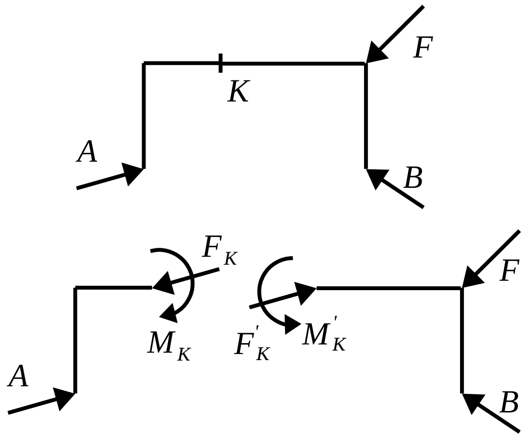
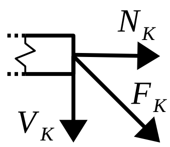
Internal forces

In the previous chapters reactions between rigid bodies or between a body and its support were focused on. Now the discussion will be concerned with forces *inside* rigid members of assemblies that are assumed to be in equilibrium. More precisely, assemblies built of long and slender members (*bars*) are considered; individual bar members are sometimes referred to as *beams* or *columns* depending on the role they play in load bearing. The direction along the largest extension of a bar is called *axial direction*, while a plane shape cut from the bar by a plane perpendicular to that axis is called *cross section*. In a more general approach, the axis of a bar that is not completely straight is usually defined as a series of centroids of successive cross sections of a bar (be careful with that definition, since cross sections were just defined using the concept of axis; sometimes it is not easy to find an axis that obeys these definitions). Once an axis is known, each cross section can be referred to by its position along the axis. Note that this new concept of bar is much wider than that of link (e.g., as a truss member), for example, hinged connection is not required, as well as loads of arbitrary distribution on bars are also allowed. From now on, *bar structures* (*frames*) are understood to be built of bars in the sense of the new definition.

According to the principle stated at compounds, global equilibrium of a structure implies the equilibrium for all its parts. The equilibrium analysis of such a part is possible by accounting for both the external loads exerted on it and forces (reactions) arising at cuts needed to isolate the respective part: those internal reaction components at cuts are termed *internal forces*. Let the bar structure be cut therefore at one of its cross sections. Due to the principle mentioned above, remaining parts at both sides of the cut still have to be in equilibrium one by one. Under the action of arbitrary loads it is only possible in the presence of (pairs of internal) reactions at the cut that correspond to the reactions of a fixed support. These are a force of arbitrary magnitude and sense (passing through the centroid of the cross section for the sake of uniqueness) as well as a torque on one side and pairs of those three on the other, obeying Newton's third law on action and reaction. For practical reasons, vectors of that force and torque are not taken by components along global coordinate directions x, y and z but rather in a way that components correspond to different mechanical effects they have on a bar.



The component of the force vector parallel to the axis of the bar is called *normal force* and denoted by N (as a reference to that the axial force is parallel to the normal direction of the cross section). A component of the force vector that lies within the plane of cross section is called *shear(ing force)* and denoted by V (but there can also be found letters T or Q for the same in literature). In a 3D problem, shear can always be resolved into two independent scalar components, for convenience, in a local frame set to the cross section itself. In a plane problem, out-of-plane shear is always zero.



The component of the torque vector parallel to the axis of the bar is called *twisting moment* and denoted by T. Twisting moments are always zero in plane problems, A component of the torque vector that lies within the plane of cross section is called *bending moment* and denoted by M. In a 3D problem, the bending moment can always be resolved into two independent scalar components, for convenience, in the same local frame as mentioned with shear. In a plane problem, in-plane vector component of the bending moment is always zero.

In summary, any cross section in plane problems has three internal force components: one normal force, one shearing force and one bending moment component.

Signs of internal forces

For the sake of uniqueness of internal force values as scalars, some rules on their signs are fixed as follows.

A *normal force* is positive if is directed outwards from the part of the bar it is exerted upon, that is, if it causes tension in the respective cross section. A negative normal force corresponds therefore to compression and is characterized by an arrow directed towards the cut. (This definition is in accordance with the sign rule applied to forces in links.)

A *twisting moment* is positive if its vector is directed outwards from the cross section (i.e., it represents a counterclockwise rotation if seen in front of the same cross section).

In a general case of 3D, there is no such a simple sign rule for shear and bending, that is why a convention valid only for plane problems is adopted here.

A shear force is defined to be positive if it is obtained by a clockwise rotation of the positive sense of the normal force by 90 degrees.

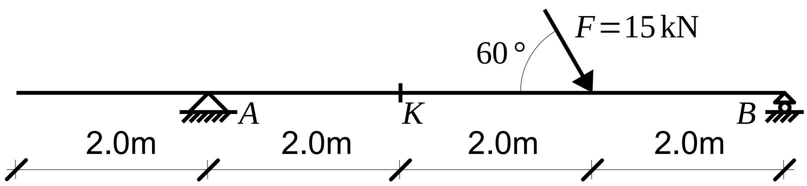
In order to decide upon the sign of a bending moment, it is necessary to set one and the other side of the axis of the bar to be positive and negative, respectively. A bending moment represented by a curved arrow on a cross section is defined to be positive if it causes tension at the positive side of the bar at the cross section (i.e., if the arrow starts at the positive and ends at the negative side of the bar). In practice, horizontal or nearly horizontal segments are mostly assumed to have their positive side at the bottom.

Calculating internal forces

Internal forces can obviously be found using the definitions given above.

Example 1

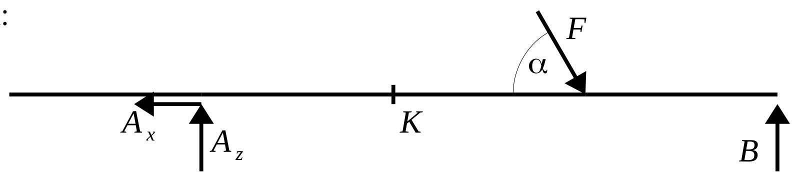
Find internal forces at section *K* of the structure given below.



Solution

The support reactions are calculated first.

Isolation:



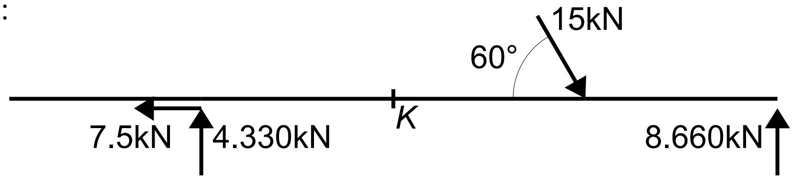
$\sum M_i^{(A)}$: $-15\sin 60°\cdot 4+B\cdot 6=0 \rightarrow B=8.660\,\text{kN}(\uparrow)$

$\sum M_i^{(B)}$: $15\sin 60°\cdot 2-A_z\cdot 6=0 \rightarrow A_z=4.330\,\text{kN}(\uparrow)$

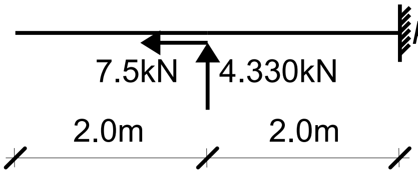
$\sum F_{ix}$: $15\cos 60°-A_x=0 \rightarrow A_x=7.5\,\text{kN}(\leftarrow)$

Check: $\sum F_{iz}=15\cdot\sin 60°-4.330-8.660=0.0004\approx 0$

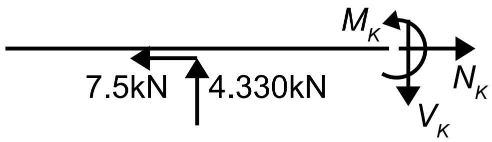
Final sketch:



Internal forces based on the equilibrium of the structural part to the left of *K*:



Isolation:

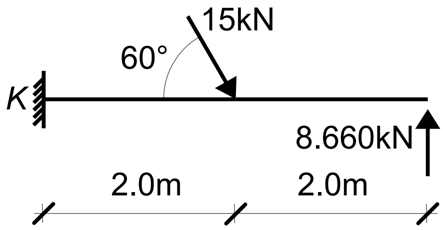


$\sum F_{ix}: -7.5+N=0 \rightarrow N=+7.5\,\text{kN}$

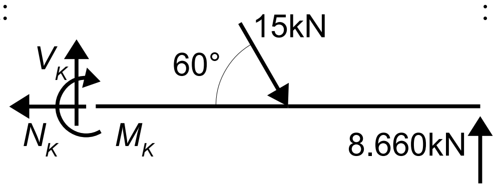
$\sum F_{iz}: -4.330+V=0 \rightarrow V=+4.330\,\text{kN}$

$\sum M_i^{(K)}: -4.330\cdot 2+M=0 \rightarrow M=+8.660\,\text{kNm}$

Internal forces based on the equilibrium of the structural part to the right of *K*:



Isolation: :



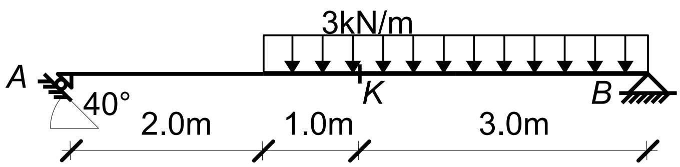
$\sum F_{ix}: -N+15\cos 60°=0 \rightarrow N=+7.5\,\text{kN}$

$\sum F_{iz}: -V+15\sin 60°-8.660=0 \rightarrow V=+4.330\,\text{kN}$

$\sum M_i^{(K)}: -M-15\sin 60°\cdot 2+8.660\cdot 4=0 \rightarrow M=+8.659\,\text{kNm}$

Exercise 1

Find internal forces at section *K* of the structure given below.



Solution

Support reactions of the beam found from equilibrium equations:

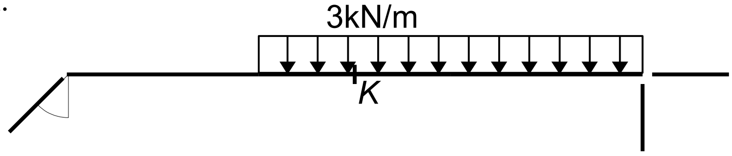
$\sum$: $\rightarrow A=$

$\sum$: $\rightarrow B_z=$

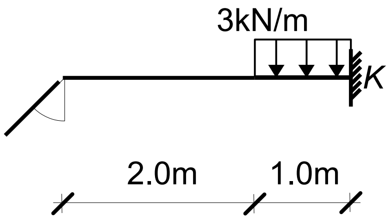
$\sum$: $\rightarrow B_x=$

Check: $\sum$:

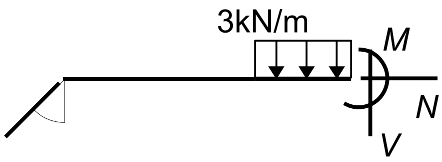
Final sketch:



Internal forces based on the equilibrium of the structural part to the left of *K*:



Isolation:

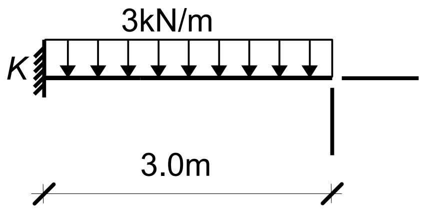


$\sum$: $\rightarrow N=$

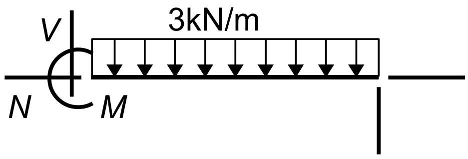
$\sum$: $\rightarrow V=$

$\sum$: $\rightarrow M=$

Internal forces based on the equilibrium of the structural part to the right of *K*:



Isolation:



$\sum$: $\rightarrow N=$

$\sum$: $\rightarrow V=$

$\sum$: $\rightarrow M=$

The examples above illustrate that there are always at least two methods for finding any internal force component in a cut that must provide the same result (any difference can only be due to round-off errors; this has already been experimented in trusses). In practice, it is sufficient to calculate the results just once.

Finding internal forces via reduction into a force-couple system

Assume that internal forces at cross section *K* are to be found. Let the resultants of external forces acting upon each side of the cut at *K* be denoted by R_1 and R_2, respectively. Because of the global equilibrium of the structure, those two resultants also maintain equilibrium:

$$(\mathrm{R}_1,\mathrm{R}_2) \doteq 0 \ .$$

Let internal forces required for the equilibrium of each side be denoted by I_1 and I_2, respectively. Because of the equilibrium of each side,

$$(\mathrm{R}_1,\mathrm{I}_1) \doteq 0 \text{ and } (\mathrm{R}_2,\mathrm{I}_2) \doteq 0 \ .$$

The three above statements of equilibrium yield that

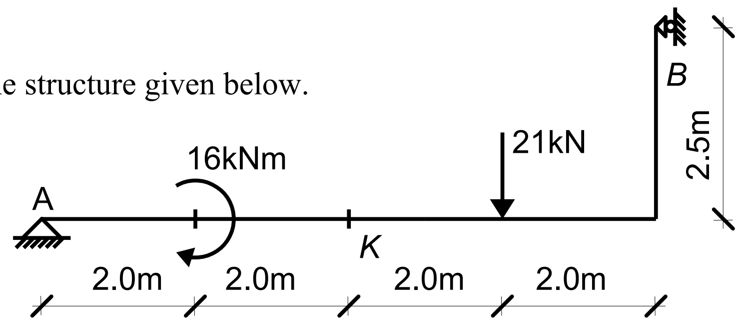
$$\mathrm{I}_1 \doteq \mathrm{R}_2 \ \ (\text{and } \mathrm{I}_2 \doteq \mathrm{R}_1) \ ;$$

meaning that internal forces at one (the other) side of the cut are equivalent to external forces acting upon the structural part at the other (the original) side. If internal forces I_1 (or I_2) are expressed with force and couple components exerted at the cross section, the problem means finding an equivalent force-couple system at the same cross section (reduction of external forces at one or the other side of *K* to the same cross section).

This procedure has a purely numeric advantage, that is, one side of the corresponding equation contains the respective internal force component alone. If the positive sense for the equation is set according to the positive internal force component, no reordering of the equation is needed (reducing so the chance of miscalculation).

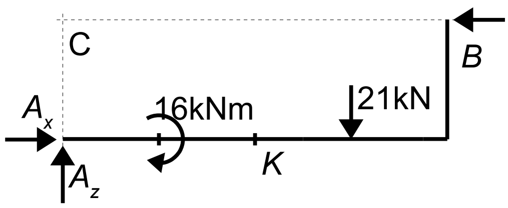
Example 2

Find internal forces at section *K* of the structure given below.



Solution

Isolation:



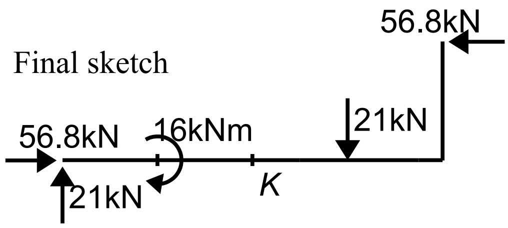
$\sum M_i^{(A)}: -16-21\cdot 6+B\cdot 2.5=0 \rightarrow B=+56.8\,\mathrm{kN}\,(\leftarrow)$

$\sum M_i^{(C)}: -16-21\cdot 6+A_x\cdot 2.5=0 \rightarrow A_x=+56.8\,\mathrm{kN}\,(\rightarrow)$

$\sum F_{iz}: -A_z+21=0 \rightarrow A_z=+21\,\mathrm{kN}\,(\uparrow)$

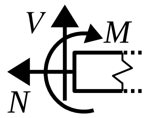
Check: $\sum F_{ix}: 56.8-56.8=0$

Final sketch



Internal forces at K based on forces to the left of K:

(now from A_x, A_z, M) Positive senses:



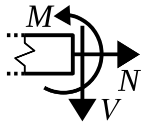
$\sum F_{i\leftarrow}$: $N = -56.8\,\text{kN}$

$\sum F_{i\uparrow}$: $V = +21\,\text{kN}$

$\sum M_{i\curvearrowright}^{(K)}$: $M = +21 \cdot 4 + 16 = +100\,\text{kNm}$

Internal forces at K based on forces to the right of K:

(now from F, B) Positive senses:



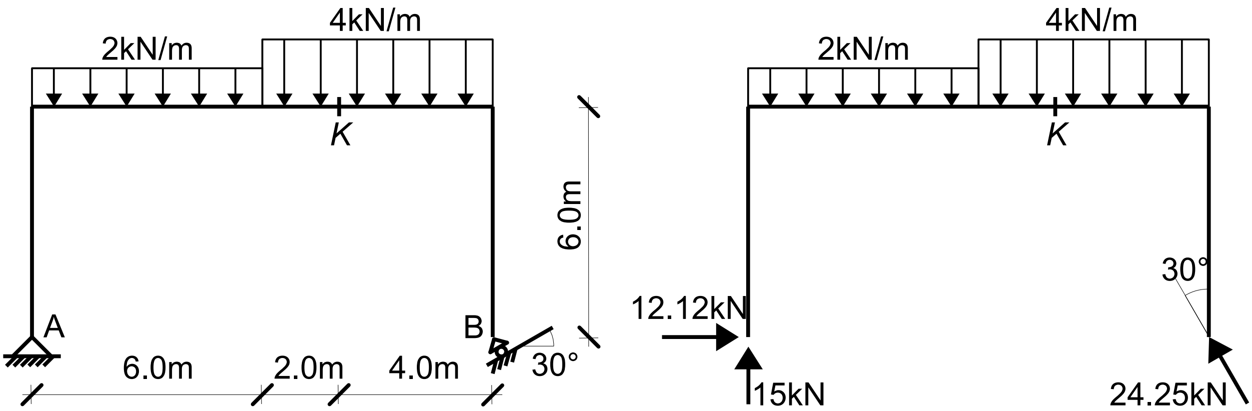
$\sum F_{i\rightarrow}$: $N = -56.8\,\text{kN}$

$\sum F_{i\downarrow}$: $V = +21\,\text{kN}$

$\sum M_{i\curvearrowleft}^{(K)}$: $M = -21 \cdot 2 + 56.8 \cdot 2.5 = +100\,\text{kNm}$

Exercise 2

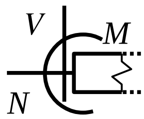
Based on known support reactions, find internal forces at section *K* of the structure.



Solution

Internal forces from left:

Forces to be considered: Positive senses:



....

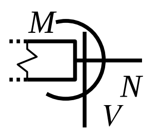
$N =$

$V =$

$M =$

Internal forces from right:

Forces to be considered: Positive senses:



....

$N =$

$V=$

$M=$

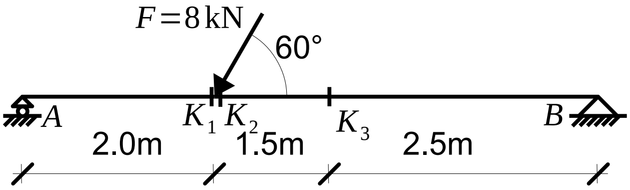
If not earlier, by completing this solution it could became obvious that active forces and reactions are not distinguished while finding internal forces: in any case, the calculation was based on all external forces acting on *either* (just one or just the other!) side of the section at *K*.

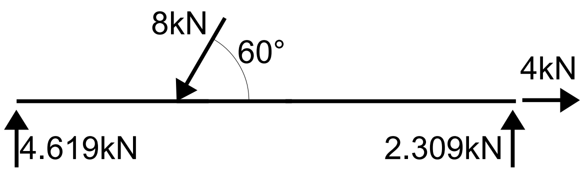
Finding internal forces from the same at another section

It is already known that if a structure is cut into two parts and both parts are acted upon by internal forces at the cut then both parts remain in equilibrium. Any new cut of such a part in equilibrium makes possible the calculation of internal forces there based on the equilibrium of the respective structural part instead of the complete structure.

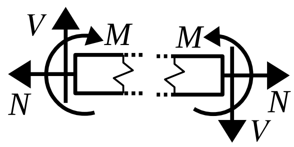
Example 3

Find internal forces at cross section K_1 located to the left of the point of application of force F by an infinitesimally small distance. Based on them, find internal forces also at cross sections K_2 (slightly to the right of the point of application of F) and K_3 .





Solution



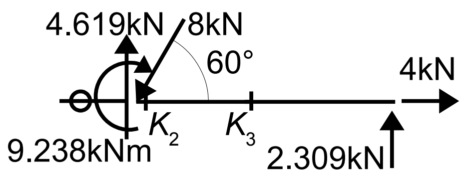
Internal forces at K_1 , for convenience, from the left:

$N_1 = 0\,\text{kN}, \quad V_1 = +4.619\,\text{kN}, \quad M_1 = +4.619 \cdot 2 = +9.238\,\text{kNm}$

These internal forces can be drawn as external forces to the structural part to the right (real directions and senses can be drawn in accordance with the sign rules).

Internal forces at K_2 from left:

(to be considered: internal forces at K_1 and force F)



$N_2 = 0 + 8\cos 60° = +4\,\text{kN}$

$V_2 = +4.619 - 8\sin 60° = -2.309\,\text{kN}$

$M_2 = 9.238 + 4.619 \cdot 0 - 8\sin 60° \cdot 0 = +9.238\,\text{kNm}$

Internal forces at K_3 from left:

(to be considered: internal forces at K_1 and force F)

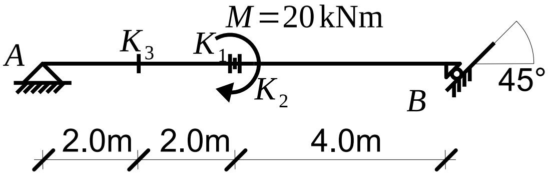
$N_3 = 0 + 8\cos 60° = +4\,\text{kN}$

$V_3 = +4.619 - 8\sin 60° = -2.309\,\text{kN}$

$M_3 = 9.238 + 4.619 \cdot 1.5 - 8\sin 60° \cdot 1.5 = +5.774\,\text{kNm}$

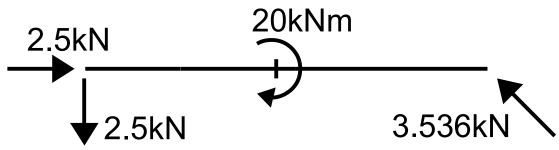
Exercise 3

Find internal forces at cross section K_1 located to the left of the point of application of torque M by an infinitesimally small distance. Based on them, find internal forces also at cross sections K_2 (slightly to the right of the point of application of M) and K_3 .

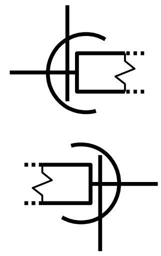


Solution

The support reactions (without details):



Internal forces at K_1 (from either side):



$N_1 =$

$V_1 =$

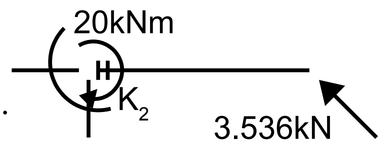
$M_1 =$

Draw actual arrows of internal force components on both stubs at the section.

Internal forces at K_2 (from left, since components at K_1 are to be used):

Draw actual arrows of internal force components at K_1 .

Force and torque components to be considered: ....



$N_2 =$

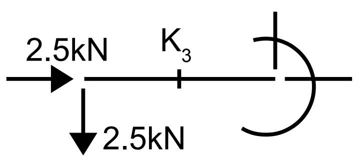
$V_2 =$

$M_2 =$

Internal forces at K_3 (from left, since components at K_1 are to be used):

Draw actual arrows of internal force components at K_1 .

Components to be considered: ....



$N_3 =$

$V_3 =$

$M_3 =$

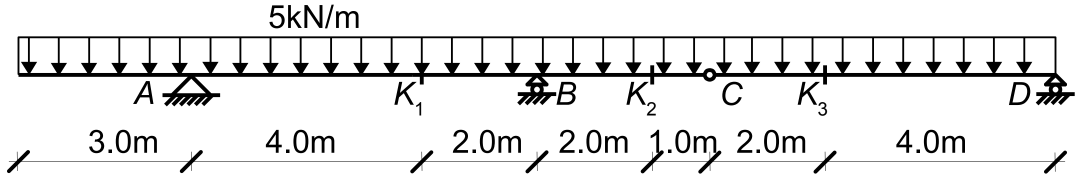
Conclusions: there is a jump between two values of the normal and shear forces at two sides of an external concentrated force: the magnitude of the jump equals the corresponding projection of the concentrated force. Likewise, there is a jump in bending moments at two sides of a concentrated external torque: the magnitude of the jump equals the magnitude of torque, while there is no change in values of other internal force components.

