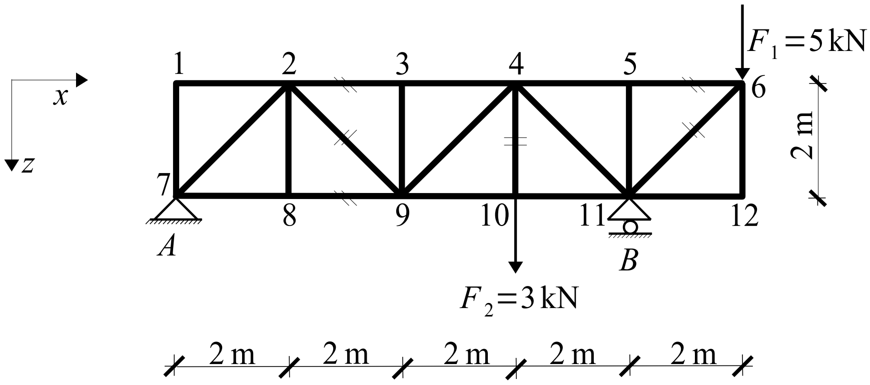
Calculated values are normally given in a table of member forces. Headings of columns refer to 1) member ID, 2) tensile forces 3) compresssive forces. Zero force members are displayed in both columns 2 and 3.

Table of member forces:

Member ID	Tensile [kN]	Compressive [kN]
(1,6)		1.25
(2,3)		1.875
(2,8)	1.563	
(4,9)		3.75
(6,7)	0	0
(7,8)	0.9375	

Exercise 1

Find zero force members (justify each of your choices by an appropriate equation). Calculate forces in all marked members of the truss shown.



Solution

Zero force members:

External reactions:

Triple section:

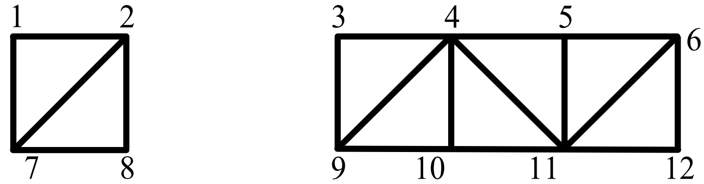


Table of member forces:

Member ID	Tensile [kN]	Compressive [kN]

Trusses and statical determinacy

Definitions of statical determinacy, indeterminacy and overdeterminacy have already been discussed in former lectures. These definitions also extend to trusses.

Sufficient but not necessary condition of statical overdeterminacy of a structure has been given in the form $u < e$, this can be rewritten for planar trusses (with $u = r + k$ and $e = 2c$) as $r + k < 2c$. Figures B7.1a and B7.1b show structural examples satisfying this sufficient condition. The assembly of Figure B7.1a cannot be balanced against the horizontal force F (the horizontal resolution equation is not satisfied), while node 2 of the truss shown in Figure B7.1b where the conditions of equilibrium of node 2 imply that it could only be balanced by a tensile force of magnitude F in member (1,2) which contradicts the condition of equilibrium of node 1. In summary, there exist a load for both assemblies which results in no solution of the equilibrium equations.