Internal forces at cross sections of a compound structure

In compounds there are even more possibilities for finding internal forces than in simple structures. It is explained by that an equilibrium not only for the complete structure but also for individual rigid bodies must hold; consequently, a substructure containing a given cross section is also suitable for finding internal forces there. At the same time, each internal force component can be calculated from right or left, making the minimal number of possible calculations to be four. It continues to hold; however, that any calculation is still sufficient to be performed only once. For this reason, the first step of the solution will be later on to find the simplest way among those four possibilities.

Example 4

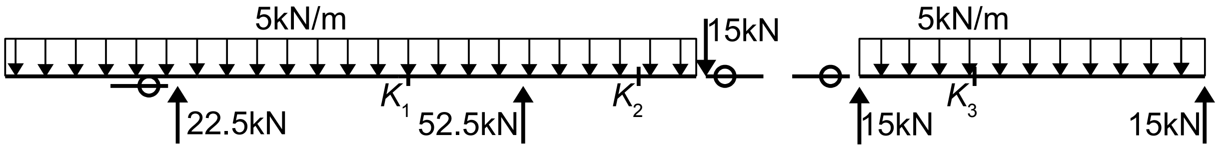
Find internal forces at cross sections K_1, K_2 and K_3.



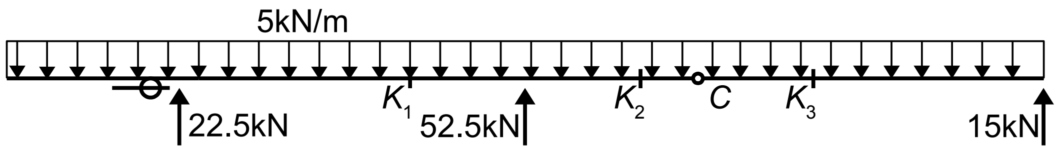
Solution

This is a problem about a Gerber beam. External and internal reactions are found first on the suspended then on the fixed part. Results without detailed calculations are given below.

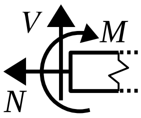
Final sketches for member I (to the left) and member II (to the right) separately:



Final sketch of the complete structure:



The axis of the beam is horizontal at each cross section, causing all normal forces to be horizontal as well. At the same time, all external (active and passive) forces are vertical, so no horizontal components could be written in formulae for normal forces; thus, $N_1 = N_2 = N_3 = 0$. Only shear forces and bending moments are dealt with henceforth in this problem.



section K_1 from left based on member I:

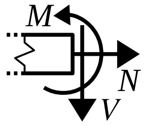
$V_1 = 22.5 - 5 \cdot 7 = -12.5\,\text{kN}$

$M_1 = 22.5 \cdot 4 - (5 \cdot 7) \cdot 3.5 = -32.5\,\text{kNm}$

K_1 from right based on the complete structure:

$V_1 = 22.5 - 5 \cdot 7 = -12.5\,\text{kN}$

$M_1 = 22.5 \cdot 4 - (5 \cdot 7) \cdot 3.5 = -32.5\,\text{kNm}$



section K_1 from right based on member I:

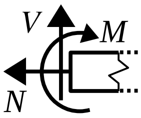
$V_1 = 15 + 5 \cdot 5 - 52{,}5 = -12.5\,\text{kN}$

$M_1 = -15 \cdot 5 - (5 \cdot 5) \cdot 2.5 + 52.5 \cdot 2 = -32.5\,\text{kNm}$

K_1 from right based on the complete structure:

$V_1 = -15 + 5 \cdot 11 - 52.5 = -12.5\,\text{kNm}$

$M_1 = 15 \cdot 11 - (5 \cdot 11) \cdot 5.5 + 52.5 \cdot 2 = -32.5\,\text{kNm}$



K_2 from left (member I):

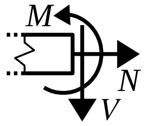
$V_2 = 22.5 - 5 \cdot 11 + 52.5 = +20\,\text{kN}$

$M_2 = 22.5 \cdot 8 - (5 \cdot 11) \cdot 5.5 + 52.5 \cdot 2 = -17.5\,\text{kNm}$

K_2 from left (complete structure):

$V_1 = 22.5 - 5 \cdot 11 + 52{,}5 = +20\,\text{kN}$

$M_1 = 22.5 \cdot 8 - (5 \cdot 11) \cdot 5.5 + 52.5 \cdot 2 = -17.5\,\text{kNm}$



K_2 from left (member I):

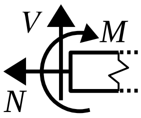
$V_2 = 15 + 5 \cdot 1 = +20\,\text{kN}$

$M_2 = -15 \cdot 1 - (5 \cdot 1) \cdot 0.5 = -17.5\,\text{kNm}$

K_2 from right (complete structure):

$V_2 = -15 + 5 \cdot 7 = +20\,\text{kNm}$

$M_2 = 15 \cdot 7 - (5 \cdot 7) \cdot 3.5 = -17.5\,\text{kNm}$



K_3 from left (member II):

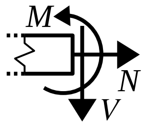
$V_3 = 15 - 5 \cdot 2 = +5\,\text{kN}$

$M_3 = +15 \cdot 2 - (5 \cdot 2) \cdot 1 = +20\,\text{kNm}$

K_3 from left (complete structure):

$V_3 = 22.5 - 5 \cdot 14 + 52.5 = +5\,\text{kN}$

$M_3 = +22.5 \cdot 11 - (5 \cdot 14) \cdot 7 + 52.5 \cdot 5 = +20\,\text{kNm}$



K_3 from right (member II):

$V_3 = -15 + 5 \cdot 4 = +5\,\text{kN}$

$M_3 = +15 \cdot 4 - (5 \cdot 4) \cdot 2 = +20\,\text{kNm}$

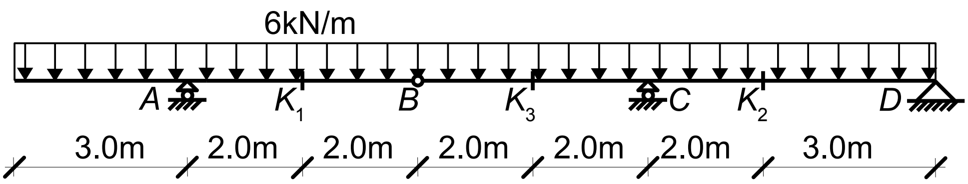
K_3 from right (complete structure):

$V_3 = -15 + 5 \cdot 4 = +5\,\text{kN}$

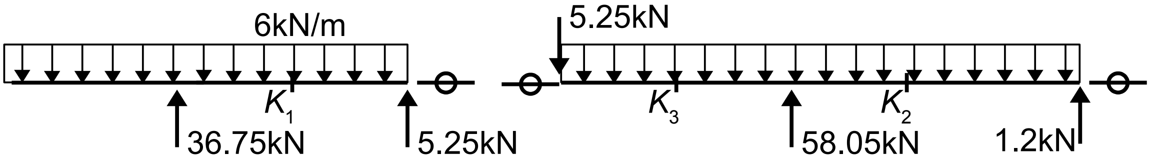
$M_3 = +15 \cdot 4 - (5 \cdot 4) \cdot 2 = +20\,\text{kNm}$

Exercise 4

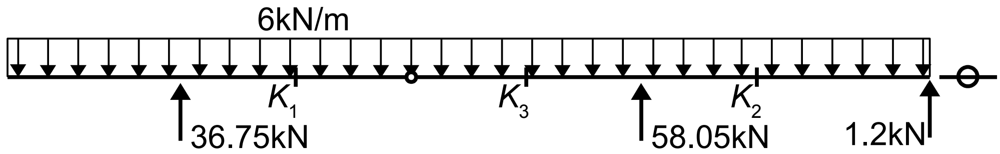
Find internal forces at cross sections K_1, K_2 and K_3 based on support reactions.



Final sketches for member I (to the left) and member II (to the right) separately:



External forces acting upon the entire assembly:



Solution

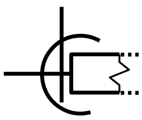
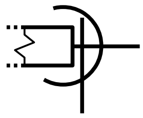
Calculation of normal forces

Which forces do contribute to the normal force of a section with horizontal (normal) axis?

....

Accounting for them all, $N_1 =$ $N_2 =$ $N_3 =$

Calculation of shear forces and bending moments

Positive senses for forces taken from left:  , for forces taken from right: 

Internal forces at K_1 based on member I:

from left: $V_1 =$

$M_1 =$

from right: $V_1 =$

$M_1 =$

Internal forces at K_1 based on the complete structure:

from left: $V_1 =$

$M_1 =$

from right: $V_1 =$

$M_1 =$

Internal forces at K_2 based on member II:

from left: $V_2 =$

$M_2 =$

from right: $V_2 =$
$M_2 =$

Internal forces at K_2 based on the complete structure:

from left: $V_2 =$
$M_2 =$

from right: $V_2 =$
$M_2 =$

Internal forces at K_3 based on member II:

from left: $V_3 =$
$M_3 =$

from right: $V_3 =$
$M_3 =$

Internal forces at K_3 based on the complete structure:

from left: $V_3 =$
$M_3 =$

from right: $V_3 =$
$M_3 =$

As illustrated by the above example, there are always more than one possibilities for calculating internal forces at a given section. For convenience, it is decided first which of the equations for the same component can be set up and solved by minimum effort and/or maximum safety. Sometimes it is necessary to consider aspects that contradict each other: a more compact expression is easier to evaluate but the less are recently obtained scalars involved, the higher is the reliability of the final result. Based on these observations, some thumb rules can be formulated as follows:

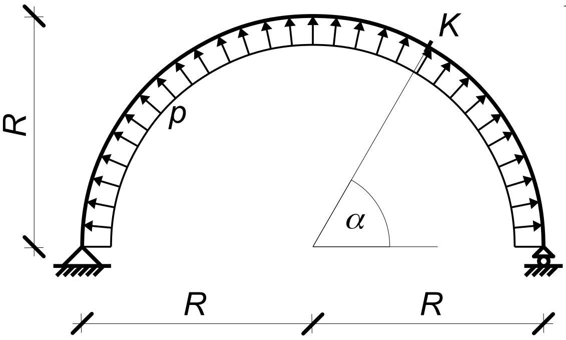
- Internal forces at a section on a cantilever beam or overhang are always obtained from the side of its free end (that is, no reactions are involved in the calculation).
- If concentrated forces or moments (either as active loads or internal / external reactions) are known at one end of a beam (which can therefore be physically attached to another member), internal force components at the same end of the beam member can be obtained directly from them. Those values are often zero; if not, only their signs are to be decided.

Appendix: Non-straight bars

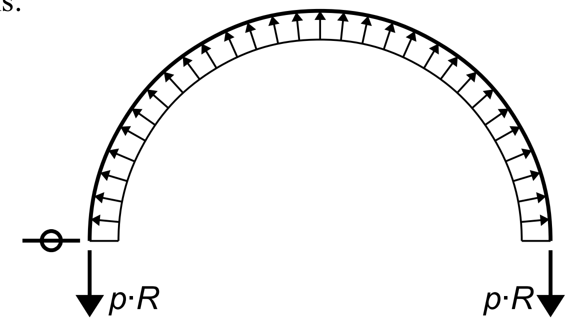
In all previous examples bars (beams) with a horizontal axis were only dealt with. In some of the forthcoming problems one will have to count with forces projected to variable directions in order to get internal force components. This procedure is demonstrated by the last example.

Example 5M

Find internal forces at cross section *K* of the semicircular arc subjected to a uniform radial load.



Reactions:



Solution

Internal forces are calculated from right. The load is distributed perpendicularly to the surface, it is transformed into projected loads according to the figure to the right.

Three forces are considered:

The reaction at the roller support: $p \cdot R$

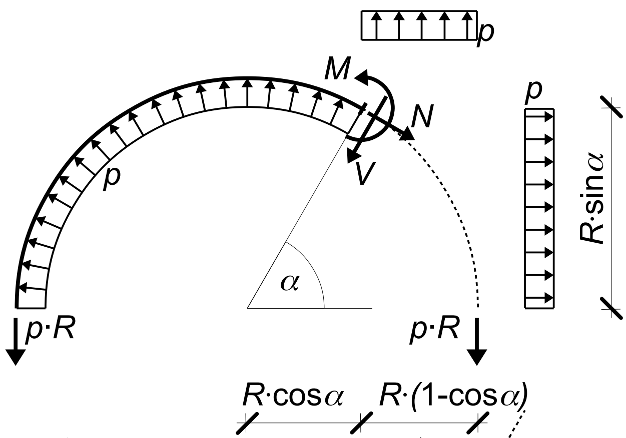
The resultant of vertical projection of the load: $P_v = p \cdot R(1-\cos\alpha)$

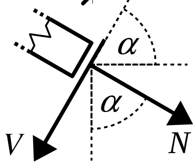
The resultant of horizontal projection of the load: $P_h = p \cdot R \sin\alpha$

In calculating normal and shear forces, first the sign for each force component then the corresponding trigonometric function is chosen:

$$N = +p \cdot R\cos\alpha - p \cdot R(1-\cos\alpha)\cos\alpha - \\ + p \cdot R\sin\alpha\sin\alpha = \\ = +p \cdot R\cos^2\alpha + p \cdot R\sin^2\alpha = +p \cdot R$$

$$V = +p \cdot R\sin\alpha - p \cdot R(1-\cos\alpha)\sin\alpha - \\ - p \cdot R\sin\alpha\cos\alpha = 0$$

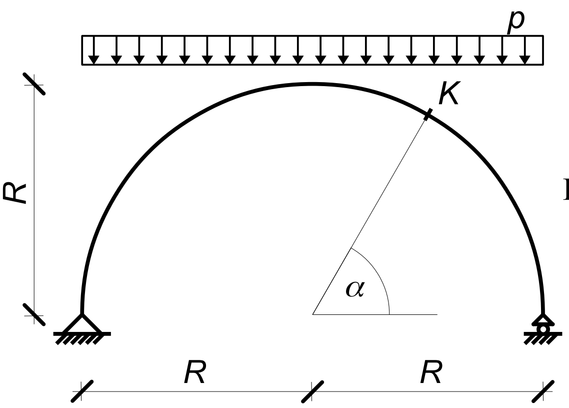




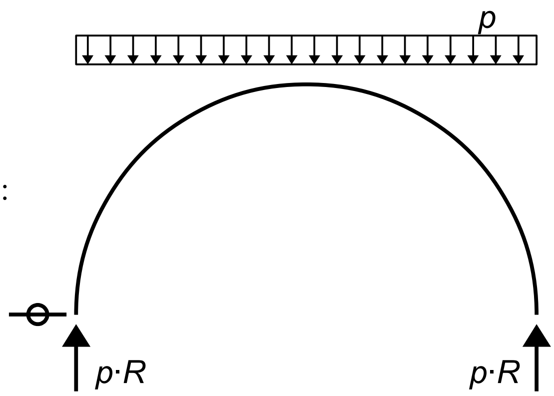
$$M = -p \cdot R \cdot R(1-\cos\alpha) + p \cdot R(1-\cos\alpha)\frac{R(1-\cos\alpha)}{2} - \\ + p \cdot R\sin\alpha\frac{R\sin\alpha}{2} = p \cdot R^2\left(-1+\cos\alpha+\frac{1}{2}-\frac{2\cos\alpha}{2}+\frac{\cos^2\alpha}{2}+\frac{\sin^2\alpha}{2}\right) \\ = 0$$

Exercise 5M

Find internal forces at cross section K of the semicircular arc subjected to a projected vertical load.



Reactions:

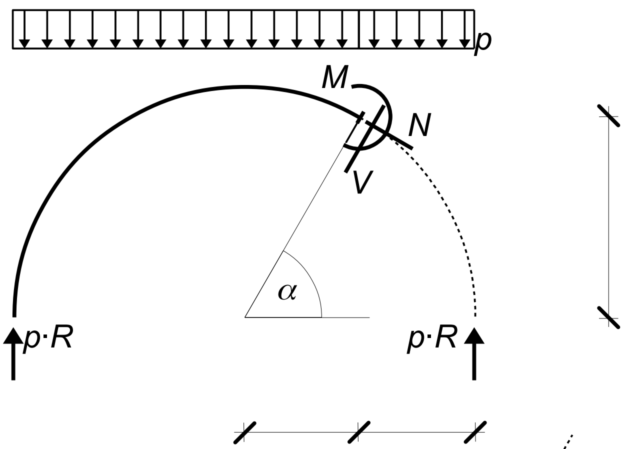


Solution

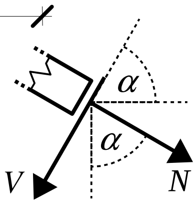
Forces on one structural part located at either side of a given cross section should be considered only.

Mark those forces in the figure and specify positive senses for all internal force components.

How much is the resultant of distributed force that should be calculated with?



Internal forces are as follows:



$N =$

$V =$

$M =$

Internal force diagrams

It has recently been shown how internal forces at an arbitrary section of a body in equilibrium can be found. By considering also the relationship between internal forces of cross sections close to each other, a qualitative information could be obtained about changes in internal force diagrams along the axis of a member. It would still be more effective to have a tool for expressing internal forces at *all* sections, that is, as a function of the position along the axis. For clarity and simplicity, those functions are plot in practice against the axes of members in a structure. Knowing that a plane problem implies the existence of three different internal force components, it is spoken about the *normal force diagram*, *shear force diagram* and *bending moment diagram*; a collective name for them is *internal force diagram*. In those diagrams, a given ordinate pertaining to a cross section K is plotted against the member axis at K. One side of the axis is considered positive in the same way as it was done in defining the sign rule of bending moment values.

It seems to be logical that internal force functions are drawn on the basis of function assignments as suggested by their definition but it is typically avoided. It is habitual in civil engineering instead that the character (order) of the function is identified at each segment in accordance with the load; then some characteristic values are calculated that are already sufficient to make the function be uniquely defined.

Such a character of the function can be *constant* when the displayed function runs parallel to the axis of the bar. Furthermore, a function can be *linear* which is defined by two values at its ends (and can be represented, of course, by a straight line segment between them) ; or it can be *parabolic* (that is, a second-order polynomial) which needs one more value to be specified in addition to those at its ends.

The relationship between loads and internal forces

If an elementary small segment of a beam is isolated and internal force components acting at both cross sections as well as (elementary) external loads are drawn in a FBD, valuable conclusions can be drawn from equilibrium equations as follows. The first derivative of the bending moment function is proportional to the value of shear; the first derivative of the shear force function is proportional to the local value of intensity of load perpendicular to the axis, whereas the first derivative of the normal force function is proportional to the axial intensity of load. These three relationships serve as a basis for deriving some more rules, although their direct application is not always possible because of *singularities* exemplified by concentrated forces or torques. Another difficulty is that calculation normally proceeds in an opposite sense, aiming at the evaluation of internal force functions from the load function. For that purpose, however, it is not sufficient to say 'integral' instead of 'derivative' since a constant C of integration is to be found from boundary conditions at each segment. Nevertheless, qualitative conclusions can easily be drawn from differential relationships listed above. For example, a bending moment diagram is flatter (steeper) under smaller (larger) values of shear and a zero shear implies a local extremum (horizontal tangent) of the bending moment diagram. Likewise, the shear diagram is flatter (steeper) under smaller (larger) intensity of distributed perpendicular load.

Relationships that are most commonly used can also be found simply by thinking over what kind of equations could we obtain those values from. This is discussed in the following paragraphs.

Normal and shear forces on straight beam segments with no external load there will be of constant value. (It goes back to that the calculation of those internal forces at two different sections from the same side is done using the same force components under the same rules for signs.)

Straight beam segments without external loads will be associated with a linear bending moment function. (It is explained by that the bending moment at any two sections on that segment is found from the same resultant from the same side and so under the same rules for signs. In such a calculation, only the moment arms for the two sections are different: it can be shown by similar triangles that the change in moments is proportional to the distance between the two sections.)

In the presence of a concentrated external torque, normal and shear forces are found from unchanged force resolutions; that is why there is no change in normal and shear force diagrams at concentrated torques. Moment diagram, however, will have a jump there as bending moments at opposite sides of an external torque are found from the same force components with the same moment arms except the torque itself which does and does not appear in the sum of moments depending on whether the section is located at the right or left hand side of the torque. (Note that no other change than a jump is produced in a moment diagram under a concentrated torque, slopes of the diagram at both sides of the jump are equal.)

In the presence of a concentrated external force, normal and shear forces are found from unchanged force resolutions except the force itself (more precisely, the force component parallel and perpendicular to the axis, respectively), which does and does not appear in the sum of forces depending on whether the section is located at the right or left hand side of the force. Thus, a normal force (shear) diagram will have a jump under a concentrated force perpendicular (parallel) to the beam axis. In the calculation of bending moments, however, the presence or absence of a concentrated force component between infinitely close sections makes no difference as its moment is written about a point on its line of action in both cases. (Despite the coincidence of moment values, the slopes on opposite sides are different and so a kink is formed if there is a jump in the shear diagram.)

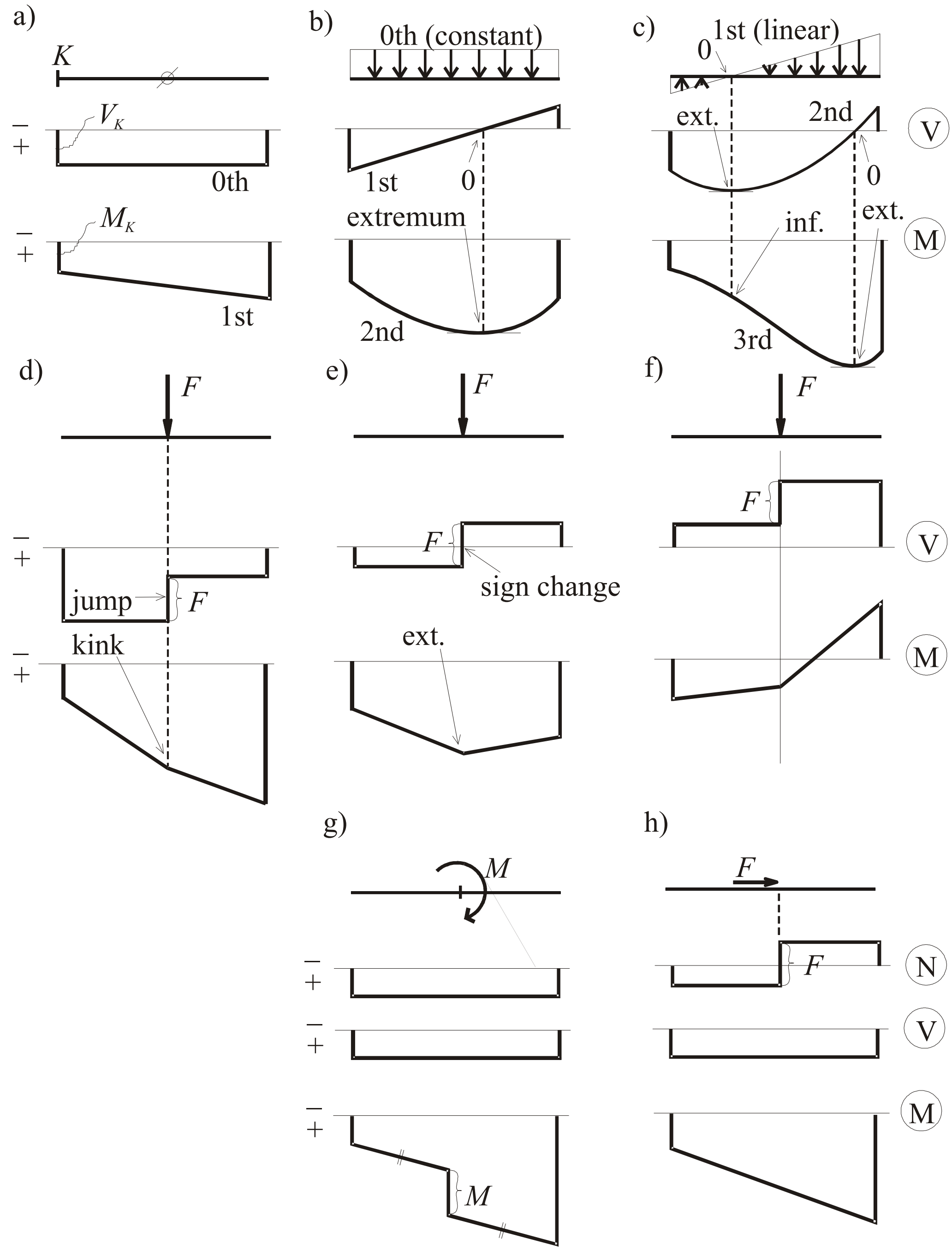
Let parallel and perpendicular components be distinguished also for distributed loads on a straight segment of a member. The parallel component has an influence on the function of normal forces, making its diagram to be linear. It is explained by that the amount of forces in axial direction to be included in calculations grows proportionally with the length. The remaining two internal force functions are independent of this load component.