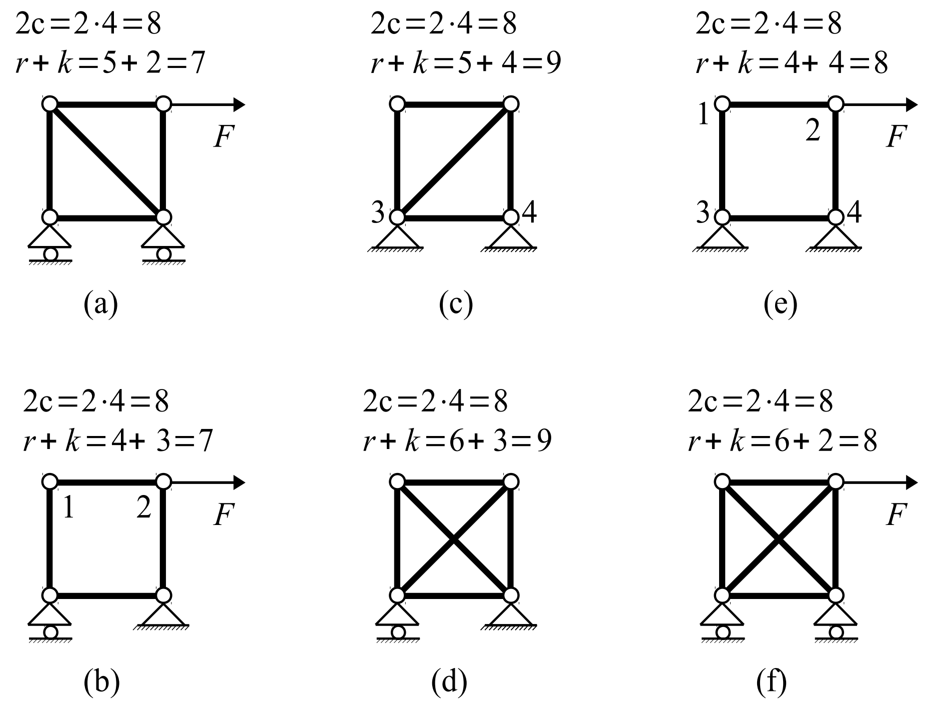


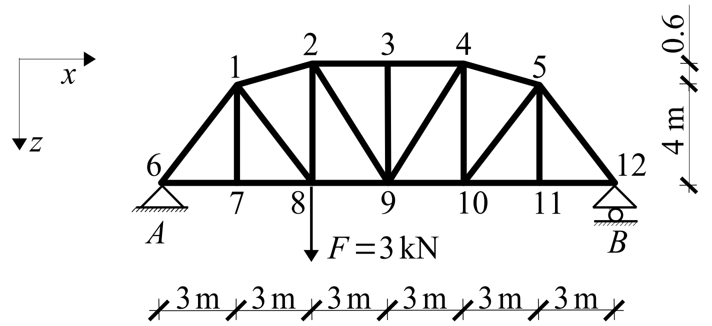
Figure B7.1 Statically overdeterminate (a,b), statically indeterminate (c,d) and statically both indeterminate and overdeterminate (e,f) trusses

Sufficient but not necessary condition of statical indeterminacy of a structure has been given in the form $u > e$, this can be rewritten for planar trusses as $r + k > 2c$. Figures B7.1c and B7.1d show structural examples satisfying this sufficient condition. Force in member (3,4) of the structure of Figure B7.1c can be of arbitrary magnitude because the lateral supports can balance any kind of load transmitted by the members. This also means that force $S_{3,4}$ cannot be uniquely determined even in the lack of any active load on the structure. The structure shown in Figure B7.1d can be made statically determinate by the removal of any single member. If the force in the removed member is applied at both adjacent joints, all remaining member forces can be determined uniquely and the structure remains in equilibrium. In other words, by prescribing one of the member forces, all other member forces can restore equilibrium, that is, member forces cannot be determined uniquely even if no active load is exerted on the truss at all.

Necessary but not sufficient condition of statical determinacy of a structure has been given in the form $u = e$, this can be rewritten for planar trusses as $r + k = 2c$. All Examples and Exercises presented in Lecture B6 and B7 are statically determinate trusses. Figures B7.1e and B7.1f show structural examples satisfying this necessary condition, too, but these are statically indeterminate and statically overdeterminate at the same time. Statical overdeterminacy of these structures is proven by applying a load F as shown in the figures: this load cannot be balanced. Assemblies under Figures B7.1e-f yield contradiction in calculating member force $S_{1,2}$ and overall horizontal equilibrium, respectively. Statical indeterminacy is proven like was done for structures of Figures B7.1c-d. Member (3,4) of the assembly of Figure B7.1e is able to carry arbitrary force even without external loading; in the assembly of Figure B7.1f, any member force can arbitrarily be prescribed and all other member forces will balance it with zero external reactions: for some loads there exist solutions that cannot be determined uniquely from equilibrium equations even in the lack of external loading.

Example 1

Find forces in members (1,2), (1,8), (7,8), (2,8) of the truss shown.