In the presence of a distributed load perpendicular to the axis of the member, successive calculations of the shear force involves more force components proportionally with the length. As a result, the shear diagram will be linear. When the bending moment is calculated, linear growth of forces to be accounted for occurs together with a linear growth in their moment arm, that is why the moment function as a product of two linear terms follows the shape of a parabola. A segment of a quadratic function can be specified, in addition to its endpoints, by its depth (which is the maximum deviation measured perpendicularly to the member axis between the parabola and a chord connecting its endpoints): it is always found in the middle of the segment. The depth of a parabola can be given by the formula $ql^2/8$ where q is the intensity of perpendicular load component and l is the length of the entire segment under a constant load.

$$\frac{ql^2}{8}$$

Use of the depth for drawing parabolas: bending moment ordinates at two endpoints are connected by a straight chord first. A line segment perpendicular to the member axis is drawn through the midpoint of the chord and a distance of $ql^2/8$ is measured *twice* along it from the chord in the direction of load. (Mind that depth is always perpendicular to the member axis, not to the chord.) By connecting this point with both ends of the chord, tangent lines to the parabola at its endpoints are obtained. According to the definition of depth, the point at $ql^2/8$ from the chord is not only incident to the parabola but its tangent is parallel to the chord. Based on these three points and tangents, the parabolic segment can easily drawn with free hand. Notice that the procedure can be continued recursively if more points are needed on a long segment: any two existing points can be used to repeat the above construction for smaller parts of the diagram.

All discussed relationships must hold also in reverse order: any jumps, kinks or curved parts in a diagram must be justified by an external effect that causes it. The most common cases of related load and internal force diagrams are shown in the following figure.



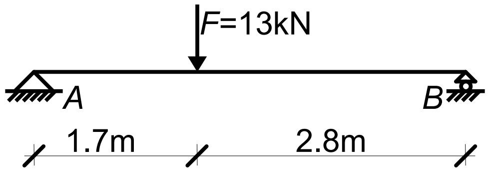
The relationship between the load and the bending moment diagram can be verified in a quite ingenuous way by imagining the load to be exerted on a rubber string. The shape of the string corresponds then to the bending moment diagram: unloaded segments remain straight, concentrated and distributed forces cause kinks and curvatures in it, respectively.

Values used for drawing internal force diagrams are written to the right of the same diagrams. The corresponding values are numbered from left according to the section they belong to (e.g., M_2 stands for the moment in the second section from the left). The order of determination of internal forces is arbitrary, hence ordinates with larger subscripts will sometimes be found first (e.g., at overhangs where internal forces do not depend on support reactions of the assembly). Any internal force diagrams must be documented so that it could uniquely be reproduced based on the results. It means, e.g., that it is not necessary to write ordinates at both ends of a constant segment if the constant property is also written out in the diagram.

Drawing internal force diagrams based on (static) calculations

Example 1

Draw internal force diagrams for the structure based on (static) calculations.



Solution

Support reactions should still be found first:

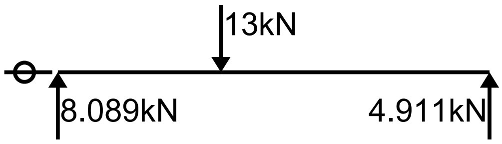
$\sum M_i^{(A)}: -13\cdot 1.7 + B\cdot 4.5 = 0 \rightarrow B = 4.911\,\text{kN}\,(\uparrow)$

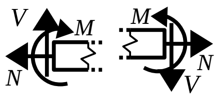
$\sum M_i^{(B)}: 13\cdot 2.8 - A_z\cdot 4.5 = 0 \rightarrow A_z = 8.089\,\text{kN}\,(\uparrow)$

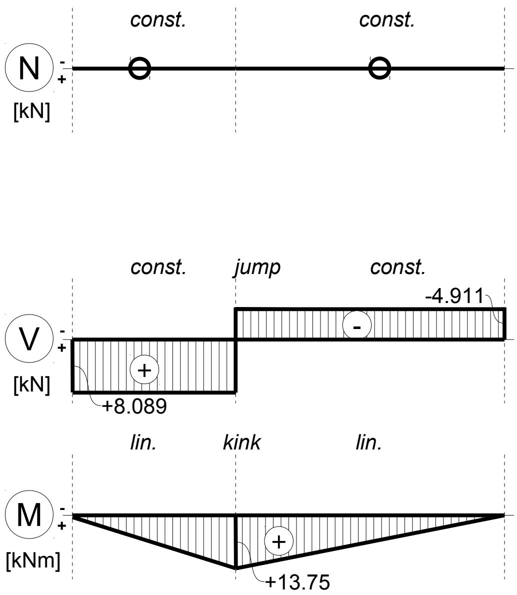
$\sum F_{ix}: A_x = 0\,\text{kN}$

Check: $\sum F_{iz}: 13 - 4.911 - 8.089 = 0$

Final sketch:







The normal force diagram consists of two constant segments. There is a jump between them of a magnitude corresponding to the horizontal component of force *F* (which is now zero). In a calculation from any side on any member there is no force having a horizontal component at all, so the diagram is a constant of zero.

Small circle drawn onto the axis means a notification that it is 'calculated' instead of being forgotten.

The shear force diagram consists of two linear segments with a jump between them. Its magnitude corresponds to the vertical component of force F:

$V_1 = +8.089\,\text{kN}$ (for convenience, from left)

$V_2 = -4.911\,\text{kN}$ (for convenience, from right)

The diagram of bending moments consists of two linear segments with a kink (but no jump) in between. At each end of the beam, the bending moment is zero. Under force F:

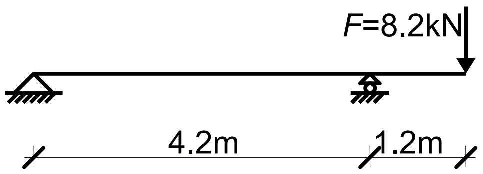
$M_1 = +8.089\cdot 1.7 = 13.75\,\text{kNm}$ (from left)

(would be $M_1 = 4.911\cdot 2.8 = 13.75\,\text{kNm}$ from right)

About the M diagram: Moment is zero at both ends (calculated from outside). Reactions point upwards that causes a kink with respect to the fictitious constant zero on inexistent overhangs. There is also a kink (but no jump in the lack of concentrated torque) at the external force *F*.

Exercise 1

Draw internal force diagrams for the structure based on calculations.



Solution

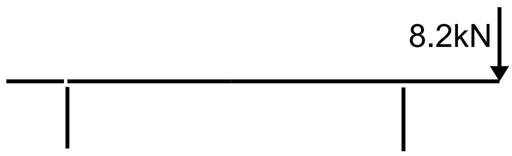
Reactions:

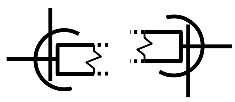
$\sum$:

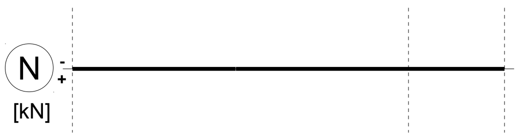
$\sum$:

$\sum$:

Final sketch (global equilibrium):



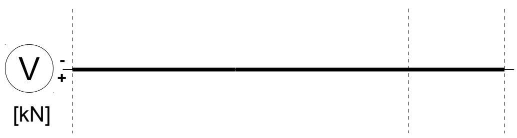




What kind of functions is the diagram composed of?
What is their connection like?

$N_1 =$

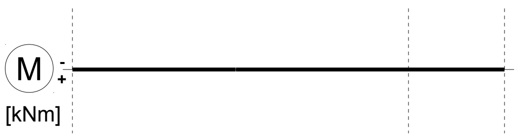
$N_2 =$



What kind of functions is the diagram composed of?
What is their connection like?

$V_1 =$

$V_2 =$



What kind of functions is the diagram composed of?
What is their connection like?

$M_1 =$

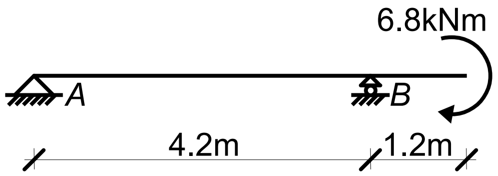
$M_2 =$

$M_3 =$

Check the existence and sense of jumps and kinks.

Example 2

Draw internal force diagrams for the structure based on calculations.



Solution

Reactions:

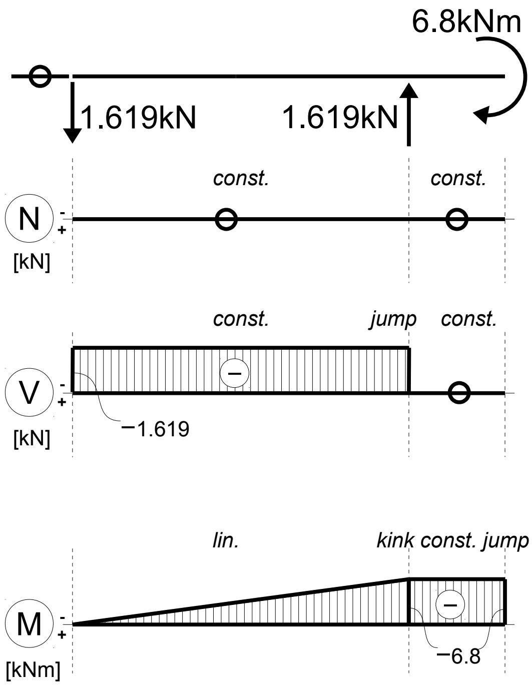
$\sum M_i^{(A)}: -6.8 + B \cdot 4.2 = 0 \rightarrow B = 1.619\,\text{kN}\,(\uparrow)$

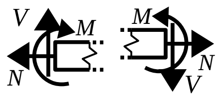
$\sum M_i^{(B)}: -6.8 - A_z \cdot 4.2 = 0 \rightarrow A_z = -1.619\,\text{kN}\,(\downarrow)$

$\sum F_{ix}: A_x = 0$

Check: $\sum F_{iz}: 1.619 - 1.619 = 0$

Final sketch:





The normal force diagram is composed of constant segments again. Their value is zero because no horizontal force components act upon the structure.

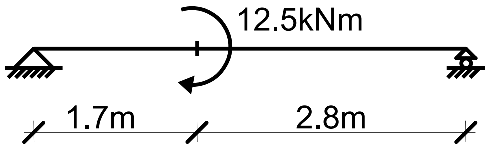
The shear force diagram consists of two linear segments with a jump corresponding to force B.
If values to the right of B are calculated from right: they are obviously zero, the shear between supports based on a calculation from either side is:
$V = -1.619\,\text{kN}$

The type of loading corresponds to two linear segments in the diagram. On the overhang it is more special with a constant value because of the zero shear. Its value obtained from right is:
$M_2 = -6.8\,\text{kNm}$ (At both ends of the segment, so those ordinates could have been connected as a linear part.)
At the right hand side of the segment between supports the value is still $-6.8\,\text{kNm}$, since there is no change in the moment under the force B.
At the left end, calculated from left we have
$M_1 = 0$ (But also from right it would be $-6.8 + 1.619 \cdot 4.2 = -0.0002 \approx 0$.)

About the M diagram: There is no force but a torque only on the overhang, so there is no shear there either. Thus, the diagram will be constant there, with an ordinate drawn to the top side because the tension appears at the top as well.

Exercise 2

Draw internal force diagrams for the structure based on calculations.



Solution

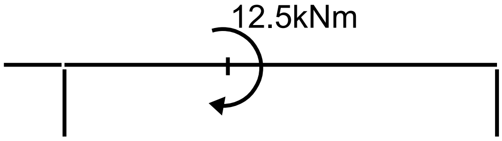
Reactions:

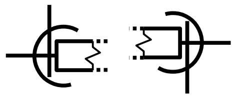
$\sum$:

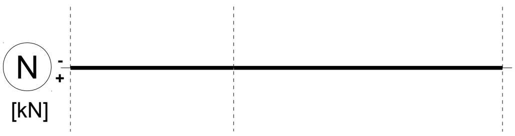
$\sum$:

$\sum$:

Final sketch (global equilibrium):

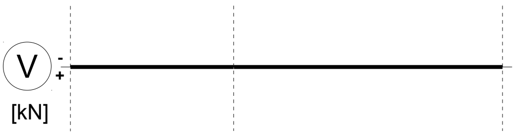






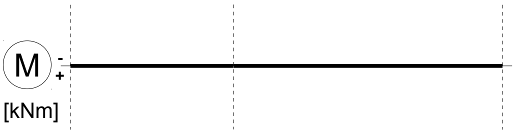
What kind of functions is the diagram composed of?
What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of?
What is their connection like?

$V_1 =$



What kind of functions is the diagram composed of?
What is their connection like?

$M_1 =$

$M_2 =$

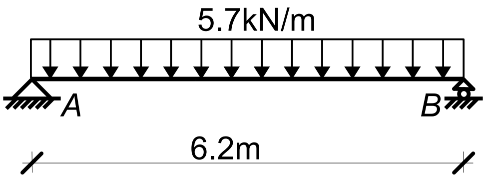
$M_3 =$

$M_4 =$

Check the existence and sense of jumps and kinks, as well as whether or not some lines are parallel to each other.

Example 3

Draw internal force diagrams for the structure based on calculations.



Solution

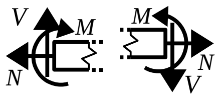
Reactions:

$\sum M_i^{(A)}: -(5.7\cdot 6.2)\cdot 3.1 + B\cdot 6.2 = 0 \rightarrow B = 17.67\text{ kN}(\uparrow)$

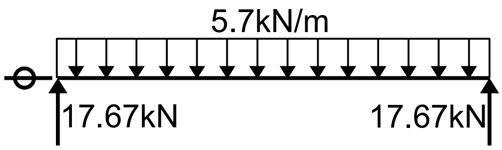
$\sum M_i^{(B)}: (5.7\cdot 6.2)\cdot 3.1 - A_z\cdot 6.2 = 0 \rightarrow A_z = 17.67\text{ kN}(\uparrow)$

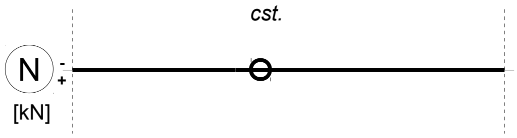
$\sum F_{ix}: A_x = 0$

Check: $\sum F_{iz}: 5.7\cdot 6.2 - 17.67 - 17.67 = 0$

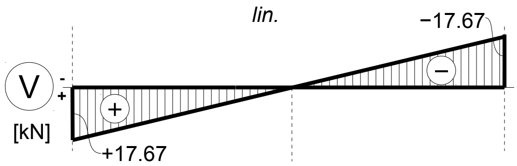


Final sketch:





The normal force diagram is composed of a single constant segment. Its value is zero because no horizontal force components act upon the structure.



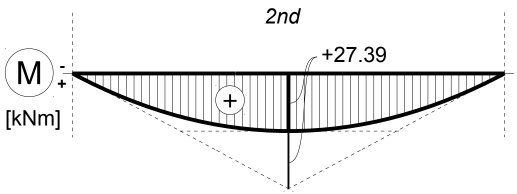
The shear force diagram consists of a single linear segment. The shear at both endpoints is obtained as follows. On the right hand side,

$V_2 = -17.67\text{ kN}$ (from right)

On the left hand side,

$V_1 = 17.67\text{ kN}$ (from left)

Zero is found exactly in the middle of the beam.



The bending moment diagram is composed of a single parabolic segment with zero values at both ends (obtained for convenience from outside in both cases). The chord between them is coincident to the axis.

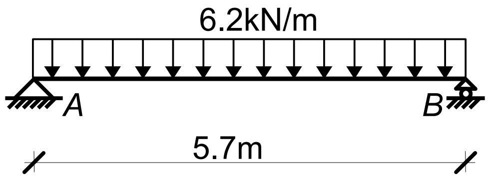
The depth of the parabola is $\frac{ql^2}{8} = \frac{5.7\cdot 6.2^2}{8} = 27.39\text{ kNm}$.

It is measured twice *downwards* (because of the sense of the load) from the midpoint of the chord. The first distance extends to a point of the parabola, the second one points to the intersection of tangents. The middle tangent is parallel to the (horizontal) chord, so the value $+27.39\text{ kNm}$ is also a maximum of the bending moment.

About the M diagram: There are zero values at both ends (in the absence of any torque there). The load is directed downwards, therefore the depth of parabola is measured downwards as well.

Exercise 3

Draw internal force diagrams for the structure based on calculations.



Solution

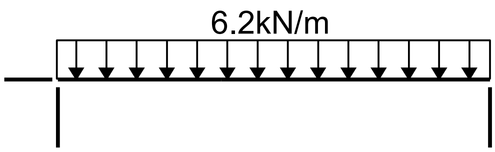
Reactions:

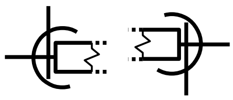
$\sum$:

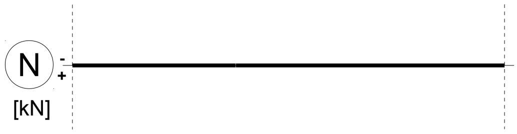
$\sum$:

$\sum$:

Final sketch:

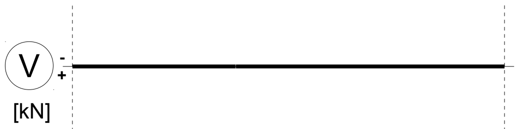






What kind of functions is the diagram composed of?
What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of?
What is their connection like?

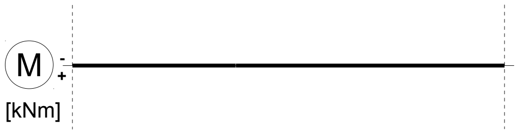
$V_1 =$

$V_2 =$

What kind of functions is the diagram composed of?
What is their connection like?

$M_1 =$

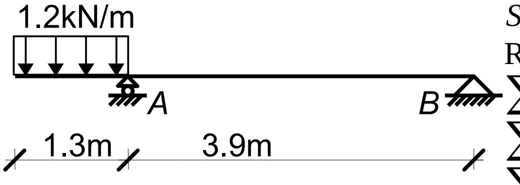
$M_2 =$



parabola: $\frac{q l^2}{8} = \underline{\qquad\qquad} =$

Example 4

Draw internal force diagrams for the structure based on calculations.



Solution

Reactions:

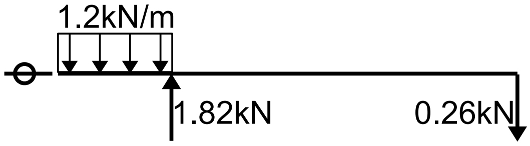
$\sum M_i^{(B)}: (1.2\cdot 1.3)\cdot 4.55 - A\cdot 3.9 = 0 \rightarrow A = 1.82\,\text{kN}\,(\uparrow)$

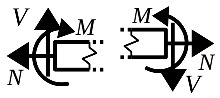
$\sum M_i^{(A)}: (1.2\cdot 1.3)\cdot 0.65 + B_z\cdot 3.9 = 0 \rightarrow B_z = -0.26\,\text{kN}\,(\downarrow)$

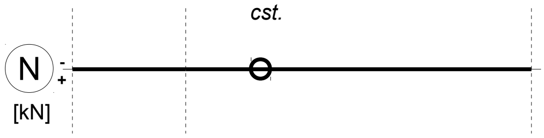
$\sum F_{ix}: B_x = 0$

Check: $\sum F_{iz}: 1.3\cdot 1.2 - 1.82 + 0.26 = 0$

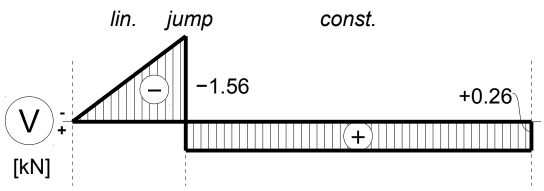
Final sketch:







The normal force diagram is composed of two constant segments. Their value is zero because no horizontal force components act upon the structure.

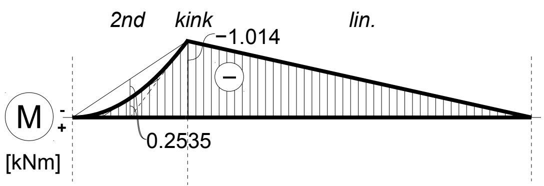


As seen from the loads, the shear diagram is composed of a linear and a constant segment. Starting and final values of the former one is found from left:

$V_1 = 0,\quad V_2 = -1.2\cdot 1.3 = -1.56\,\text{kN}$

The value of the constant part is got better from right: $V_3 = +0.26\,\text{kN}$ (from B.)

(There is a jump corresponding to A between them.)



The moment diagram is composed of a parabolic and a linear segment. Starting and final values of the parabolic part:

$M_1 = 0,\quad M_2 = -1.2\cdot 1.3\cdot 0.65 = -1.014\,\text{kNm},$

its depth is $\dfrac{ql^2}{8} = \dfrac{1.2\cdot 1.3^2}{8} = 0.2535\,\text{kNm}.$

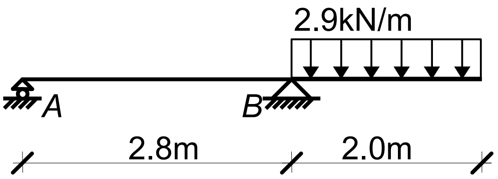
(The double of the depth just reaches the baseline, making the tangent to the rightmost point be horizontal as shown also by the zero shear there.)

At the left end of the linear segment the moment equals $-1.014\,\text{kNm}$, which is not modified locally by force A. At the right end (seen from right), $M_3 = 0\,\text{kNm}$.

About the M diagram: There is no concentrated force or torque at the end of the overhang, hence the (parabolic) segment starts from zero and with a zero slope there. The parabola is convex from below because the load is directed downwards. Due to the support reaction, the diagram has a kink at the right hand side of parabolic segments and continues linearly to the rightmost point. The right endpoint of the diagram is zero because of the absence of any concentrated torque again.

Exercise 4

Draw internal force diagrams for the structure based on calculations.



Solution

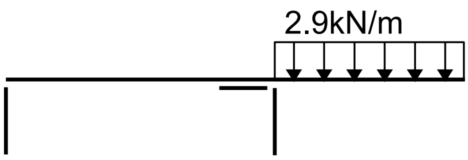
Reactions:

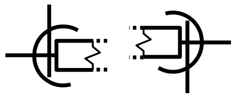
$\sum$:

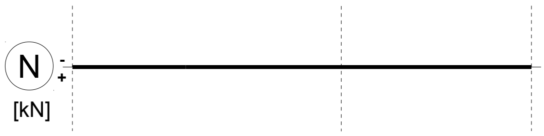
$\sum$:

$\sum$:

Final sketch:



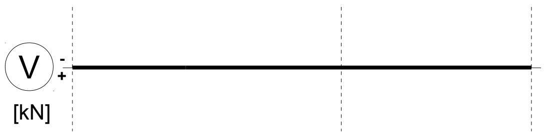




What kind of functions is the diagram composed of?
What is their connection like?

$N_1 =$

$N_2 =$

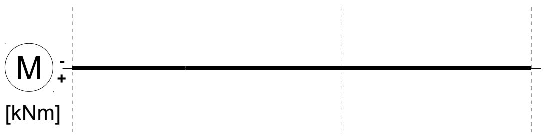


What kind of functions is the diagram composed of?
What is their connection like?

$V_1 =$

$V_2 =$

$V_3 =$



What kind of functions is the diagram composed of?
What is their connection like?

$M_1 =$

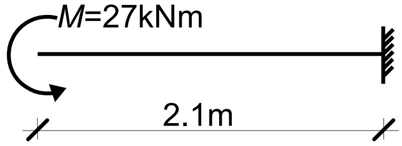
$M_2 =$

$M_3 =$

parabola: $\frac{q l^2}{8} = \frac{\qquad}{} =$

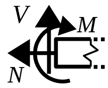
Example 5

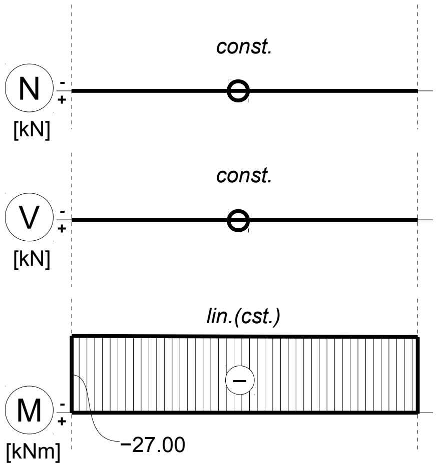
Draw internal force diagrams for the structure based on calculations.