Solution

The first task is to determine external reactions. The entire truss is regrded as a simple rigid body in this procedure, reactions are obtained as for a simply supported beam (but the FBD of the simply supported beam is not drawn yet). The equilibrium of the entire truss is maintained by active force F and two passive forces A and B:

Str: $(F, A, B) \doteq 0$

Unknown components of reactions are obtained from moment equilibrium equations about the supports as well as from a horizontal resolution:

Str $\sum M_i^{(A)}: -3\cdot 6 + B\cdot 18 = 0 \quad \rightarrow \quad B = 1\ \text{kN}(\uparrow)$

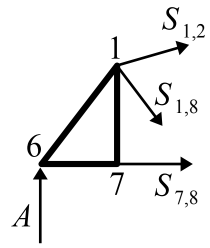
Str $\sum M_i^{(B)}: 3\cdot 12 - A_z\cdot 18 = 0 \quad \rightarrow \quad A_z = 2\ \text{kN}(\uparrow)$

Str $\sum F_{ix}: A_x = 0$

Verification of results is made by a resolution equation along z:

Str $\sum F_{iz}: 3 - 2 - 1 = 0$

In view of these reactions, member forces $S_{1,2}$, $S_{1,8}$, $S_{7,8}$ can be calculated from the same triple section. Let those members be removed from the structure and replaced by their member forces; this modification does not affect the equilibrium of substructures both to the left and right of the section.
Let the substructure rather on left hand side be analysed, since it is only acted upon by reaction A (beyond, of course, the member forces considered). Make a FBD for that part and state its equilibrium as follows:



The equilibrium statement reads:

left: $(S_{1,2}, S_{1,8}, S_{7,8}, A) \doteq 0$

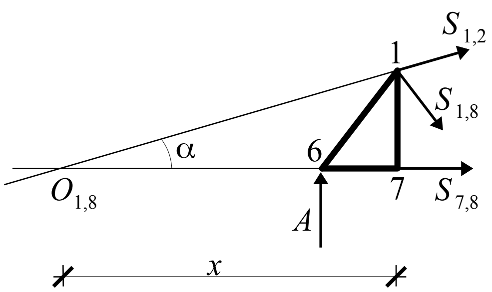
Here reaction A is balanced by three unknown member forces. As already known for such cases, there always can be written three equations in one variable each to gst froces in the triple section. Le t the method of principal points be applied, that is, a member force will be calculated from a momrent equilibrium about the point of intersection of lines of action of the remaining two forces. Let the angle of member (1,2) to the horizontal be denoted by α. Its numeric value will NOT be required in calculations but its sine and cosine will, let both be expressed therefore by proportions of horizontal and vertical legs and the hypotenuse (1-2) of a right triangle. For this purpose, we need the length of diagonal segment $l_{1,2}$:

$$l_{1,2}=\sqrt{3^2+0.6^2}=3.059\,\mathrm{m}$$

Member force $S_{1,2}$ is obtained therefore from a moment equilibrium equation written about joint 8, accounting for a resolution of force $S_{1,2}$ at joint 2 into horizontal and vertical components:

$$\text{left } \sum M_i^{(8)}: -S_{1,2}\cdot\frac{3}{3.059}\cdot 4.6-2\cdot 6=0 \quad \rightarrow \quad S_{1,2}=-2.660\ \mathrm{kN}(\mathrm{c})$$

For calculating member force $S_{1,8}$, principal point $O_{1,8}$ (intersection of lines (1,2) and (7,8)) should be determined:



The right triangle defined by nodes (1), (7) and principal point $O_{1,8}$ is similar to that considered previously, that is,

$$\frac{x}{4\,\mathrm{m}}=\frac{3\,\mathrm{m}}{0.6\,\mathrm{m}} \quad \rightarrow \quad x=20\,\mathrm{m} \quad .$$

It is now possible to write the balance of moments about point $O_{1,8}$, with force $S_{1,8}$ being resolved into components at node 8:

$$\text{left} \sum M_i^{(O_{1,8})} : -S_{1,8} \cdot \frac{4}{5} \cdot 23 + 2 \cdot 17 = 0 \quad \rightarrow \quad S_{1,8} = 1.848 \text{ kN} (\text{t})$$