Solution

Reactions are not needed in a cantilever problem, calculations are done from outside (now from left).





There is no force with horizontal component, that is why the normal force is all zero.

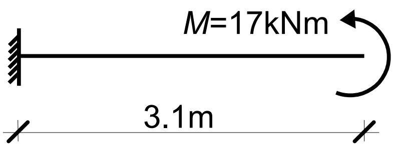
The shear force diagram is composed of a single straight segment of value $V=0\,\text{kN}$.

The bending moment diagram is composed of a single linear segment in the absence of load; here it is simplified to a constant due to the zero shear. Its value is $M=-27\,\text{kNm}$.

About the M diagram: There is no force at all on the cantilever, hence the shear is zero everywhere. Following from that, the moment diagram starts with a horizontal tangent at the free end and is drawn up to the support on the side of tension.

Exercise 5

Draw internal force diagrams for the structure based on calculations.

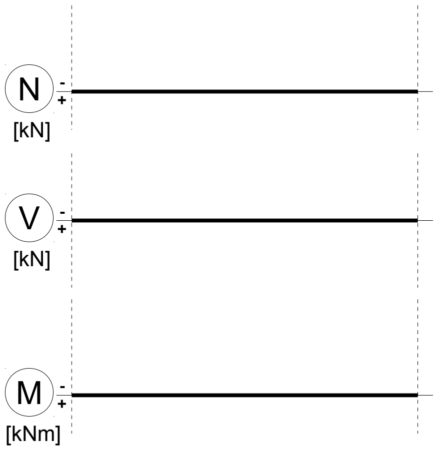


Solution

Reactions?

Which side the values are got from?

Positive senses for internal forces:



What kind of functions is the diagram composed of?
What is their connection like?

$N_1 =$

What kind of functions is the diagram composed of?
What is their connection like?

$V_1 =$

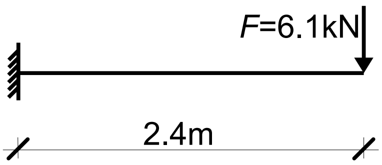
What kind of functions is the diagram composed of?
What is their connection like?

$M_1 =$

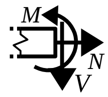
$M_2 =$

Example 6

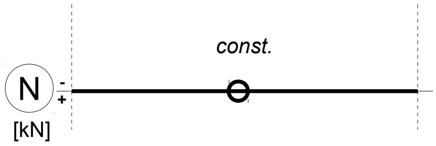
Draw internal force diagrams for the structure based on calculations.



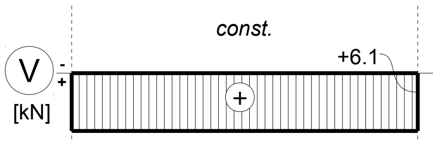
Solution



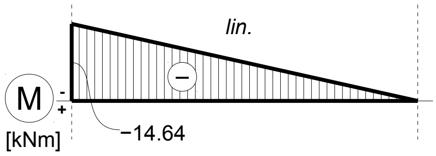
Reactions are not needed in a cantilever problem, calculations are done from outside (now from right).



There is no force with horizontal component, that is why the normal force is all zero.

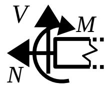


The shear force diagram is composed of a single straight segment of value $V=+6.1\,\text{kN}$.



The bending moment diagram is composed of a single linear segment. Values at both its ends found from outside (from right):
$M_2=0, \quad M_1=-6.1\cdot 2.4=-14.64\,\text{kNm}.$

Support reactions can be obtained by reinterpreting internal forces at the leftmost section from left:

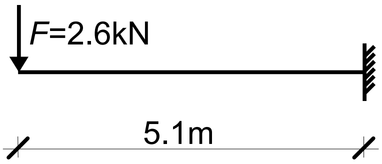


$N=0 \rightarrow A_x=0$
$V=6.1\,\text{kN} \rightarrow A_y=6.1\,\text{kN}\,(\uparrow)$
$M=-14.64\,\text{kNm} \rightarrow M_A=14.64\,\text{kNm}\,(\curvearrowleft)$

About the M diagram: The diagram will be linear with a zero value at the free end of the cantilever. There must be a (fictitious) kink there with respect to the (fictitious) continuation of the diagram outside the member such that the kink and the arrowhead of the force causing it should match. For that reason, the diagram should increase on the top side until the support.

Exercise 6

Draw internal force diagrams for the structure based on calculations.

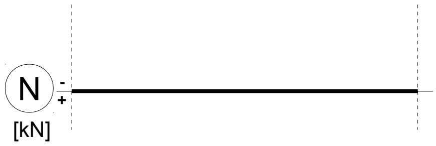


Solution

Reactions?

Which side the values are got from?

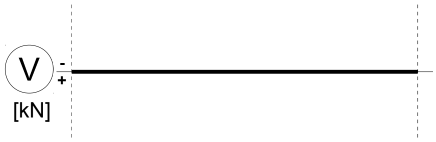
Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

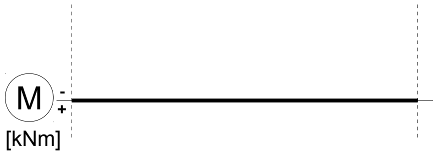
$N_1 =$



What kind of functions is the diagram composed of?

What is their connection like?

$V_1 =$



What kind of functions is the diagram composed of?

What is their connection like?

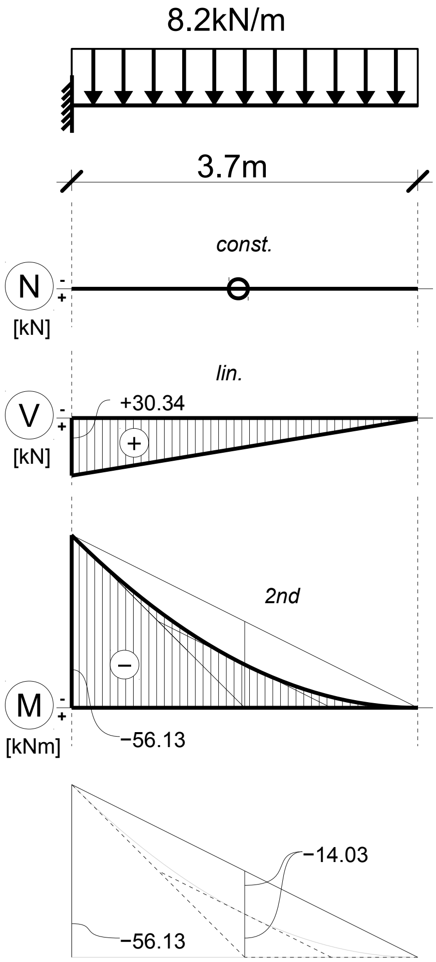
$M_1 =$

$M_2 =$

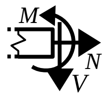
Determine support reactions from internal forces at the cross section next to the support.

Example 7

Draw internal force diagrams for the structure based on calculations.



Solution



Reactions are not needed in a cantilever problem, calculations are done from outside (now from right).

There is no force with horizontal component, that is why the normal force is zero everywhere.

The shear force diagram has a single linear segment. Starting and final values are:
$V_2 = 0\,\text{kN},\ V_1 = +8.2 \cdot 3.7 = +30.34\,\text{kN}.$

The moment diagram is composed of a single parabolic segment. Starting and final values of the parabolic part:
$M_2 = 0,\ \ M_1 = -8.2 \cdot 3.7 \cdot 1.85 = -56.13\,\text{kNm},$

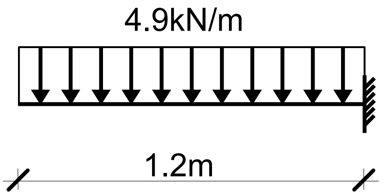
its depth is $\dfrac{q l^2}{8} = \dfrac{8.2 \cdot 3.7^2}{8} = 14.03\,\text{kNm}.$

The double of the depth just reaches the baseline, making the tangent to the rightmost point be horizontal as shown also by the zero shear there. (Construction lines and distances to be measured are shown in the bottom figure.)

About the M diagram: There is no concentrated force or torque at the free end of the cantilever, hence the parabolic segment starts from zero and with a zero slope there. The parabola is convex from below because the load is directed downwards.

Exercise 7

Draw internal force diagrams for the structure based on calculations

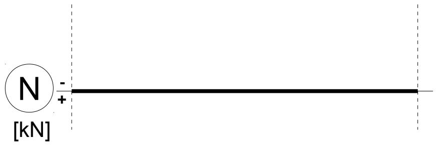


Solution

Reactions?

Which side the values are got from?

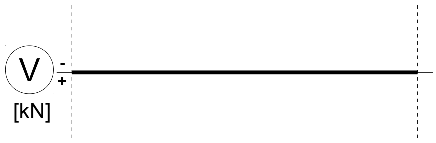
Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$

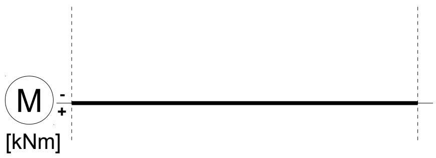


What kind of functions is the diagram composed of?

What is their connection like?

$V_1 =$

$V_2 =$



What kind of functions is the diagram composed of?

What is their connection like?

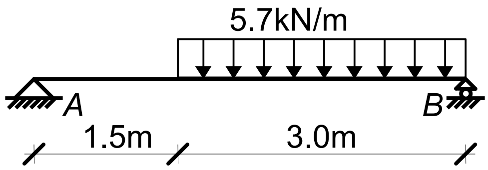
$M_1 =$

$M_2 =$

parabola: $\frac{q l^2}{8} = \underline{\qquad\qquad} =$

Example 8

Draw internal force diagrams for the structure based on calculations



Solution

Reactions:

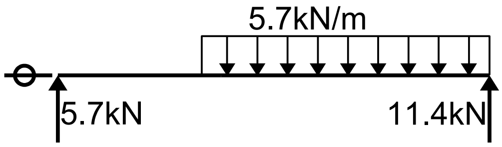
$\sum M_i^{(A)}: -(5.7\cdot 3.0)\cdot 3.0 + B\cdot 4.5 = 0 \rightarrow B = 11.4\,\text{kN}\,(\uparrow)$

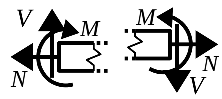
$\sum M_i^{(B)}: (5.7\cdot 3.0)\cdot 1.5 - A_z\cdot 4.5 = 0 \rightarrow A_z = 5.7\,\text{kN}\,(\uparrow)$

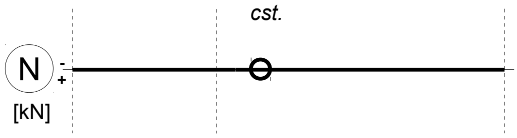
$\sum F_{ix}: A_x = 0$

Check: $\sum F_{iz}: 5.7\cdot 3.0 - 5.7 - 11.4 = 0$

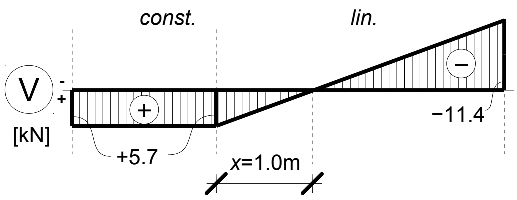
Final sketch:



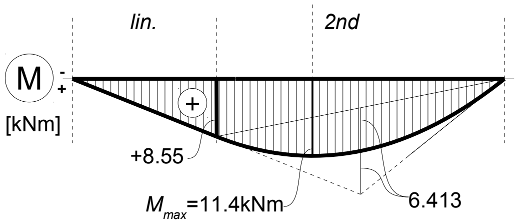




There is no force with horizontal component, that is why the normal force is zero everywhere.



As seen from the loads, the shear diagram is composed of a constant and a linear segment.
The value of the constant part is found from left: $V_1 = +5.7\,\text{kN}$.
It equals the starting value of the linear segment.
The rightmost value found from right: $V_2 = -11.4\,\text{kN}$ (from B).
The linear part will intersect the baseline at some point. Let its position relative to the left end of distributed forces be marked by x. The shear here (from left) is $V(x) = +5.7 - 5.7\cdot x = 0 \quad \rightarrow \quad x = 1.0\,\text{m}$.



The bending moment diagram is composed of a linear and a parabolic segment. Starting and final values of the linear part are:
$M_1 = 0, \quad M_2 = +5.7\cdot 1.5 = 8.55\,\text{kNm}$.
This latter one equals the leftmost value of the parabola; at its other end, $M_3 = 0$.

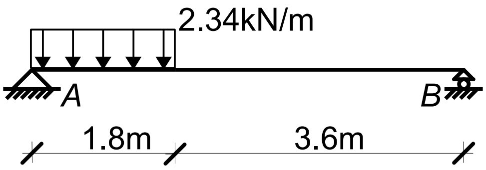
The depth of the parabola: $\dfrac{q\,l^2}{8} = \dfrac{5.7\cdot 3.0^2}{8} = 6.413\,\text{kNm}$.

Slopes of final tangents are opposite: where it becomes zero, a local maximum is found.
Its position has already been obtained at the shear diagram. The maximum of moment there is found as $M_{max} = 5.7\cdot 2.5 - (5.7\cdot 1.0)\cdot 0.5 = +11.4\,\text{kNm}$.

Check: the tangent to the leftmost point of the parabola coincides with the linear segment.

Exercise 8

Draw internal force diagrams for the structure based on calculations



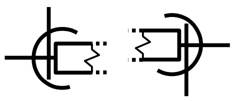
Solution

Reactions:

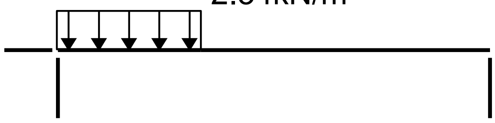
$\sum \quad :$

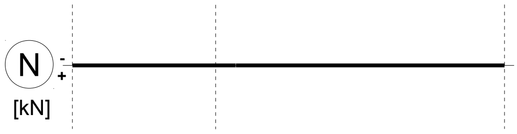
$\sum \quad :$

$\sum \quad :$



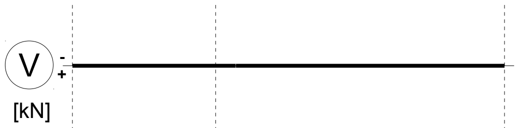
Final sketch:





What kind of functions is the diagram composed of?
What is their connection like?

$N_1 =$

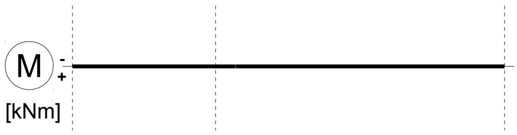


What kind of functions is the diagram composed of?
What is their connection like?

$V_1 =$

$V_2 =$

$V_3 =$



What kind of functions is the diagram composed of?
What is their connection like?

$M_1 =$

$M_2 =$

$M_3 =$

parabola: $\frac{q l^2}{8} = \underline{\qquad\qquad} =$

Is there a maximum? → Has the shear diagram a zero?
Where?

$V_m = \qquad\qquad\qquad\qquad = 0$

$x_m =$

How much the moment is here?

$M_{max} =$

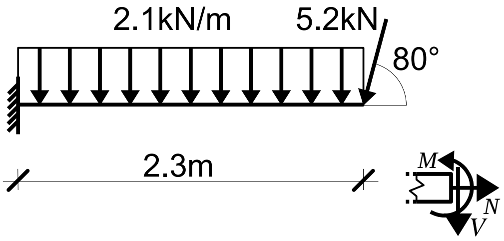
Internal force diagrams of cantilever beams

Any previous method for producing internal force diagrams obviously applies for cantilever beams as well. However, calculations are essentially simplified by the fact that any internal force value can be found from the free end without the need of calculating any components of reaction. Although the whole procedure still begins with the identification of segments and their connections, the entire diagram is drawn started at the free end towards the support by accounting for concentrated and gradual changes in ordinates. This technique makes even possible to draw qualitative diagrams with no numbers but of a correct shape.

Successive ordinates calculated for any diagram in the following examples are still numbered from left to right but their determination always proceeds from the free end towards the support.

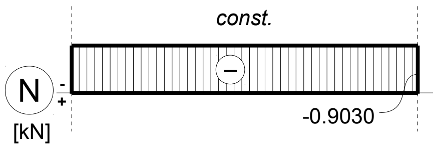
Example 1

Draw internal force diagrams for the structure based on calculations.

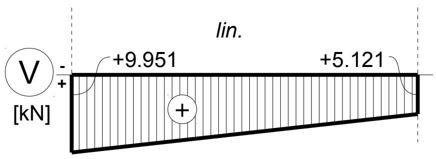


Solution

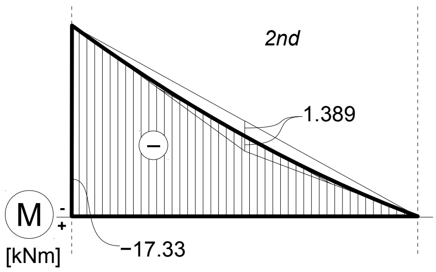
Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from right).

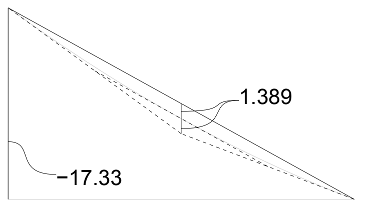


Normal force in a horizontal beam can only be due to horizontal force components. The distributed force is vertical, so the diagram is constant with a value of $N_1 = -5.2\cos 80° = -0.9030\,\text{kN}$.



The shear force diagram has a single linear segment. Starting and final values are:
$V_2 = +5.2\sin 80° = +5.121\,\text{kN}$,
$V_1 = +5.121 + 2.1 \cdot 2.3 = +9.951\,\text{kN}$.





The moment diagram is composed of a single parabolic segment. Starting and final values of the parabolic part:

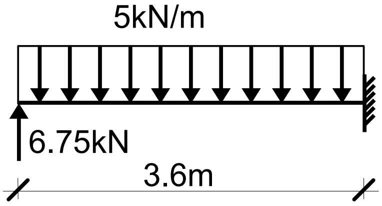
$$M_2 = 0\,\text{kNm}$$

$$M_1 = -5.2\sin 80° \cdot 2.3 - (2.1 \cdot 2.3) \cdot \frac{2{,}3}{2} = -17.33\,\text{kNm}$$

The depth of the parabola is $\frac{2.1 \cdot 2.3^2}{8} = 1.389\,\text{kNm}$

Exercise 1

Draw internal force diagrams for the structure based on calculations.

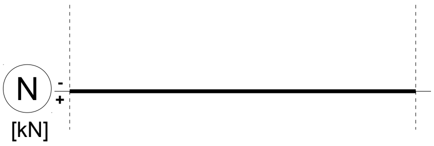


Solution

Reactions?

Which side the values are got from?

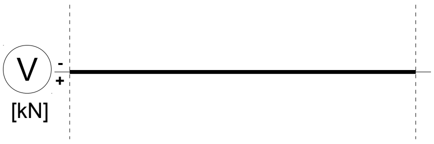
Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$



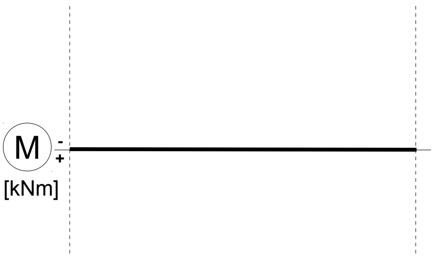
What kind of functions is the diagram composed of?

What is their connection like?

$V_1 =$

$V_2 =$

zero: $V_m = 0 =$



What kind of functions is the diagram composed of?

What is their connection like?

$M_1 =$

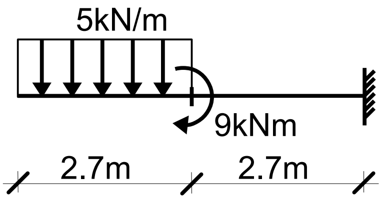
$M_2 =$

parabola: $\frac{q l^2}{8} = \underline{\qquad\qquad} =$

maximum: $M_{max} =$

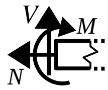
Example 2

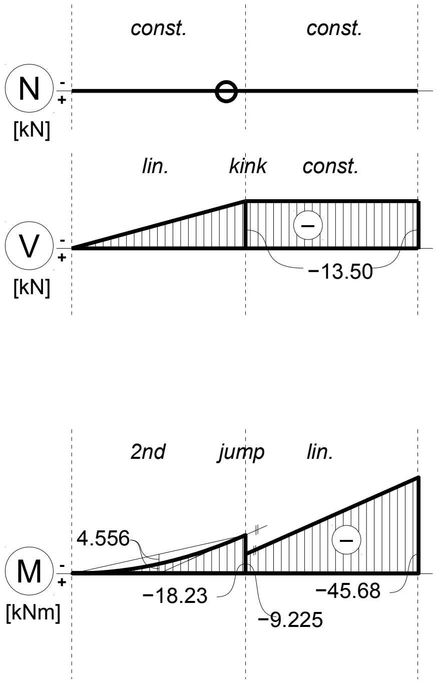
Draw internal force diagrams for the structure based on calculations.



Solution

Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from left).





Normal forces can change due to horizontal forces here. In their absence the diagram is constant with a value of $N_1 = 0\,\text{kN}$.

The shear diagram is composed of a linear and a constant segment with no jump between. Values at the boundaries of segments are as follows:
$V_1 = 0\,\text{kN}$, $V_2 = -5 \cdot 2.7 = -13.5\,\text{kN}$.

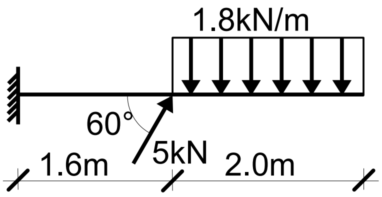
The bending moment diagram is composed of a parabolic and a linear segment. There is a jump between them but slopes of tangents at both sides are equal.
Values at the boundaries of segments are as follows:
$M_1 = 0\,\text{kNm}$,
$M_2 = -(5 \cdot 2.7) \cdot 1.35 = -18.23\,\text{kNm}$ (parabolic part),
$M_3 = -(5 \cdot 2.7) \cdot 1.35 + 9 = -9.225\,\text{kNm}$,
$M_4 = -(5 \cdot 2.7) \cdot 4.05 + 9 = -45.68\,\text{kNm}$ (linear part).

The depth of the parabola is $\frac{5 \cdot 2.7^2}{8} = 4.556\,\text{kNm}$

The double of the depth just reaches the baseline, making the tangent to the rightmost point be horizontal as shown also by the zero shear there.

Exercise 2

Draw internal force diagrams for the structure based on calculations.

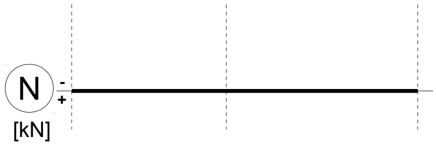


Solution

Reactions?

Which side the values are got from?

Positive senses for internal forces:

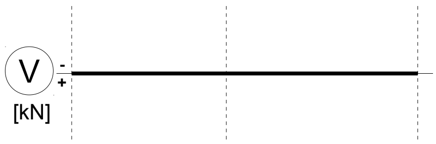


What kind of functions is the diagram composed of?

What is their connection like?

$N_2 =$

$N_1 =$



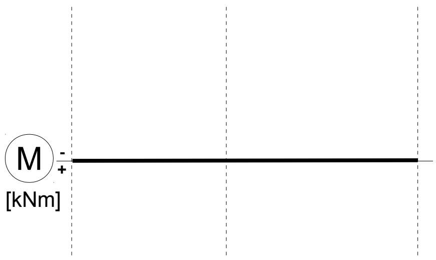
What kind of functions is the diagram composed of?

What is their connection like?

$V_3 =$

$V_2 =$

$V_1 =$



What kind of functions is the diagram composed of?

What is their connection like?