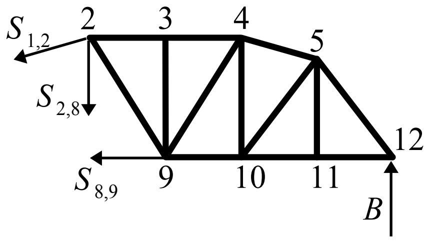
Member force $S_{7,8}$ ris calculated from a moment equation written about the point 1 of intersection of two remaining members as follows:

$$\text{left} \sum M_i^{(1)} : S_{7,8} \cdot 4 - 2 \cdot 3 = 0 \quad \rightarrow \quad S_{7,8} = 1.5 \text{ kN} (\text{t})$$

In order to check all forces in the current triple section, let a resolution along x be considered:

$$\text{left} \sum F_{ix} : (-2.660) \cdot \frac{3}{3.059} + (1.848) \cdot \frac{3}{5} + (1.5) = 0.0005 \approx 0 \cdot$$

Member force $S_{2,8}$ is calculated from a section involving members (1,2), (2,8) and (8,9). Use now the equilibrium of the right hand side structural unit for calculation.
Here is the free body diagram:



The equilibrium is stated as:

$$\text{right:} \ (S_{1,2}, S_{2,8}, S_{8,9}, B) \doteq 0$$

Member force $S_{2,8}$ is determined from the balance of moments about principal point $O_{2,8}$ (coincident with point $O_{1,8}$), that is, about the point of intersection of members $S_{1,2}$ and $S_{8,9}$:

$$\text{right} \sum M_i^{(O_{2,8})} : -S_{2,8} \cdot 23 + 1 \cdot 35 = 0 \quad \rightarrow \quad S_{2,8} = 1.522 \text{ kN} (\text{t})$$

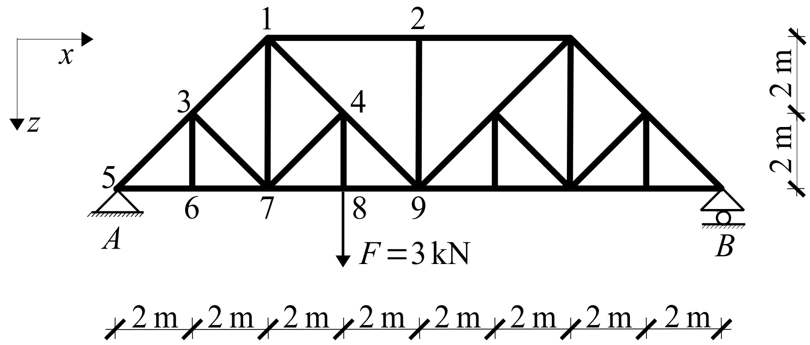
Let member forces $S_{1,8}$ and $S_{2,8}$ be checked by a vertical resolution equation based on the equilibrium of joint 8:

$$8 \sum F_{iz} : 3 - (1.848) \cdot \frac{4}{5} - (1.522) = -0{,}0004 \approx 0$$

All calculated results are finally given in a table of member forces as follows:

Member ID	Tensile [kN]	Compressive [kN]
(1,2)		2.660
(1,8)	1.848	
(2,8)	1.522	
(7,8)	1.5	

Exercise 1
Find forces in members between numbered joints of the truss shown.

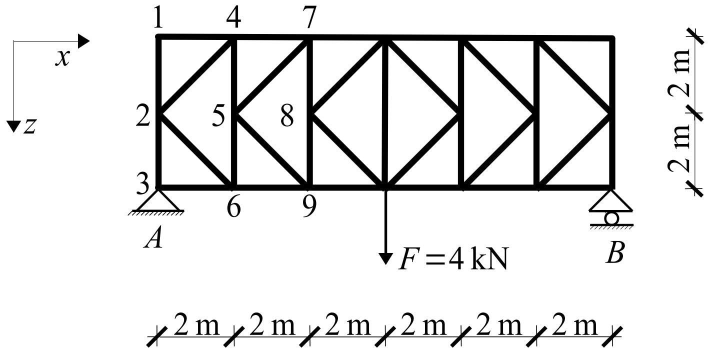


Solution

Table of member forces:

Member ID	Tensile [kN]	Compressive [kN]
(1,2)		
(1,3)		
(1,4)		
(1,7)		
(2,9)		
(3,5)		
(3,6)		
(3,7)		
(4,7)		
(4,8)		
(4,9)		
(5,6)		
(6,7)		
(7,8)		
(8,9)		

Example 2
Determine forces in members (4,7), (5,7), (5,9), (6,9), (7,8) and (8,9) of the K-truss shown.



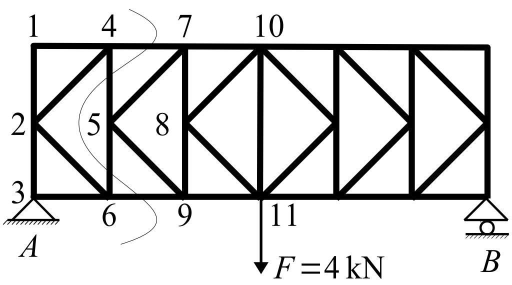
Solution

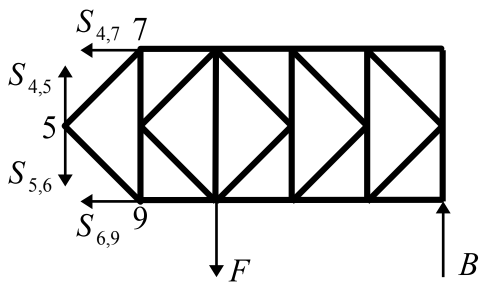
The first step of solution is still the determination of external reactions (yet without any details presented):

$A = 2\,\mathrm{kN}\,(\uparrow);\ B = 2\,\mathrm{kN}\,(\uparrow)$

The method of solution to a K-truss is pecial, as notriple sections can be found for that structure (by the removal of three members, the truss is not yet split into two disjoint parts). In order to have such a complete separation of two parts, at least four members should be „cut". This means, however, that there will be found four unknown member forces in the corresponding equilibrium statement, while the number of independent scalar equations continues to be three. Nevertheless, due to the special geometry of the web, it is still possible to obtain some member forces appearing in a four-fold section.

If member forces $S_{4,7}$ and $S_{6,9}$ are aimed at, let members (4,7), (4,5), (5,6) and (6,9) be removed. An advantage of this section, in contrast to another one involving two chord members and two oblique ones, is that two out of four members in the same section share a single line of action. This fact makes possible to write two moment equations involving just one unknown member force each. Let the structural unit to the left of the section be analysed:





Equilibrium statement for the analysed structural part reads:

right: $(F, B, S_{4,7}, S_{4,5}, S_{5,6}, S_{6,9}) \doteq 0$

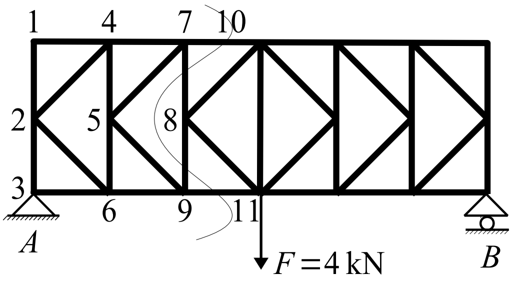
Member force $S_{4,7}$ is obtained from a balance of moments about the point of intersection (at node 6) of three other members in the section:

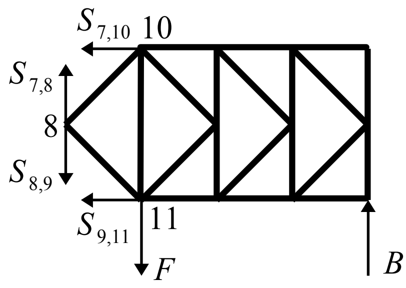
right $\sum M_i^{(6)}: -4 \cdot 4 + 2 \cdot 10 + S_{4,7} \cdot 4 = 0 \quad \rightarrow \quad S_{4,7} = -1\ \mathrm{kN}(\mathrm{c})$