$M_3 =$

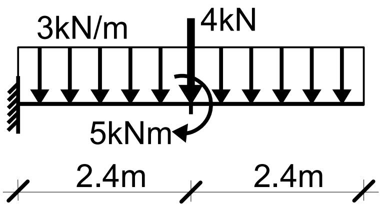
$M_2 =$

$M_1 =$

parabola: $\frac{q l^2}{8} = \frac{\qquad}{} =$

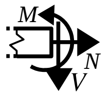
Example 3

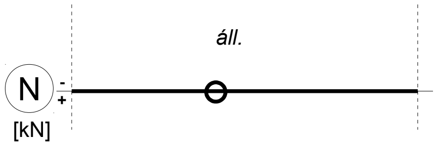
Draw internal force diagrams for the structure based on calculations.



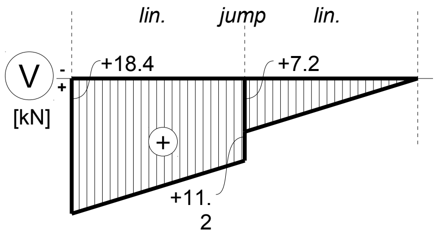
Solution

Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from right).





Normal forces can change due to horizontal forces here. In their absence the diagram is constant with a value of $N_1 = 0\,\text{kN}$.

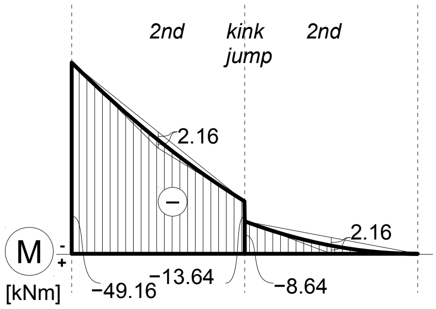


The shear force diagram is composed of two linear segments parallel to each other and with a jump between them due to the vertical load there.

Values at the boundaries of segments are as follows:

$V_4 = 0\,\text{kN}$, $V_3 = +3 \cdot 2.4 = +7.2\,\text{kN}$,

$V_2 = +7.2 + 4 = 11.2\,\text{kN}$, $V_1 = +11.2 + 3 \cdot 2.4 = 18.4\,\text{kN}$.



The bending moment diagram is composed of two parabolic segments with a jump between them; slopes of tangents at two sides of the 4-kN force are also different.

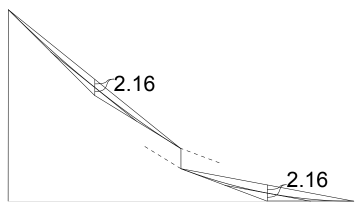
Values at the boundaries of segments are as follows:

$M_4 = 0\,\text{kNm}$,

$M_3 = -(3 \cdot 2.4) \cdot 1.2 = -8.64\,\text{kNm}$,

$M_2 = -(3 \cdot 2.4) \cdot 1.2 - 5 = -13.64\,\text{kNm}$,

$M_1 = -(3 \cdot 4.8) \cdot 2.4 - 5 = -49.16\,\text{kNm}$.

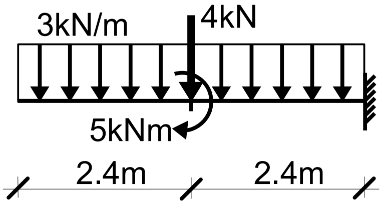


The depth of each parabola is $\dfrac{3 \cdot 2.4^2}{8} = 2.16\,\text{kNm}$

(Tangents to the curves *opposite* the jump are drawn in dashed lines. It makes clearly visible that not only a jump due to the concentrated torque but also a kink caused by the concentrated force appears in the figure.)

Exercise 3

Draw internal force diagrams for the structure based on calculations.

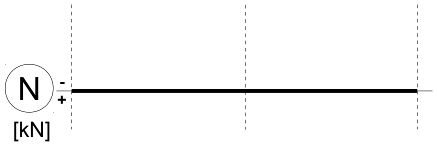


Solution

Reactions?

Which side the values are got from?

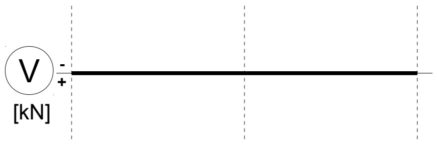
Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of?

What about the slope of each segment?

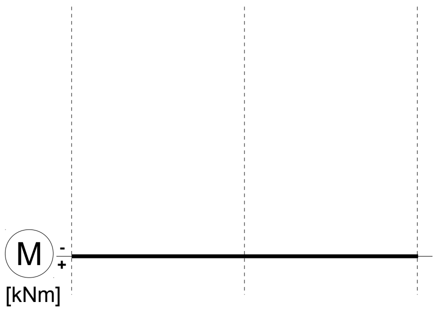
What is their connection like?

$V_1 =$

$V_2 =$

$V_3 =$

$V_4 =$



What kind of functions is the diagram composed of?

What is their connection like?

$M_1 =$

$M_2 =$

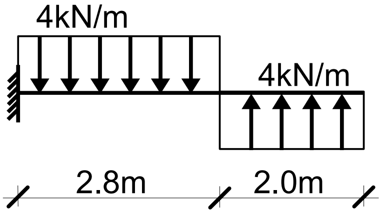
$M_3 =$

$M_4 =$

parabola: $\frac{q l^2}{8} = \underline{\qquad\qquad} =$

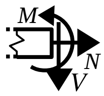
Example 4

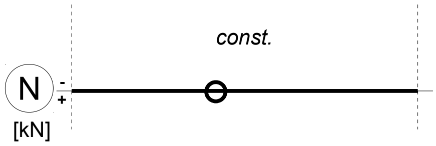
Draw internal force diagrams for the structure based on calculations.



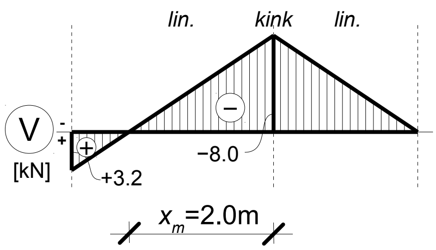
Solution

Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from left).





Normal forces can change due to horizontal forces here. In their absence the diagram is constant with a value of $N_1 = 0$ kN.



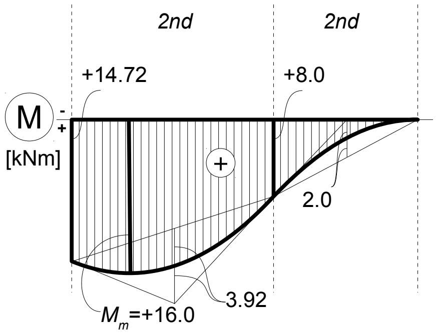
The shear diagram is composed of two linear segments with no jump between.
Values at the boundaries of segments are as follows:

$V_3 = 0$ kN,

$V_2 = -4 \cdot 2.0 = -8.0$ kN,

$V_3 = -8.0 + 4 \cdot 2.8 = +3.2$ kN.

zero value: $T_m = -4 \cdot 2 + 4 \cdot x_m = 0 \rightarrow x_m = 2.0$ m.



The moment diagram is composed of two parabolic segments with a common tangent.
(The parabola to the right is convex from above.)
Values at the boundaries of segments are as follows:

$M_3 = 0$ kNm,

$M_2 = +(4 \cdot 2.0) \cdot 1.0 = +8.0$ kNm,

$M_1 = +(4 \cdot 2.0) \cdot 3.8 - (4 \cdot 2.8) \cdot 1.4 = +14.72$ kNm.

Depths of parabolas: $\frac{4 \cdot 2.0^2}{8} = 2.00$ kNm (upwards)

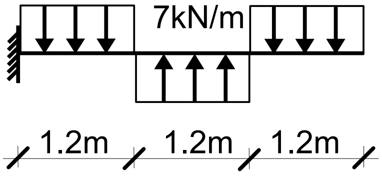
$\frac{4 \cdot 2.8^2}{8} = 3.92$ kNm (downwards)

The maximum value is

$M_{max} = +(4 \cdot 2.0) \cdot 3.0 - (4 \cdot 2.0) \cdot 1.0 = +16.00$ kNm

Exercise 4

Draw internal force diagrams for the structure based on calculations.

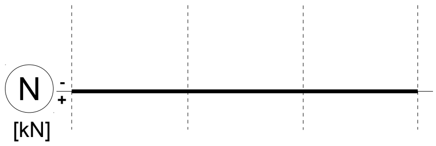


Solution

Reactions?

Which side the values are got from?

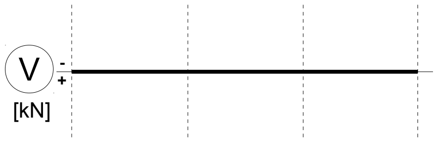
Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of?

What is their connection like?

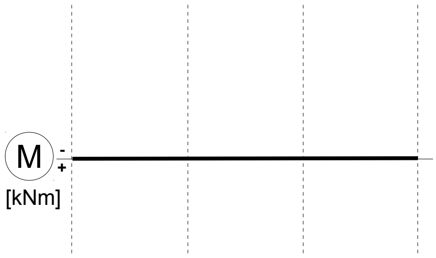
$V_4 =$

$V_3 =$

$V_2 =$

$V_1 =$

change of signs?



What kind of functions is the diagram composed of?

What is their connection like?

$M_4 =$

$M_3 =$

$M_2 =$

$M_1 =$

parabolas: $\frac{q l^2}{8} = \frac{\qquad}{} =$

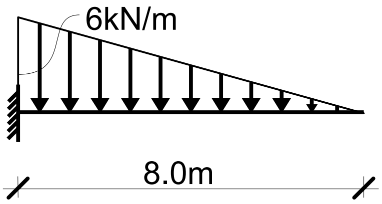
maximum: ?

Effect of generally distributed loads on internal force diagrams

The intensity of a load along a beam is not always constant. A typical example of that can be the load of wind load where the velocity of wind and so the intensity of load increases with height; or the distributed weight of piles of granular material on the ground which varies approximately with the local height within the pile. There are several ways to draw diagrams from loads of given function of distribution. If such a function is a polynomial, then the shear force and bending moment will be described by polynomials of higher order by one and two, respectively. Although no general drawing rules for higher-order polynomials are introduced here, note that tangents at ends of a segment intersect on the line of action of the resultant of the load on the same segment.

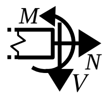
Example 5

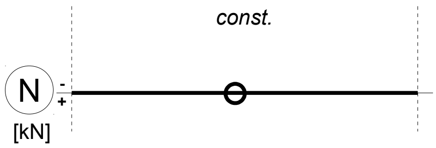
Draw internal force diagrams for the structure based on calculations.



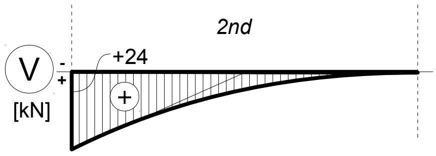
Solution

Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from right).





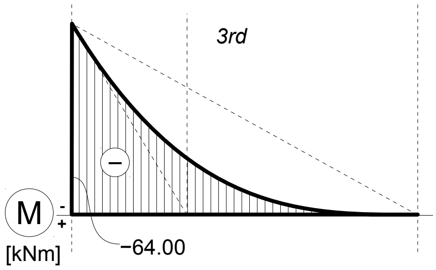
Normal forces can change due to horizontal forces here. In their absence the diagram is constant with a value of $N_1 = 0$ kN.



The shear force diagram is quadratic because of the linear distributed load. Its tangent is horizontal at the free end because of the zero load intensity there; it gets steeper monotonically when proceeding to the left. Final ordinates:

$$V_2 = 0\ \text{kN}, \quad V_1 = +\frac{6 \cdot 8}{2} = +24\ \text{kN}.$$

(Since tangents to a parabola at its final points intersect in the middle of the horizontal extension of the parabola, the tangent to the leftmost point can then be constructed to get a more precise figure.)



As seen from the linear distributed load, the moment diagram will be cubic. Its values at endpoints are

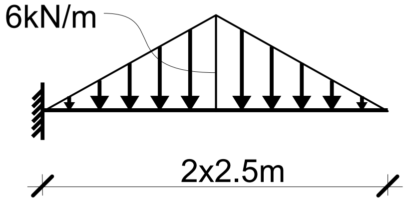
$M_2 = 0$ kNm,

$$M_1 = -\frac{6 \cdot 8}{2} \cdot \frac{8.0}{3} = -64\ \text{kNm}.$$

The tangent at the free end is horizontal because of the zero shear there. Due to the force directed downwards, the diagram is convex from below. Further help for drawing is that the resultant of the load passes through the centroid of triangle in the load diagram; the leftmost tangent must pass through the intersection of the resultant and the rightmost tangent.

Exercise 5

Draw internal force diagrams for the structure based on calculations.

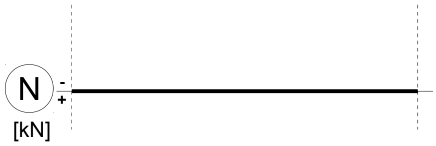


Solution

Reactions?

Which side the values are got from?

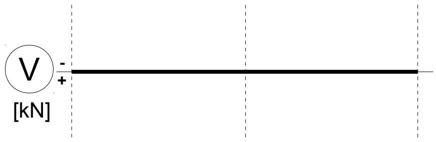
Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of?

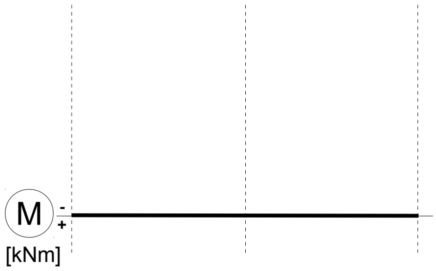
What is their connection like?

Where will the tangent be horizontal?

$V_3 =$

$V_2 =$

$V_1 =$



What kind of functions is the diagram composed of?

What is their connection like?

Where are different segments convex from?

How do slopes change along the segments?

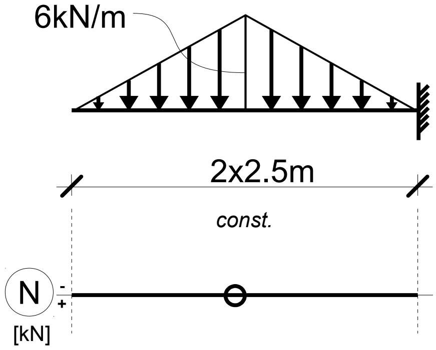
$M_3 =$

$M_2 =$

$M_1 =$

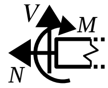
Example 6

Draw internal force diagrams for the structure based on calculations.

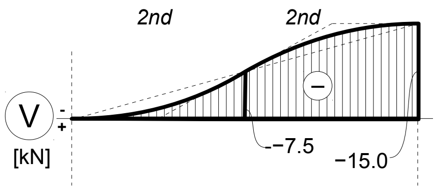


Solution

Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from left).



Normal forces can change due to horizontal forces here. In their absence the diagram is constant with a value of $N_1 = 0\,\text{kN}$.



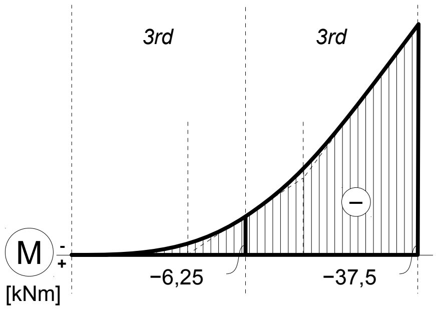
The shear force diagram is composed of two parabolic segments. Values at the boundaries are as follows:

$V_1 = 0\,\text{kN}$,

$$V_2 = -\frac{6\cdot 2.5}{2} = -7.5\,\text{kN},$$

$$V_3 = -7.5 - \frac{6\cdot 2.5}{2} = -15\,\text{kN}.$$ A két parabola közös

The parabolas have a common tangent, whereas tangents at zero load intensities are horizontal. Tangents at both ends intersect exactly in the middle of the beam.



The bending moment diagram is composed of two cubic. segments; both are convex from below. Ordinates at the boundaries are as follows:

$M_1 = 0\,\text{kNm}$,

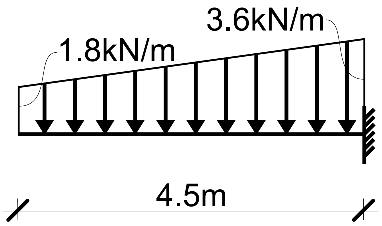
$$M_2 = -\frac{6\cdot 2.5}{2}\cdot\frac{2.5}{3} = -6.25\,\text{kNm},$$

$$M_3 = -\frac{6\cdot 2.5}{2}\cdot\left(2.5 + \frac{2.5}{3}\right) - \frac{6\cdot 2{,}5}{2}\cdot 2\,\frac{2.5}{3} = -37.5\,\text{kNm}.$$

The leftmost tangent is horizontal because of the zero shear there; another one in the middle of the beam intersects it exactly under the centroid of the triangular load to the left. This tangent is common to both segments and intersects the rightmost tangent under the centroid of the triangular load to the right.

Exercise 6

Draw internal force diagrams for the structure based on calculations.

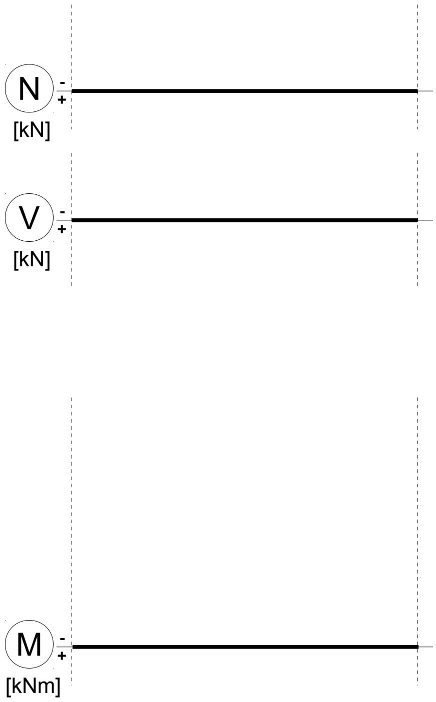


Solution

Reactions?

Which side the values are got from?

Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$

What kind of functions is the diagram composed of?

How do the slopes change?

Is there a known slope somewhere?

$V_1 =$

$V_2 =$

What kind of functions is the diagram composed of?

How do the slopes change?

Is there a known slope somewhere?

$M_1 =$

$M_2 =$

Where do tangents to endpoints intersect?

$R_1 =$ $\qquad x_{R1} =$

$R_2 =$ $\qquad x_{R2} =$

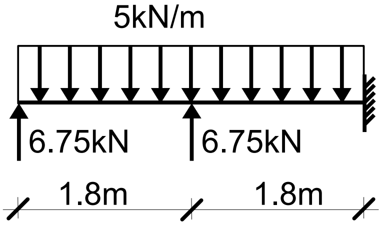
$R =$

$$R_1 x_{R1} + R_2 x_{R2} = R x_R$$

$x_R =$

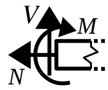
Example 7

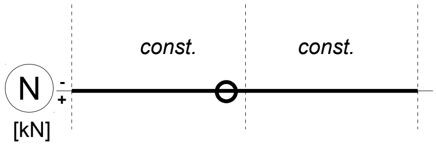
Draw internal force diagrams for the structure based on calculations.



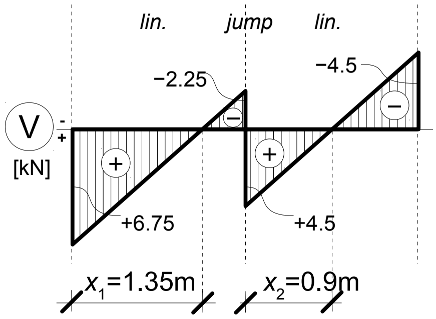
Solution

Reactions are not needed in a cantilever problem. Internal forces are found proceeding from the free end to the support, calculations are done from outside (now from left).





Normal forces can change due to horizontal forces here. In their absence the diagram is constant with a value of $N_1 = 0\,\text{kN}$.



The shear diagram is composed of two parallel linear segments with a jump in between.
Values at the boundaries of segments are as follows:

$V_1 = +6.75\,\text{kN}$,

$V_2 = +6.75 - 5 \cdot 1.8 = -2.25\,\text{kN}$,

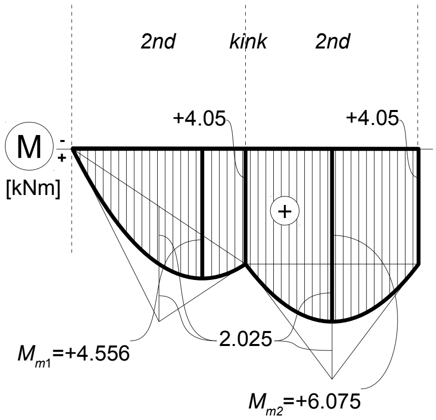
$V_3 = -2.25 + 6.75 = +4.5\,\text{kN}$,

$V_4 = +6.75 - 5 \cdot 3.6 + 6.75 = -4.5\,\text{kN}$.

There are two zeroes:

$V_{m1} = +6.75 - 5 \cdot x_1 = 0 \rightarrow x_1 = 1.35\,\text{m}$

$V_{m2} = +4.5 - 5 \cdot x_2 = 0 \rightarrow x_2 = 0.9\,\text{m}$



The bending moment diagram is composed of two parabolic segments with a kink in between.
Values at the boundaries of segments are as follows:

$M_1 = 0\,\text{kNm}$,

$M_2 = +6.75 \cdot 1.8 - (5 \cdot 1.8) \cdot 0{,}9 = +4.05\,\text{kNm}$,

$M_3 = +6.75 \cdot 3.6 - (5 \cdot 3.6) \cdot 1.8 + 6.75 \cdot 1.8 = +4.05\,\text{kNm}$.

Depths of parabolas: $\frac{5 \cdot 1.8^2}{8} = 2.025\,\text{kNm}$ (both)

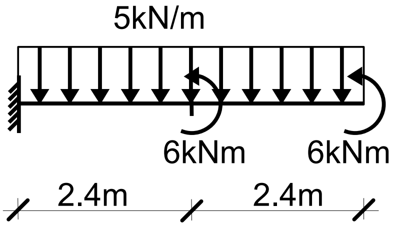
Maximum values are

$$M_{m1} = +6.75 \cdot 1.35 - (5 \cdot 1.35)\frac{1.35}{2} = +4.556\,\text{kNm}$$

$$M_{m2} = +6.75 \cdot 2.7 - (5 \cdot 2.7)\frac{2.7}{2} + 6.75 \cdot 0.9 = +6.075\,\text{kNm}$$

Exercise 7

Draw internal force diagrams for the structure based on calculations.

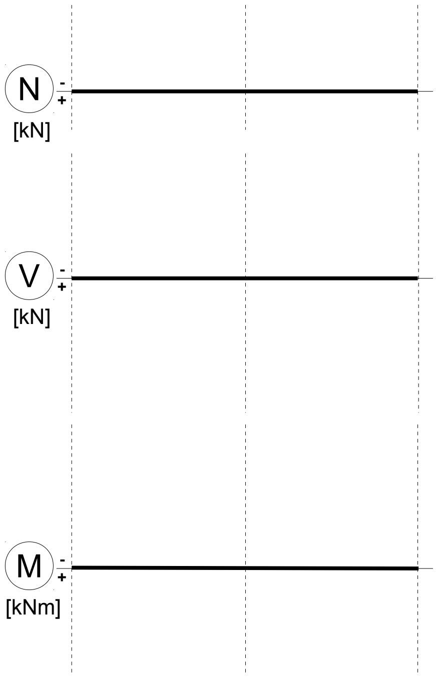


Solution

Reactions?

Which side the values are got from?

Positive senses for internal forces:



What kind of functions is the diagram composed of?

What is their connection like?

$N_1 =$

What kind of functions is the diagram composed of?

What is their connection like?

$V_2 =$

$V_1 =$

What kind of functions is the diagram composed of?

What is their connection like?