Member force $S_{6,9}$ is obtained from a balance of moments about node 4:

right $\sum M_i^{(4)}: -4 \cdot 4 + 2 \cdot 10 - S_{6,9} \cdot 4 = 0 \quad \rightarrow \quad S_{6,9} = 1\ \mathrm{kN}(\mathrm{t})$

Forces in members $S_{5,7}$, $S_{5,9}$ will be determined from the equilibrium of joints 7 and 9, but before that, forces in members $S_{7,10}$, $S_{9,11}$ should be obtained from another four-fold section (this preliminary step is necessary because currently there are three unknown member forces in the equilibrium statement of both joint 7 and 9). Let a section through members (7,10), (7,8), (8,9) and (9,11) be considered and the equilibrium of the structural part at the right hand side analysed:





Equilibrium statement of the structural unit analysed:

right: $(F, B, S_{7,10}, S_{7,8}, S_{8,9}, S_{9,11}) \doteq 0$

Both chord member forces are obtained from equilibrium of moments as follows:

right $\sum M_i^{(9)}: -4 \cdot 2 + 2 \cdot 8 + S_{7,10} \cdot 4 = 0 \quad \rightarrow \quad S_{7,10} = -2\ \text{kN}(\text{c})$

right $\sum M_i^{(7)}: -4 \cdot 2 + 2 \cdot 8 - S_{9,11} \cdot 4 = 0 \quad \rightarrow \quad S_{9,11} = 2\ \text{kN}(\text{t})$

Forces in members (5,7) and (7,8) can now be obtained using the method of joints. Equilibrium staement for node 7 reads:

7: $(S_{4,7}, S_{5,7}, S_{7,8}, S_{7,10}) \doteq 0$

Member force $S_{5,7}$ is the only unknown variable in a horizontal resolution equation:

7 $\sum F_{ix}: 1 - S_{5,7} \cdot \cos(45°) - 2 = 0 \quad \rightarrow \quad S_{5,7} = -1.414\ \text{kN}(\text{c})$

Using this latter result, the vertical resolution equation includes only one unknown:

7 $\sum F_{iz}: (-1.414) \cdot \sin(45°) + S_{7,8} = 0 \quad \rightarrow \quad S_{7,8} = 1\ \text{kN}(\text{t})$

Forces in members (5,9), (8,9) are calculated accordingly, with respect to the equilibrium statement for node 9,

9: $(S_{5.9}, S_{6.9}, S_{8,9}, S_{9,11}) \doteq 0$:

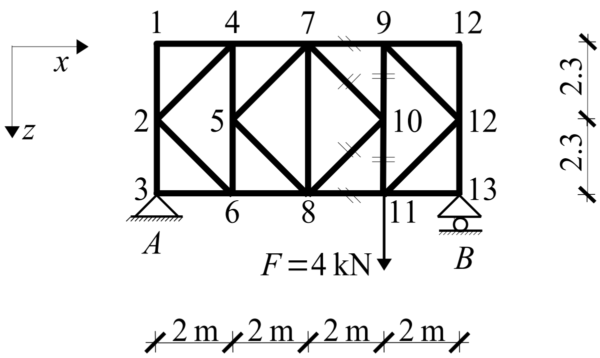
9 $\sum F_{ix}: -S_{5,9} \cdot \cos(45°) - 1 + 2 = 0 \quad \rightarrow \quad S_{5,9} = 1.414\ \text{kN}(\text{t})$

9 $\sum F_{iz}: -(1.414) \cdot \sin(45°) - S_{8,9} = 0 \quad \rightarrow \quad S_{8,9} = -1\ \text{kN}(\text{c})$

In the end, prepare the table of member forces:

Member ID	Tensile [kN]	Compressive [kN]
(4,7)		1
(5,7)		1.414
(5,9)	1.414	
(6,9)	1	
(7,8)	1	
(8,9)		1

Exercise 2
Find (and justify by equations) zero force members. Find forces in marked members of the K-truss shown.



Solution

Table of member forces:

Member ID	Tensile [kN]	Compressive [kN]