$M_4 =$

$M_3 =$

$M_2 =$

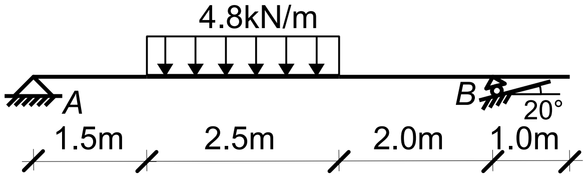
$M_1 =$

parabolas: $\frac{q l^2}{8} = \frac{\quad}{\quad} =$

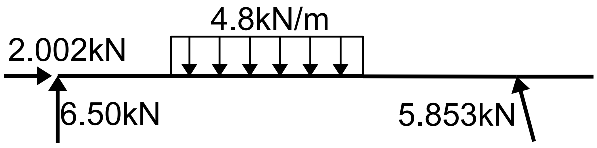
Internal force diagrams of simply supported beams

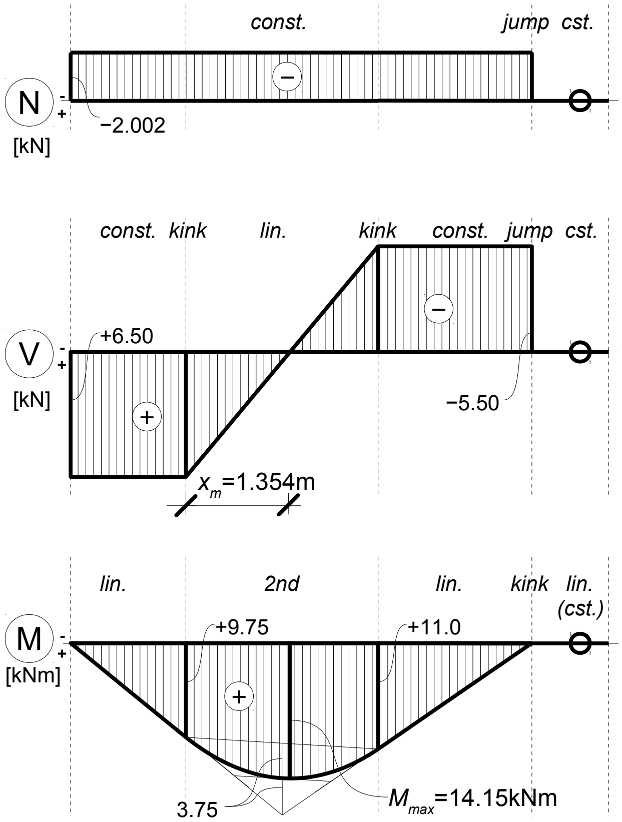
Example 1

Draw internal force diagrams for the structure based on calculations.



Final sketch:





Solution

Reactions:

$\sum M_i^{(A)}: -(4.8\cdot 2.5)\cdot 2.75 + B\cos 20°\cdot 6.0 = 0$

$\rightarrow\ B = 5.853\,\text{kN}(\nwarrow)$

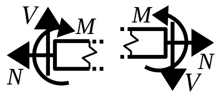
$\sum M_i^{(B)}: (4.8\cdot 2.5)\cdot 3.25 - A_z\cdot 6.0 = 0$

$\rightarrow\ A_z = 6.50\,\text{kN}(\uparrow)$

$\sum F_{ix}: A_x - 5.853\sin 20° = 0 \rightarrow A_x = 2.002\,\text{kN}(\rightarrow)$

Check:

$\sum F_{iz}: 4.8\cdot 2.5 - 5.853\cdot\cos 20° - 6.5 = -2\cdot 10^{-5} \approx 0$



Normal forces are only influenced by A_x and the horizontal component of B, the diagram is composed of two constant segments:

$N_1 = -2.002\,\text{kN},\quad N_2 = 0\,\text{kN}$

The shear force is linear under the constant load and there is a jump at force B in the diagram. Other three segments are constant.

$V_3 = 0\,\text{kN},$

$V_2 = -5.583\cos 20° = -5.50\,\text{kN}$

$V_1 = -5.50 + 4.8\cdot 2.5 = +6.50\,\text{kN}$

The zero on the linear part is found at x_m:

$V_m = +6.50 - 4.8\cdot x_m = 0 \rightarrow x_m = 1.354\,\text{m}$

The moment diagram is composed of one parabolic and two linear parts between the supports and a constant segment on the overhang (it is specifically zero here).

Moment values at boundaries: $M_4 = 0\,\text{kNm},$

$M_3 = +5.853\cos 20°\cdot 2.0 = +11.00\,\text{kNm}$

$M_2 = +6.5\cdot 1{,}5 = +9.75\,\text{kNm}, M_1 = 0\,\text{kNm}$

The depth is $\dfrac{4.8\cdot 2.5^2}{8} = 3.75\,\text{kNm}$

The value of maximum:

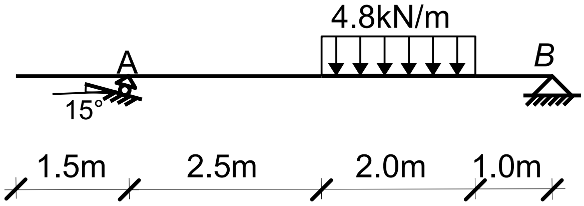
$$M_{max} = +6.5\cdot 2.854 - (4.8\cdot 1.354)\frac{1.352}{2} =$$

$$= +14.15\,\text{kNm}$$

Notice that all internal force diagrams on the overhang are zero.

Exercise 1

Draw internal force diagrams for the structure based on calculations.



Solution

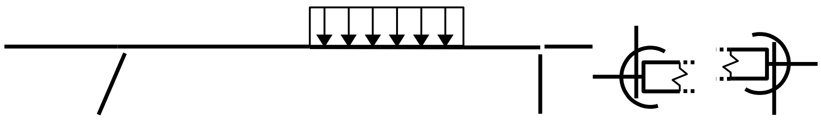
Reactions:

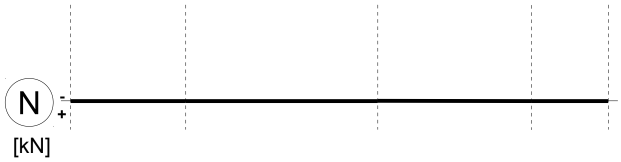
$\sum$:

$\sum$:

$\sum$:

Final sketch:

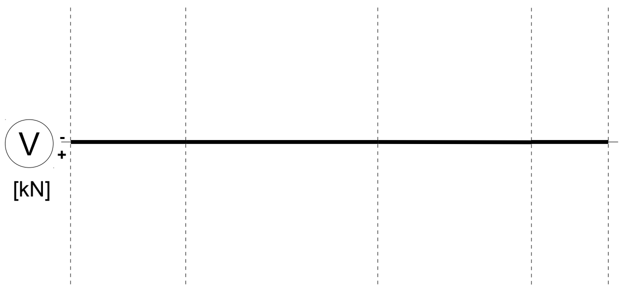




What kind of functions is the diagram composed of? What is their connection like?

$N_1 =$

$N_2 =$



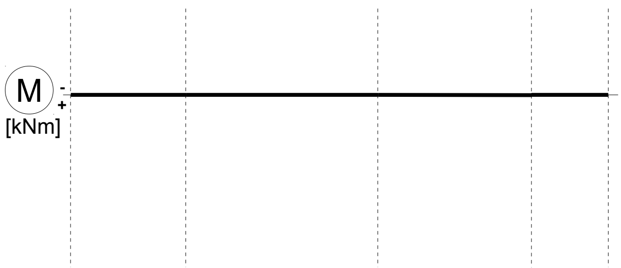
What kind of functions is the diagram composed of? What is their connection like?

$V_1 =$

$V_2 =$

$V_3 =$

zero: $V_m = 0 =$



What kind of functions is the diagram composed of? What is their connection like?

$M_1 =$

$M_2 =$

$M_3 =$

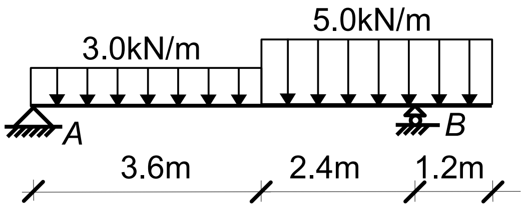
$M_4 =$

parabola: $\frac{ql^2}{8} = \frac{\qquad}{} =$

maximum: $M_{max} =$

Example 2

Draw internal force diagrams for the structure based on calculations.



Solution

Resultants: $R_1 = 3.0 \cdot 3.6 = 10.8\,\text{kN},\ R_2 = 5.0 \cdot 3.6 = 18\,\text{kN}$

Reactions:

$\sum M_i^{(A)}: -10.8 \cdot 1.8 - 18 \cdot 5.4 + B \cdot 6.0 = 0$

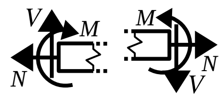
$\rightarrow \quad B = 19.44\,\text{kN}\,(\uparrow)$

$\sum M_i^{(B)}: 10.8 \cdot 4.2 + 18 \cdot 0.6 - A_z \cdot 6.0 = 0$

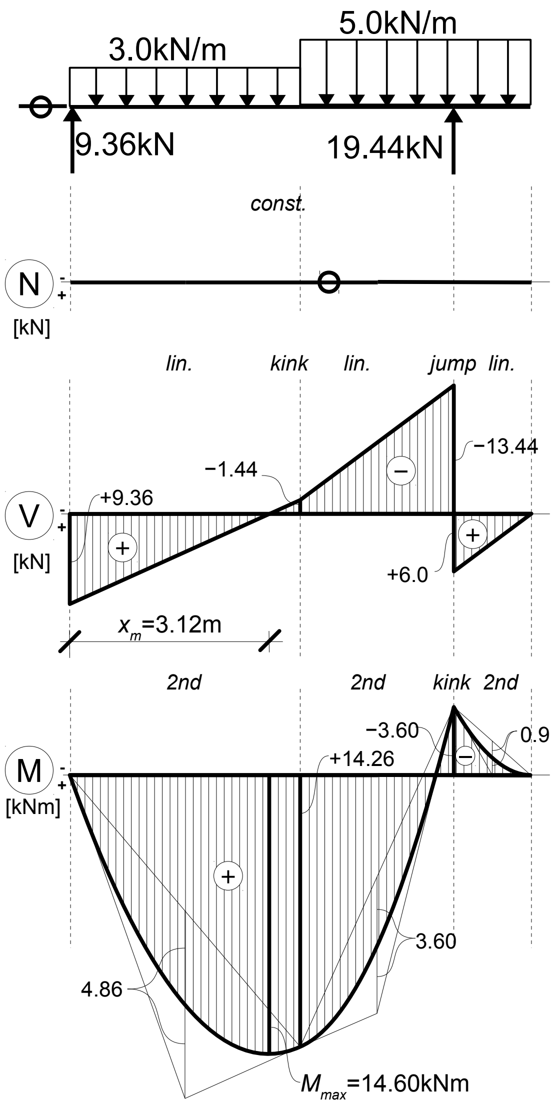
$\rightarrow \quad A_z = 9.36\,\text{kN}\,(\uparrow)$

$\sum F_{ix}: A_x = 0$

Check: $\sum F_{iz}: 3.0 \cdot 3.6 + 5.0 \cdot 3.6 - 9.36 - 19.44 = 0$



Final sketch:



There are no normal forces in a horizontal beam in the absence of horizontal external forces.

The shear force diagram is composed of three linear segments connected by a kink and a jump (from left to right); the last two are parallel to each other, the first one has a smaller slope.

Values at the boundaries are as follows:

$V_5 = 0\,\text{kN},\ V_4 = +5.0 \cdot 1.2 = 6.0\,\text{kNm}$

$V_3 = +5.0 \cdot 1.2 - 19.44 = -13.44\,\text{kNm}$

$V_2 = +5.0 \cdot 3.2 - 19.44 = -1.44\,\text{kNm},\ V_1 = +9.36\,\text{kNm}$

The zero is on the first segment:

$V_m = +9.36 - 3.0 \cdot x_m = 0 \rightarrow x_m = 3.12\,\text{m}$

The moment diagram is composed of three parabolic segments with a continuous and kinked connection on the left and right hand side, respectively. Values at boundaries: $M_4 = 0\,\text{kNm}$,

$M_3 = -\frac{5 \cdot 1.2^2}{2} = -3.6\,\text{kNm},\ M_1 = 0\,\text{kNm}$

$M_2 = 9.36 \cdot 3.6 - 10.8 \cdot 1.8 = +14.26\,\text{kNm}.$

Depths of parabolas: $\frac{3.0 \cdot 3.6^2}{8} = 4.86\,\text{kNm},$

$\frac{5{,}0 \cdot 2.4^2}{8} = 3.60\,\text{kNm},\ \frac{5.0 \cdot 1.2^2}{8} = 0.9\,\text{kNm}$

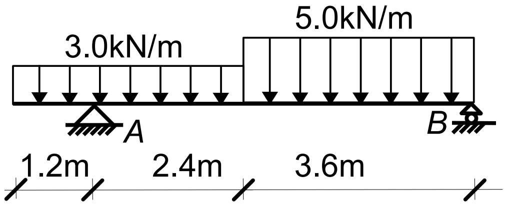
The value of maximum is

$M_{max} = +9.36 \cdot 3.12 - (3.0 \cdot 3.12)\frac{3.12}{2} = +14.60\,\text{kNm}$

Remark: the bracketed product in the formula of the maximum equals 9.36 just based on the method used in the calculation of x_m = 3.12 m; thus, the maximum could also be obtained as $+9.36 \cdot 3.12/2$.

Exercise 2

Draw internal force diagrams for the structure based on calculations.



Solution

Resultants:

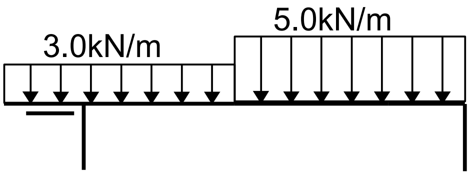
Reactions:

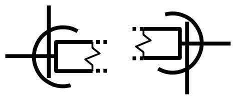
$\sum$:

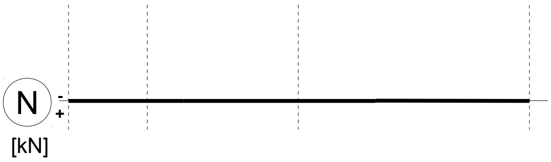
$\sum$:

$\sum$:

Final sketch:

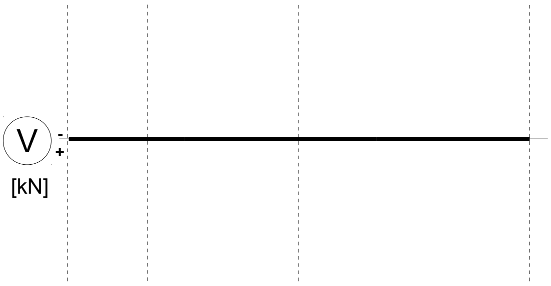






What kind of functions is the diagram composed of? What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of? What is their connection like?

$V_1 =$

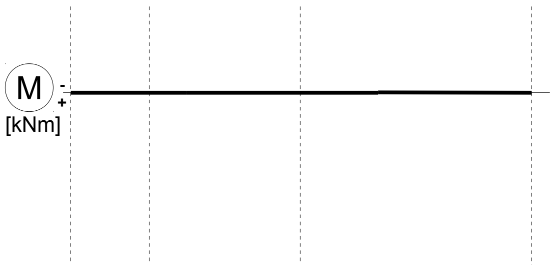
$V_2 =$

$V_3 =$

$V_4 =$

$V_5 =$

zero: $V_m = 0 =$



What kind of functions is the diagram composed of? What is their connection like?

$M_1 =$

$M_2 =$

$M_3 =$

$M_4 =$

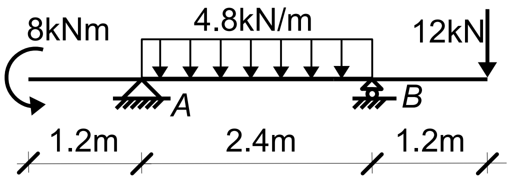
Maximum value:

$M_{max} =$

parabolas: $\frac{ql^2}{8} = \underline{\qquad} =$ $\underline{\qquad} =$ $\underline{\qquad} =$

Example 3

Draw internal force diagrams for the structure based on calculations.



Solution

Reactions:

$\sum M_i^{(A)}: 8-(4.8\cdot 2.4)\cdot 1.2-12\cdot 3.6+B\cdot 2.4=0$

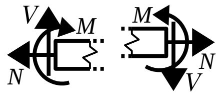
$\rightarrow B=20.43\,\text{kN}\,(\uparrow)$

$\sum M_i^{(B)}: 8+(4.8\cdot 2.4)\cdot 1.2-12\cdot 1.2-A_z\cdot 2.4=0$

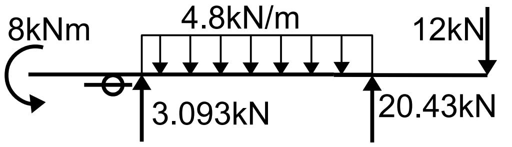
$\rightarrow A_z=3.093\,\text{kN}\,(\uparrow)$

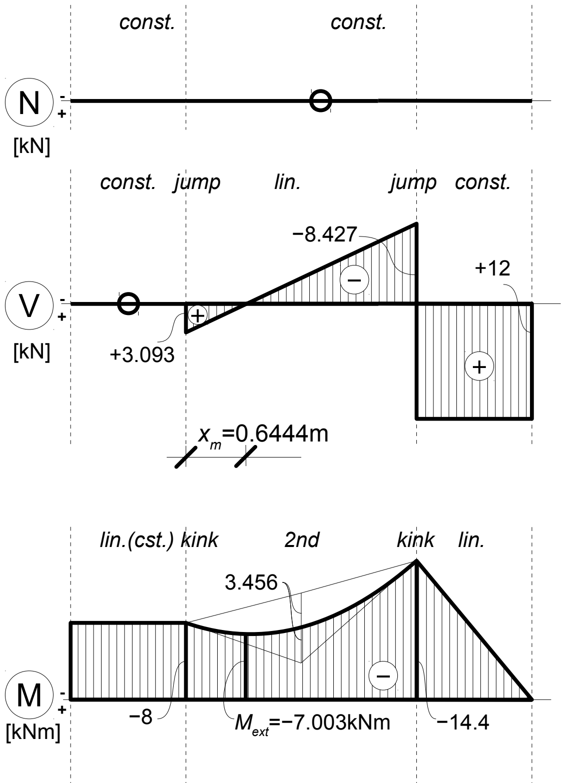
$\sum F_{ix}: A_x=0$

Chk: $\sum F_{iz}: 4.8\cdot 2.4+12-3.093-20.43=-0.003\approx 0$



Final sketch:





Normal forces could arise from horizontal forces; in their absence the diagram is constant with a zero value.

The shear force diagram is composed of two constant segments and a linear one between them; any segmants are connected by jump

Important values (first on overhangs!):

$V_1=0\,\text{kN},\ V_4=+12\,\text{kN},\ V_2=+3.093\,\text{kN},$

$V_3=+3.093-4.8\cdot 2.4=-8.427\,\text{kN}.$

A zero will be found at the linear segment:

$V_m=+3.093-4.8\cdot x_m=0\rightarrow x_m=0.6444\,\text{m}$

The moment diagram is composed of a constant (see the zero shear to the right), a quadratic and a linear segment. Values at the boundaries:

$M_1=-8\,\text{kNm},$

$M_2=-12\cdot 1.2=-14.4\,\text{kNm},\ M_3=0\,\text{kNm}.$

The depth of the parabola: $\dfrac{4.8\cdot 2.4^2}{8}=3.456\,\text{kNm}$

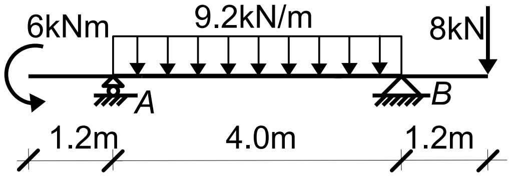
There is a local minimum:

$$M_{ext}=-8+3.093\cdot 0.6444-(4.8\cdot 0.6444)\frac{0.6444}{2}=$$
$$=-7.003\,\text{kNm}$$

Remark: While preparing the moment diagram, reactions were only used in finding local minimum only. Such a moment diagram could then be used to derive also the shear force diagram: slopes at overhangs can easily be calculated; moreover, the height of the point of intersection of two final tangents to the parabola can also be found as $(-8-14.4)/2+2\cdot 3.456=-4.288\,\text{kNm}$. Slopes of the same tangents (that is, shear forces at the ends of the middle segment) are $\left|\dfrac{-8-(-4.288)}{1.2}\right|=3.093$ and $\left|\dfrac{-14.4-(-4.288)}{1.2}\right|=8.427$ (signs can rather be found from inspection based on the senses of slopes).

Exercise 3

Draw internal force diagrams for the structure based on calculations.



Solution

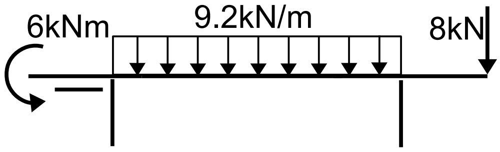
Reactions:

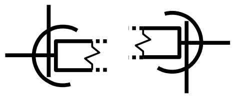
$\sum$:

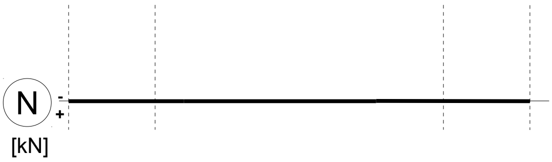
$\sum$:

$\sum$:

Final sketch:

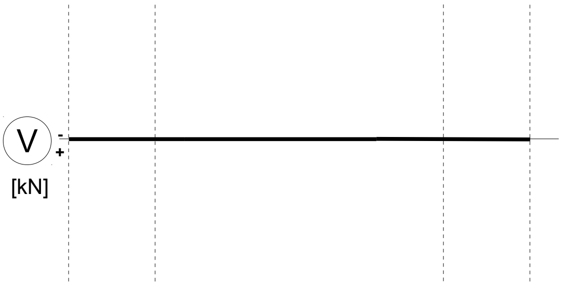






What kind of functions is the diagram composed of? What is their connection like?

$N_1 =$



What kind of functions is the diagram composed of? What is their connection like?

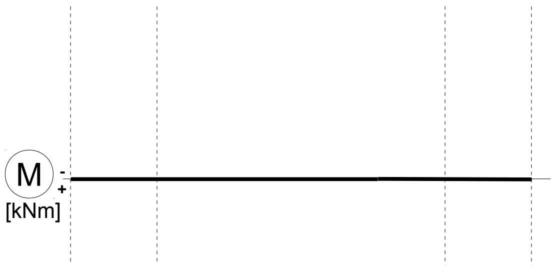
$V_1 =$

$V_4 =$

$V_2 =$

$V_3 =$

zero: $V_m = 0 =$



What kind of functions is the diagram composed of? What is their connection like?

$M_1 =$

$M_3 =$

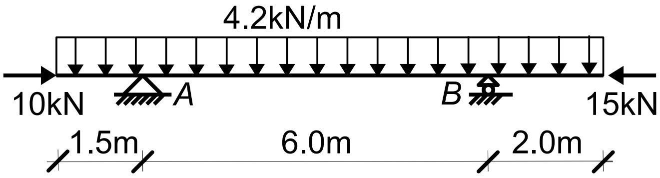
$M_2 =$

parabola: $\frac{q l^2}{8} = \frac{\qquad}{} =$

maximum: $M_{\_} =$

Example 4

Draw internal force diagrams for the structure based on calculations.



Solution

Reactions:

$\sum M_i^{(A)}: -(4.2 \cdot 9.5) \cdot 3.25 + B \cdot 6.0 = 0$

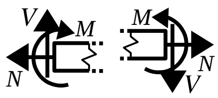
$\rightarrow B = 21.61\,\text{kN}(\uparrow)$

$\sum M_i^{(B)}: (4.2 \cdot 9.5) \cdot 2.75 - A_z \cdot 6.0 = 0$

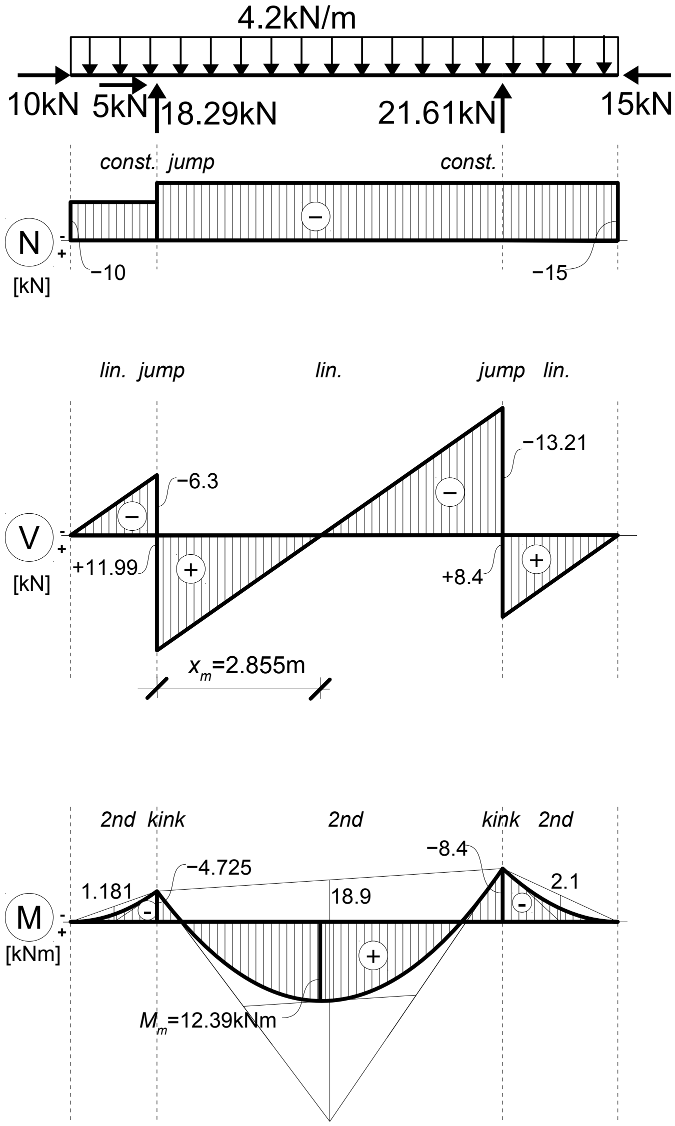
$\rightarrow A_z = 18.29\,\text{kN}(\uparrow)$

$\sum F_{ix}: 10 + A_x - 15 = 0 \rightarrow A_x = 5\,\text{kN}(\rightarrow)$

Check: $\sum F_{iz}: 4.2 \cdot 9.5 - 18.29 - 21.61 = 0$



Final sketch:



The normal force diagram is composed of a 1.5-m and an 8.0-m long segment with a jump in between. Important values (found from oustide on overhangs first):

$N_1 = -10\,\text{kN},\ N_2 = -15\,\text{kN}.$

The shear force diagram is composed of three parallel linear segments with jumps at connections according to the reactions. Values at boundaries: $V_1 = 0\,\text{kN}$,

$V_2 = -4.2 \cdot 1.5 = -6.3\,\text{kN},\ V_6 = 0\,\text{kN},$

$V_5 = +4.2 \cdot 2.0 = +8.4\,\text{kN},$

$V_4 = +8.4 - 21.61 = -13.21\,\text{kN},$

$V_3 = -13.21 + 4.2 \cdot 6.0 = +11.99\,\text{kN}.$

A zero will be found on the middle segment:

$V_m = +11.99 - 4.2 \cdot x_m = 0 \quad \rightarrow x_m = 2.855\,\text{m}$

The moment diagram is composed of three parabolas connected through kinks. Values at the boundaries: $M_1 = 0\,\text{kNm},\ M_4 = 0\,\text{kNm},$

$$M_2 = -(4.2 \cdot 1.5) \cdot \frac{1.5}{2} = -4.725\,\text{kNm},$$

$$M_3 = -(4.2 \cdot 2.0) \cdot \frac{2.0}{2} = -8.4\,\text{kNm}.$$

Depths: $\frac{4.2 \cdot 1.5^2}{8} = 1.181\,\text{kNm},$

$$\frac{4.2 \cdot 2.0^2}{8} = 2.1\,\text{kNm},\ \frac{4.2 \cdot 6.0^2}{8} = 18.9\,\text{kNm}$$

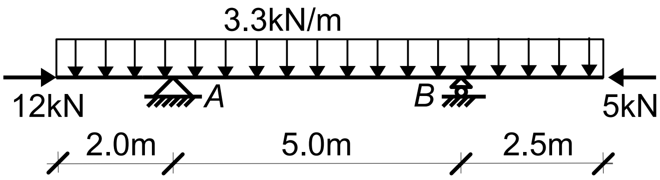
The maximum value is

$$M_{max} = +18.29 \cdot 2.855 - (4.2 \cdot 4.355) \frac{4.355}{2} =$$

$$= +12.39\,\text{kNm}$$

Exercise 4

Draw internal force diagrams for the structure based on calculations.



Solution

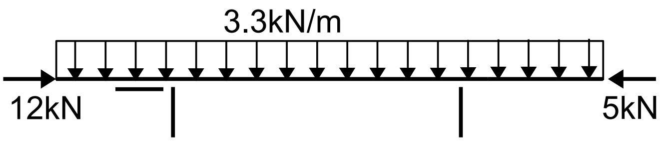
Reactions:

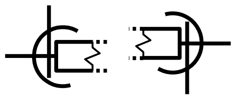
$\sum$:

$\sum$:

$\sum$:

Final sketch:

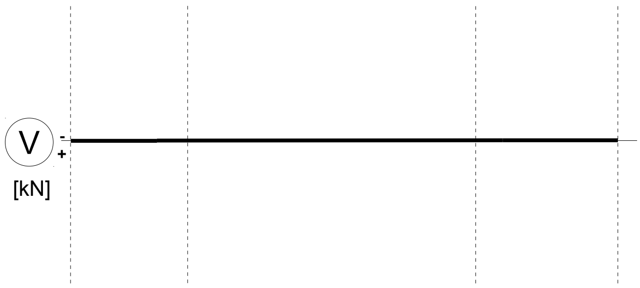




What kind of functions is the diagram composed of? What is their connection like?

$N_1 =$

$N_2 =$



What kind of functions is the diagram composed of? What is their connection like?

$V_1 =$

$V_2 =$

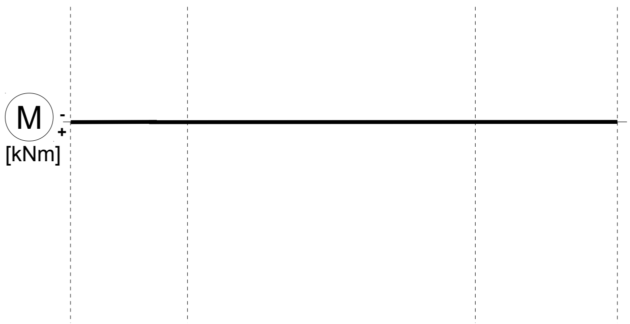
$V_6 =$

$V_5 =$

$V_3 =$

$V_4 =$

zero: $V_m = 0 =$



What kind of functions is the diagram composed of? What is their connection like?

$M_1 =$ $M_4 =$

$M_2 =$

$M_3 =$

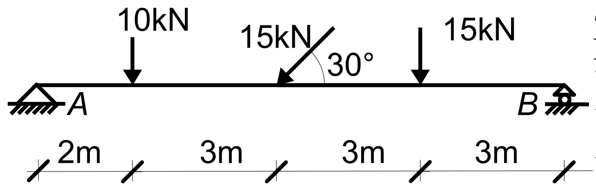
parabolas: $\frac{ql^2}{8} = \underline{\qquad\qquad} =$

$\underline{\qquad\qquad} =$ $\underline{\qquad\qquad} =$

maximum: $M_{max} =$

Example 5

Draw internal force diagrams for the structure based on calculations.



Solution

Reactions:

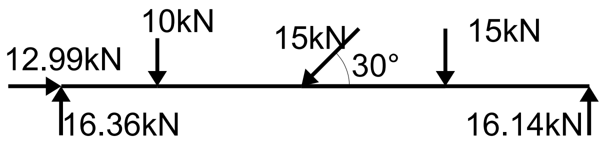
$\sum M_i^{(A)}: -10\cdot 2-15\sin 30°\cdot 5-15\cdot 8+B\cdot 11=0$
$\rightarrow B=16.14\,\mathrm{kN}(\uparrow)$

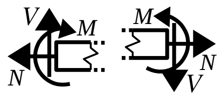
$\sum M_i^{(B)}: 10\cdot 9+15\sin 30°\cdot 6+15\cdot 3-A_z\cdot 11=0$
$\rightarrow A_z=16.36\,\mathrm{kN}(\uparrow)$

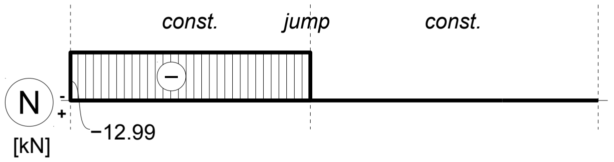
$\sum F_{ix}: A_x-15\cos 30°=0\rightarrow A_x=12.99\,\mathrm{kN}(\rightarrow)$

Chk: $\sum F_{iz}: 10+15\sin 30°+15-16.36-16.14=0$

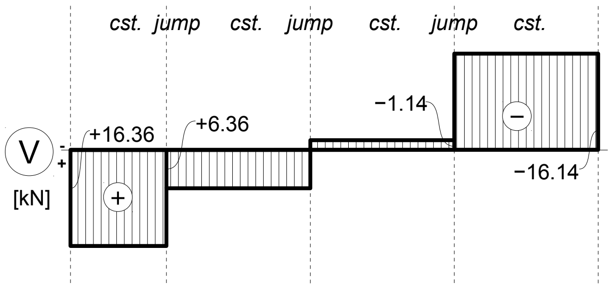
Final sketch:



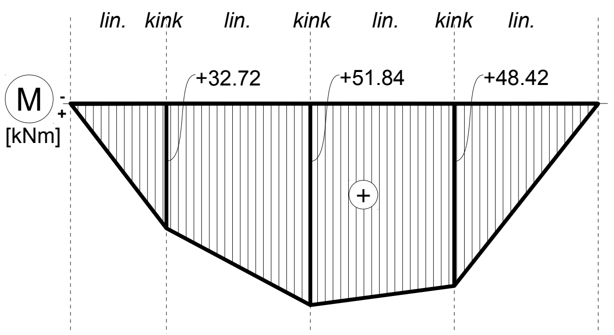




The normal force diagram is composed of two constant segments connected by a jump equal to the horizontal component of the oblique force. The values are as follows:
$N_1=-12.99\,\mathrm{kN}$, $N_2=0\,\mathrm{kN}$.



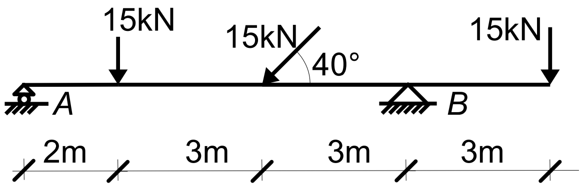
The shear force diagram is composed of four constant segments separated by jumps defined by vertical force components. Values:
$V_1=+16.36\,\mathrm{kN}$, $V_2=+16.36-10=+6.36\,\mathrm{kN}$,
$V_3=+6.36-15\sin 30°=-1.14\,\mathrm{kN}$,
$V_4=-16.14\,\mathrm{kN}$.



The moment diagram is composed of four linear segments connected through kinks:
$M_1=0\,\mathrm{kNm}$, $M_2=+16.36\cdot 2=+32.72\,\mathrm{kNm}$,
$M_3=+16.36\cdot 5-10\cdot 3=+51.84\,\mathrm{kNm}$,
$M_4=+16.14\cdot 3=+48.42\,\mathrm{kNm}$, $M_5=0\,\mathrm{kNm}$.

Exercise 5

Draw internal force diagrams for the structure based on calculations.



Solution

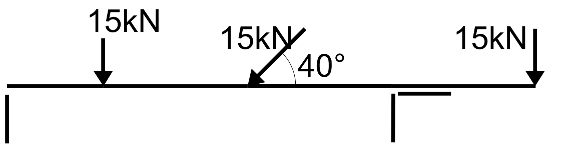
Reactions:

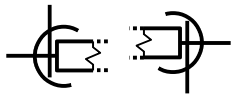
$\sum$:

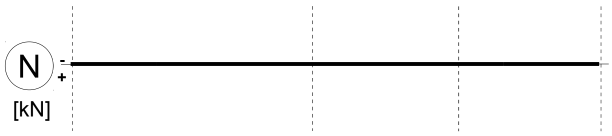
$\sum$:

$\sum$:

Final sketch:





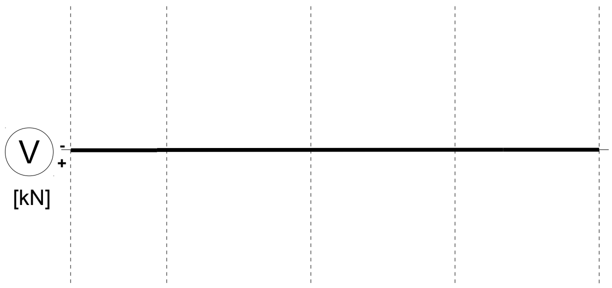


What kind of functions is the diagram composed of? What is their connection like?

$N_3 =$

$N_1 =$

$N_2 =$



What kind of functions is the diagram composed of? What is their connection like?

$V_4 =$

$V_1 =$

$V_2 =$

$V_3 =$

What kind of functions is the diagram composed of? What is their connection like?

$M_5 =$

$M_4 =$

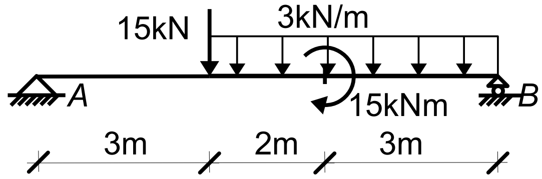
$M_1 =$

$M_2 =$

$M_3 =$

Example 6

Draw internal force diagrams for the structure based on calculations.



Solution

Reactions:

$\sum M_i^{(A)}: -15\cdot 2 - 3\cdot 5\cdot 5.5 - 15 + B\cdot 8 = 0$

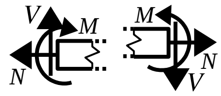
$\rightarrow B = 17.81\,\text{kN}(\uparrow)$

$\sum M_i^{(B)}: 15\cdot 5 + 3\cdot 5\cdot 2.5 - 15 - A_z\cdot 8 = 0$

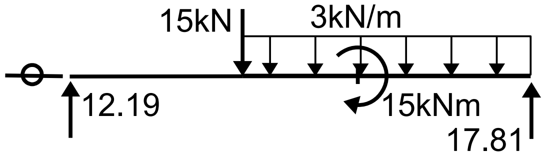
$\rightarrow A_z = 12.19\,\text{kN}(\uparrow)$

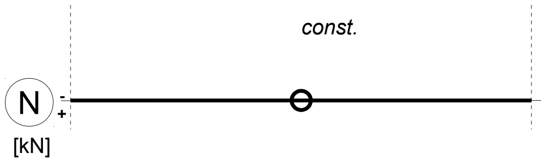
$\sum F_{ix}: A_x = 0$

Check: $\sum F_{iz}: 15 + 3\cdot 5 - 12.19 - 17.87 = 0$

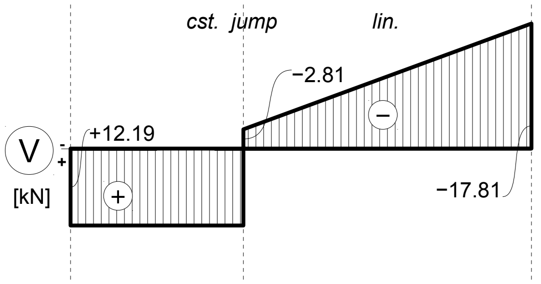


Final sketch:





Normal forces could arise from horizontal forces; in their absence the diagram is constant with a zero value.

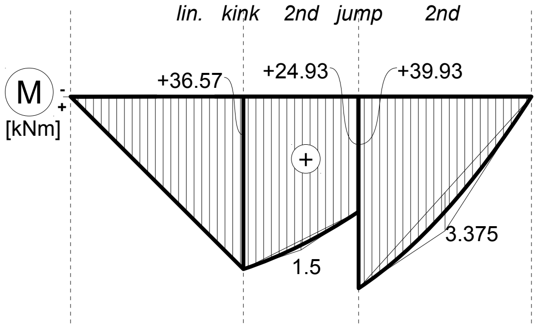


The shear force diagram is composed of a constant and a linear segment separated by a jump of magnitude of the concentrated load:

$V_1 = +12.19\,\text{kN}$,

$V_2 = +12.19 - 15 = -2.81\,\text{kN}$,

$V_3 = -17.81\,\text{kN}$.



The moment diagram is composed of one linear and two quadratic segments. The linear part is separated by a jump, while there is a jump between parabolas (their tangents are parallel at the jump). Values:

$M_1 = 0\,\text{kNm}$,

$M_2 = +12.19\cdot 3 = +36.57\,\text{kNm}$,

$M_5 = 0\,\text{kNm}$,

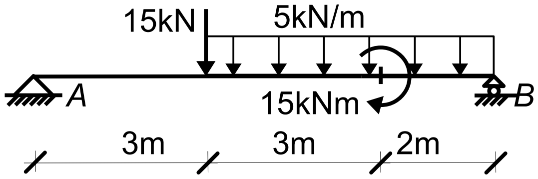
$M_4 = 17.81\cdot 3 - 3\cdot 3\cdot 1.5 = +39.93\,\text{kNm}$,

$M_3 = 17.81\cdot 3 - 3\cdot 3\cdot 1.5 - 15 = +24.93\,\text{kNm}$,

depths: $\frac{3\cdot 2^2}{8} = 1.5\,\text{kNm}, \frac{3\cdot 3^2}{8} = 3.375\,\text{kNm}$

Exercise 6

Draw internal force diagrams for the structure based on calculations.



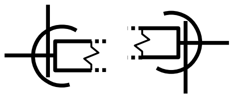
Megoldás

Reakciók:

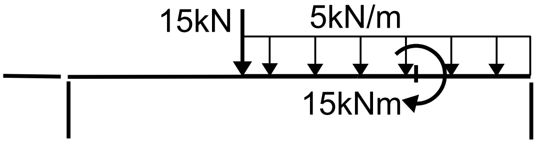
$\sum$:

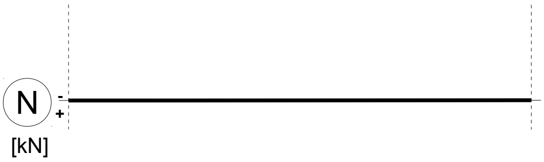
$\sum$:

$\sum$:



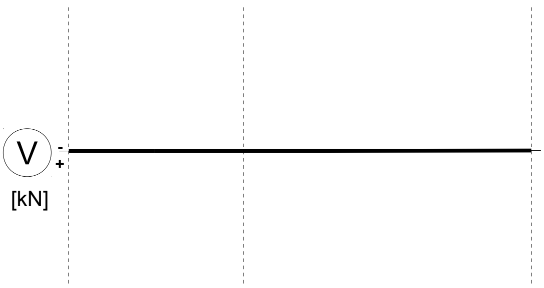
Final sketch:





What kind of functions is the diagram composed of? What is their connection like?

$N_1 =$



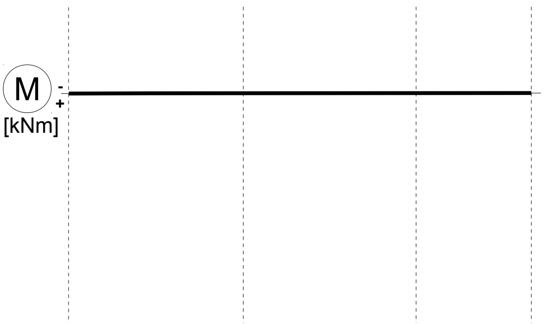
What kind of functions is the diagram composed of? What is their connection like?

$V_1 =$

$V_2 =$

$V_3 =$

zero: $V_m = 0 =$



What kind of functions is the diagram composed of? What is their connection like?

$M_1 =$

$M_2 =$

$M_3 =$

$M_4 =$

$M_5 =$

parabolas: $\frac{q l^2}{8} = \frac{\quad}{} =$

$\frac{\quad}{} =$

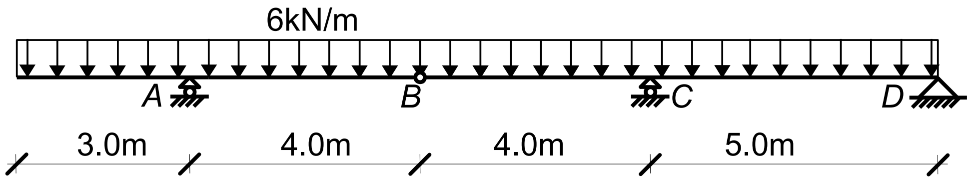
maximum: $M_{max} =$

Internal force diagrams of Gerber beams

Observations made on the calculation of individual values of internal forces in compounds can be generalized to diagrams. In a compound structure, there are even more ways of finding internal force values than in a simple one. It resides in the fact that equilibrium must be hold not only for the entire compound but for any of its components; that is why any substructure containing the cross section in case can be used for calculation. Moreover, values can be found from two sides for any such substructure or even the entire compound. However, it still holds that one calculation is sufficient for any value in practical problems. Thus, the first step still remains to be the decision on the simplest way possible of the calculation.

Example 1

Draw internal force diagrams for the structure based on calculations.



Solution

Calculation of the reactions: susp. $\sum F_{ix}: B_x = 0\,\text{kN}$, fixed $\sum F_{ix}: D_x = 0\,\text{kN}$

susp. $\sum M_i^{(B)}: 6\cdot 7.0\cdot 3.5 - A\cdot 4.0 = 0 \rightarrow A = 36.75\,\text{kN}(\uparrow)$

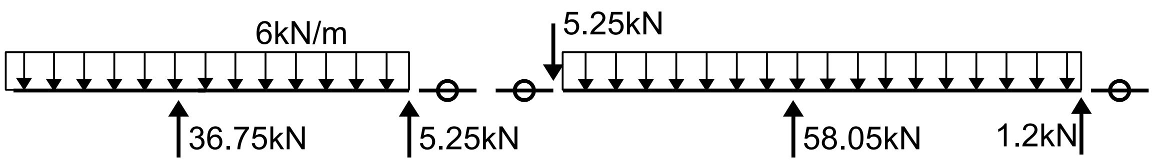
susp. $\sum M_i^{(A)}: -6\cdot 7.0\cdot 0.5 + B_z\cdot 4.0 = 0 \rightarrow B_z = 5.25\,\text{kN}(\uparrow)$

fixed $\sum M_i^{(D)}: 5.25\cdot 9.0 + 6\cdot 9.0\cdot 4.5 - C\cdot 5.0 = 0 \rightarrow C = 58.05\,\text{kN}(\uparrow)$

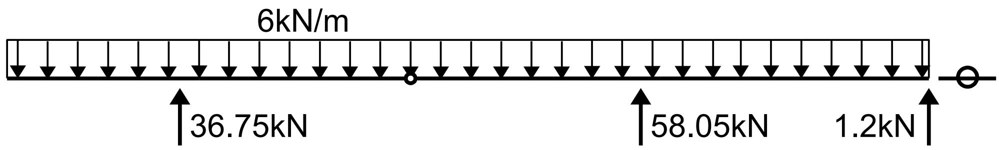
fixed $\sum M_i^{(C)}: 5.25\cdot 4.0 - 6\cdot 9.0\cdot 0.5 + D_z\cdot 5.0 = 0 \rightarrow D_z = 1.2\,\text{kN}(\uparrow)$

Check: str $\sum F_{iz}: 6\cdot 16 - 36.75 - 58.05 - 1.2 = 0$

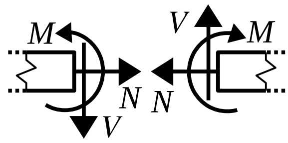
Final sketches by members:



It is also possible to make a final sketch for the entire structure as follows:

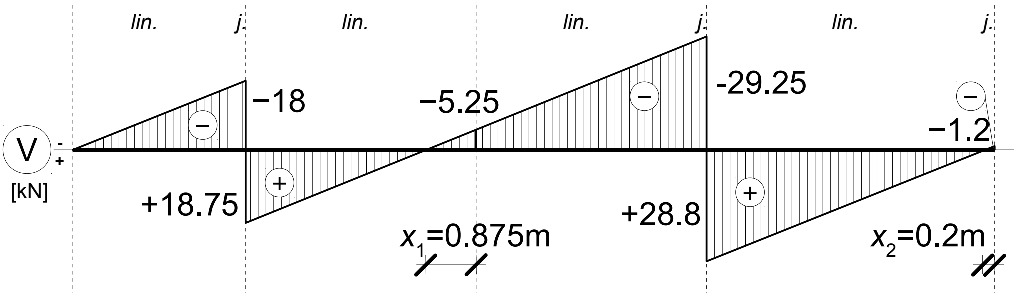


In the forthcoming procedure, internal forces will be obtained from both sides, so both reference frames are given in the figure to the right. Since there are only vertical force components acting on the beam, the normal force diagram is a constant zero:



With reference to the final sketch, the shear force diagram is composed of three linear segments with jumps at supports *A*, *C* and *D*. In addition, the shear at the internal hinge is also commonly specified, which means the determination of seven values. Ordinates on the overhanging part of the suspended (drop-in) beam are dealt with first: $V_1=0\,\text{kN}$, $V_2=-6\cdot 3.0=-18\,\text{kN}$, $V_3=-6\cdot 3.0+36.75=18.75\,\text{kN}$, $V_4=-6\cdot 7.0+36.75=-5.25\,\text{kN}$. (The last value could have been obtained even more quickly from right on the drop-in beam.) Shear force at the left end of the fixed part is the same $V_4=-5.25\,\text{kN}$ as calculated above (it can be seen more directly looking at the entire structure, since there is no concentrated active or passive force there). The remaining three values are got for convenience from the fixed part (one from left and two from right) as follows: $V_5=-5.25-6\cdot 4.0=-29.25\,\text{kN}$, $V_6=-1.2+6\cdot 5.0=+28.8\,\text{kN}$, $V_7=-1.2\,\text{kN}$.

There are two points of change of signs in the diagram. Their positions are looked for as distances to the left of points *B* and *D* such that the shear force is expressed in terms of those distances (based on the isolated parts, for convenience, from right): $V(x_1)=-5.25+6\cdot x_1=0 \rightarrow x_1=0.875\,\text{m}$, $V(x_2)=-1.2+6\cdot x_2=0 \rightarrow x_2=0.2\,\text{m}$.



As seen again from the final sketch of the entire assembly, the moment diagram is composed of three parabolic segments with kinks at supports *A*, *C* and *D*. Based on the equilibrium of the entire assembly, no kink or jump can occur at internal hinge *B* that means a smooth connection there; moreover, the moment is zero there as seen from individual final sketches of adjacent separate members.

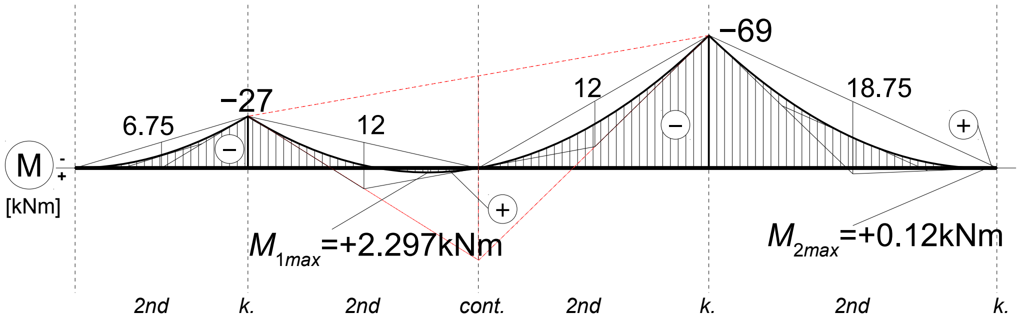
The calculation is started again on the overhang of the suspended part: $M_1=0\,\text{kNm}$, $M_2=-6\cdot 3.0\cdot 1.5=-27\,\text{kNm}$, the depth of the parabola is: $6\cdot 3.0^2/\ 8=6.75\,\text{kNm}$. In order to draw the parabola on the overhang, the bending moment at point *B* is required: it is got from right based on the suspended part as $M_3=0\,\text{kNm}$ with a depth of parabola of $6\cdot 4.0^2/\ 8=12\,\text{kNm}$.

The solution continues at the overhang of the fixed part. The moment at point *B* is obtained again as $M_3=0\,\text{kNm}$ from outside. At support *C*, $M_4=-5.25\cdot 4.0-6\cdot 4.0\cdot 2.0=-69\,\text{kNm}$ based on the isolated member; the depth of of segment *BC* is $6\cdot 4.0^2/\ 8=12\,\text{kNm}$. Finally, the moment at support *D* is obtained from right as $M_5=0\,\text{kNm}$. The depth of the parabola here is $6\cdot 5.0^2/\ 8=18.75\,\text{kNm}$.

Maxima of moment are calculated at zeros of shear from right, based on the isolated part:

$$M_{1max} = +5.25 \cdot 0.875 - 6 \cdot 0.875 \cdot \frac{0.875}{2} = +2.297\,\text{kNm}\ ,$$

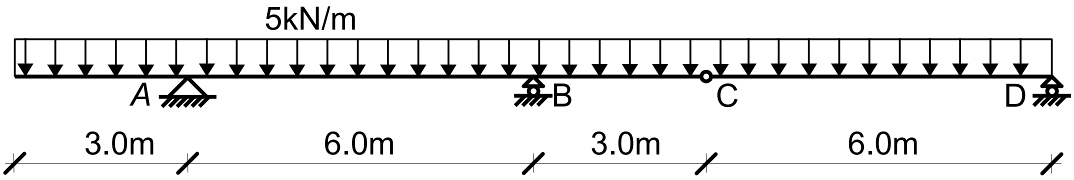
$$M_{2max} = +1.2 \cdot 0.2 - 6 \cdot 0.2 \cdot \frac{0.2}{2} = +0.12\,\text{kNm}\ .$$



Remark: as concluded from the equilibrium of the entire assembly, there is a single distributed load only between supports *A* and *C*, that is why the moment diagram there is a single parabolic segment. It could also be drawn between ordinates of −27 and −69 kNm. The depth is now $6 \cdot 8^2/8 = 48\,\text{kNm}$, construction lines pertaining to this parabola are drawn in red. Recall that a parabola obtained in this way must coincide to that is drawn in two separate parts (with an emphasis on the zero value of moment at the internal hinge).

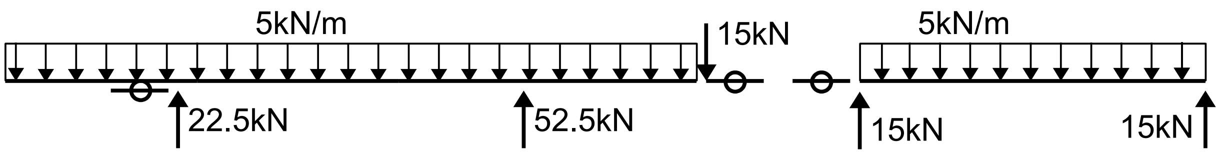
Exercise 1

Draw internal force diagrams for the structure based on calculations.

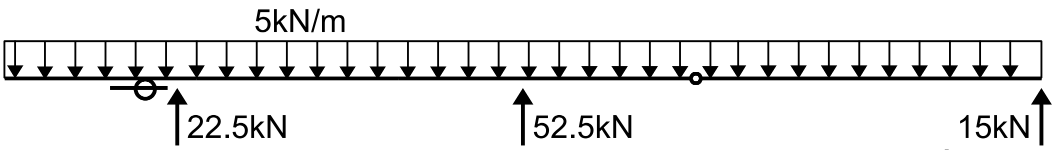


Solution

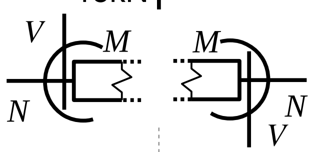
All forces external to each isolated body:

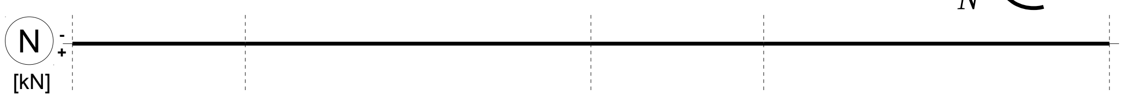


All external forces exerted on the structure:



Normal forces on a horizontal beam can only produced by horizontal forces. Here all forces are vertical, so thediagram is a constant zero:





The shear diagram is piecewise .... The diagram is drawn first on the suspended part.

The arrangement of forces resembles that of a simply supported beam with a constant distributed load. The shear force values needed to draw a linear segment are as follows:

$V_6 =$ $V_7 =$

The shear force at the end of the right overhang of the fixed part just became known; to the right of support *B* we have

$V_5 =$

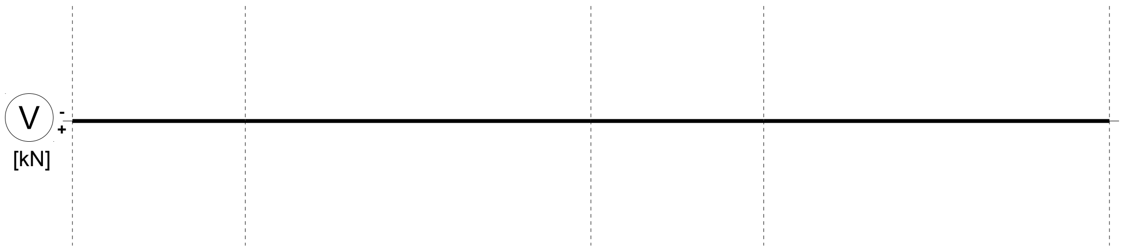
Shear forces at two ends of segment between supports *A* and *B*:

$V_4 =$ $V_3 =$

The shear on the left overhang is found rather from left, independently from all preceding values.

$V_1 =$ $V_2 =$

Check whether the jump at support *A* corresponds to the concentrated force there.
Check the slopes of the diagram for uniformity.



Zeros correspond to possible maxima of moments. Determine their positions:

$V(x_1) =$ $\rightarrow x_1 =$

$V(x_2) =$ $\rightarrow x_2 =$

The moment diagram is piecewise .... The diagram is drawn first on the suspended part. The arrangement of forces resembles that of a simply supported beam with a constant distributed load. The bending moment values needed to draw a parabolic segment are as follows:

$M_4 =$ $M_5 =$

The depth of the parabola: ——— =

The bending moment at the end of the right overhang of the fixed part just became known; at support *B* we have

$M_3 =$

The depth of the parabola to the right of *B* is ——— =

The bending moment on the left overhang is found rather from left again.

$M_1 =$ $M_2 =$

The depth of the parabola between them: $\frac{}{} =$

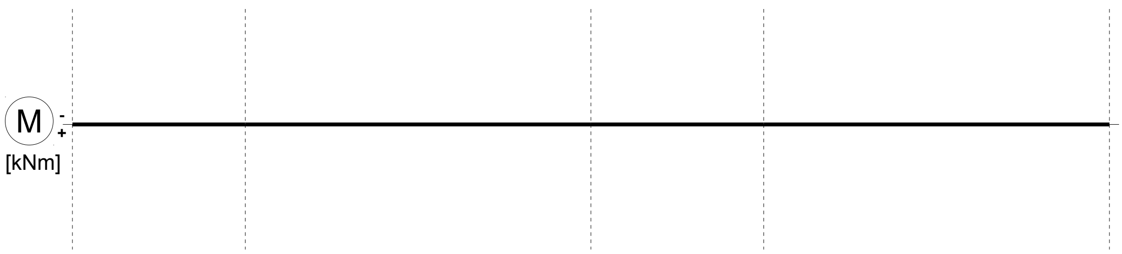
Moments at both supports A and B are already known. The depth of the parabola between them is $\frac{}{} =$

Two extreme values at positions determined earlier are

$M_{1max} =$

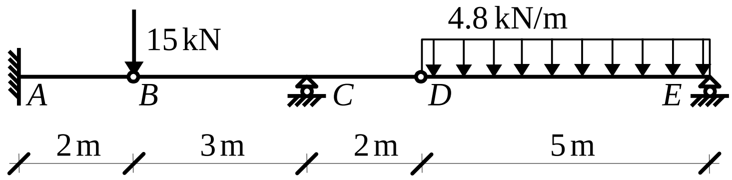
$M_{2max} =$

Check the connection of segments at the internal hinge for smoothness.



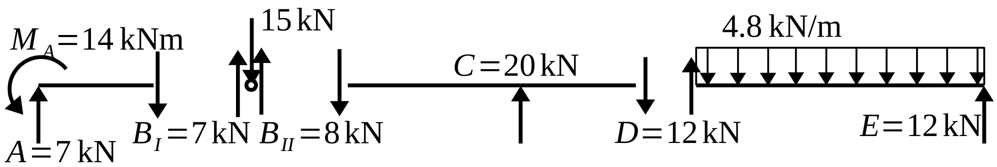
Example 2

Draw internal force diagrams for the structure based on calculations.

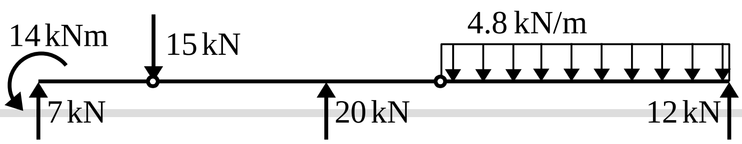


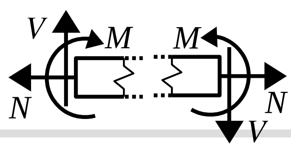
Solution

External and internal reactions on each rigid member (horizontal components of reactions are all zero, so they are not drawn in the figure):



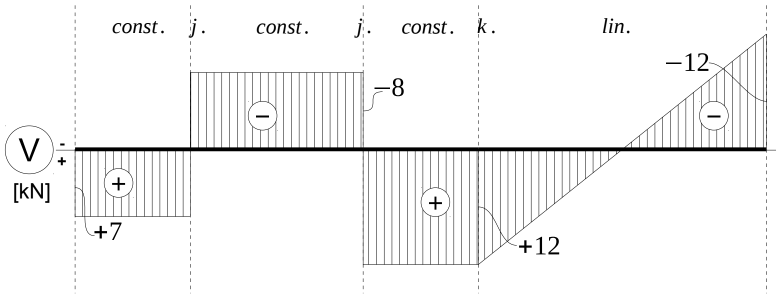
All external forces exerted on the structure:



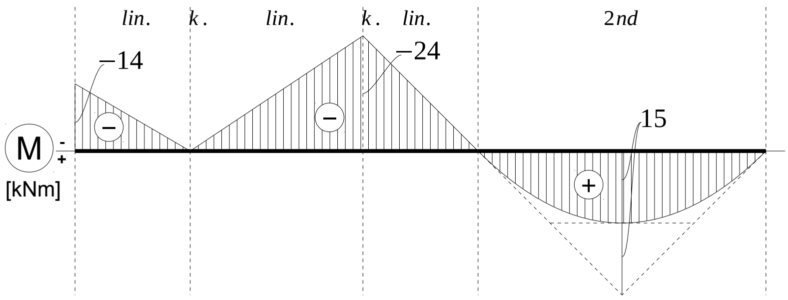


Normal forces could only arise from horizontal force components here. In their absence:

The shear force diagram is composed of three constant and a linear segment. The diagram on the rightmost suspended part corresponds to that of a simply supported beam with final values $V_4=-12\,\text{kN}$ and $V_3=+12\,\text{kN}$. The zero is in the middle of the segment. As can be seen from the forces exerted on the entire beam, V_3 is also the value of the constant segment between *C* and *D*. The remaining two constant segments have values $V_2=8\,\text{kN}$ and $V_1=+7\,\text{kN}$.

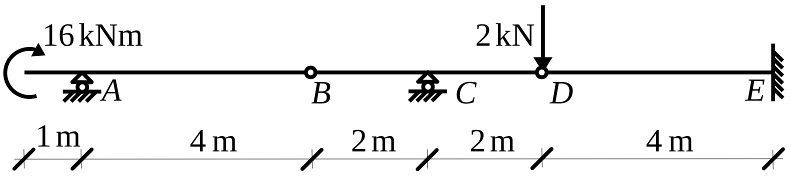


The moment diagram is composed of three linear and a parabolic segment. The parabola over the suspended part looks as in simply supported beams with final values $M_4=M_5=0\,\text{kNm}$ and depth $4.8\cdot 5^2/8=15\,\text{kNm}$, which now corresponds also to a local maximum. The linear segment between *C* and *D* is connected smoothly (without a jump or kink) to the parabola, bending moment at the support equals $M_3=-12\cdot 2=-24\,\text{kNm}$. At the right end of the linear segment between *B* and *C*, the same value is found, while at hinge *B* $M_2=0\,\text{kNm}$ is obtained from forces acting on the member *BD*. Moment at hinge *B* is also zero on the leftmost member as clearly seen from its isolation; finally, the moment at the support is $M_1=-7\cdot 2=-14\,\text{kNm}$.



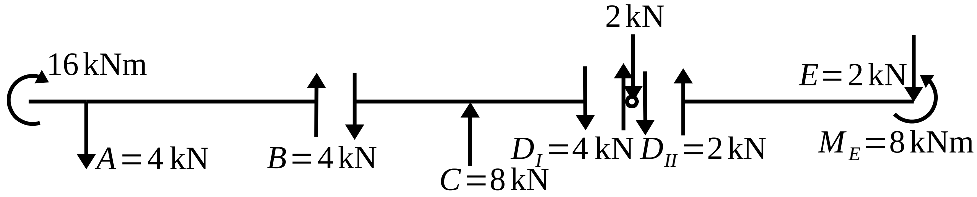
Exercise 2

Draw internal force diagrams for the structure based on calculations.

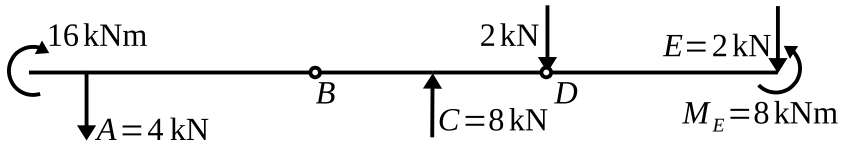


Solution

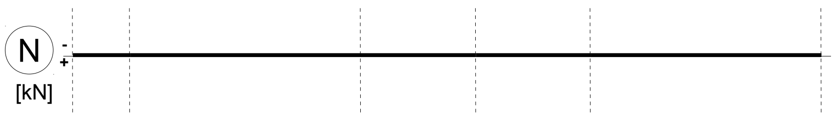
Active forces and (nonzero) reactions by members:



External forces and (nonzero) reactions acting on the entire structure:



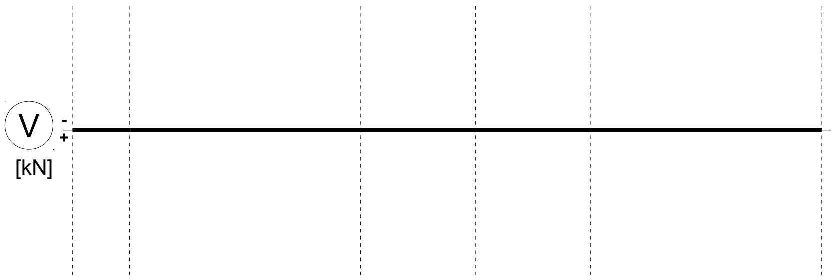
What kind of segments is the normal force diagram composed of?



After identifying types and relative connections of segments, characteristic values of the shear force diagram are as follows:

$V_1=$ $V_2=$

$V_3=$ $V_4=$



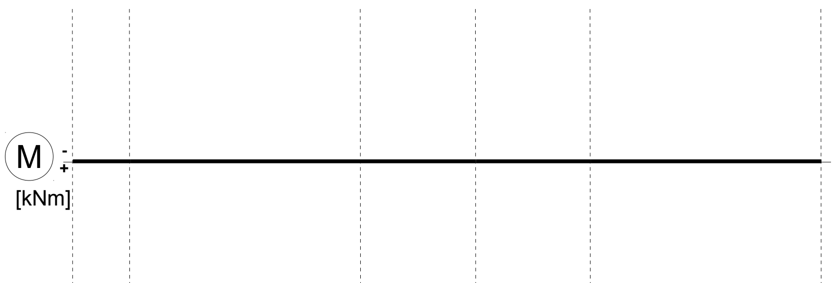
After identifying types and relative connections of segments, characteristic values of the bending moment diagram are as follows:

$M_1=$ $M_2=$

$M_3=$ $M_4=$

$M_5=$

How the slopes of linear segments are related to each other?

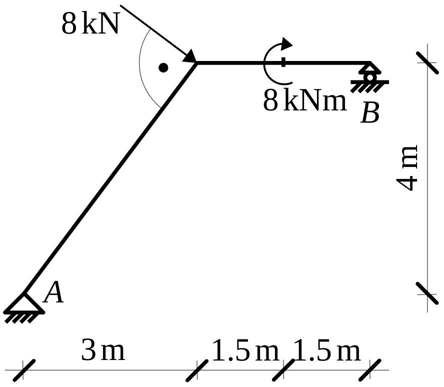


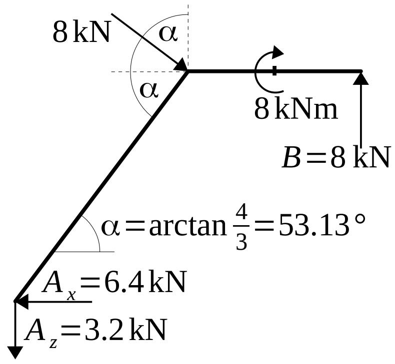
Frames

Frames are composed of members of variable axial directions. As an immediate consequence, the direction of both the normal and shear forces change whenever such a change in the axial direction occurs: such corner points are always points of division between segments of those diagrams. A corner point (with two small stubs of beams there) has still to be in equilibrium. This makes possible to check the equilibrium of moments by a simple look on the diagram, with reference to the drawing rule that ordinates are always shown on the tensile side. On oblique parts of beams, both the normal and shear force have to be calculated, of course, from oblique resolutions.

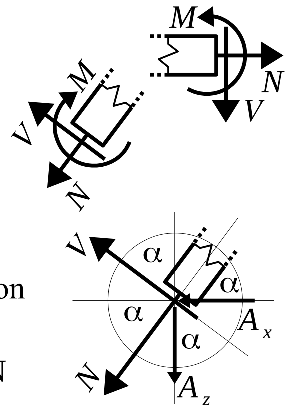
Example 3

Draw internal force diagrams for the structure based on calculations. (Reactions are given in the figure to the right.)





Solution



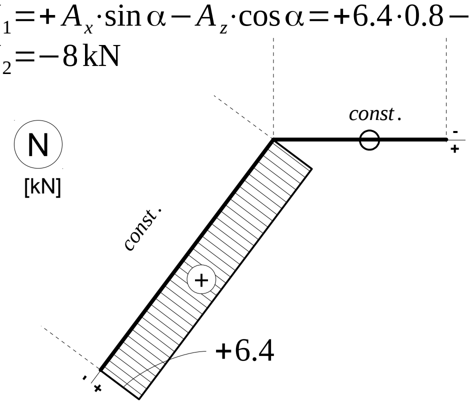
Let the 'left-to-right' orientation of beam segments in internal force calculations be set from *A* to *B* consistently: in other words, let the positive side of the beam be set to the bottom right side of the axis. It makes positive senses of internal forces to be as drawn in the top right figure. Just below that, angles and components A_x and A_z are drawn in order to help finding adequate signs and trigonometric functions.

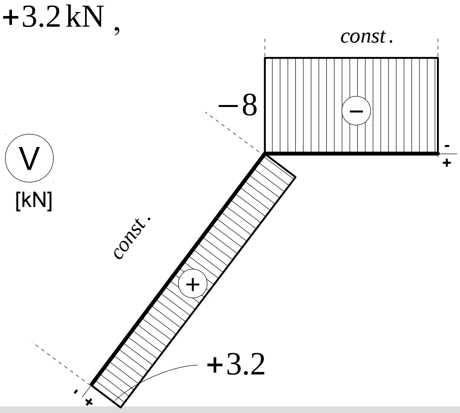
The normal force diagram is composed of a single constant segment on both the horizontal and inclined parts. The normal forces are as follows:

$$N_1=+A_x\cdot\cos\alpha+A_z\cdot\sin\alpha=6.4\cdot0.6+3.2\cdot0.8=+6.4\text{ kN}\ , \qquad N_2=0\text{ kN}$$

The shear force diagram is also composed of a single constant segment on both the horizontal and inclined parts. Their values are

$$V_1=+A_x\cdot\sin\alpha-A_z\cdot\cos\alpha=+6.4\cdot0.8-3.2\cdot0.6=+3.2\text{ kN}\ ,$$
$$V_2=-8\text{ kN}$$

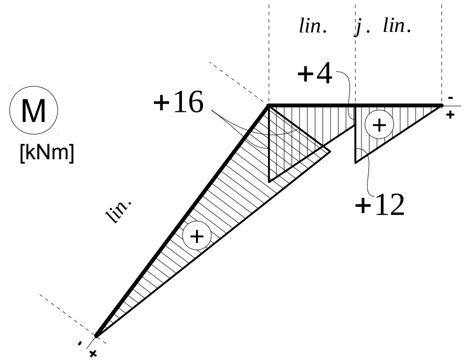




The moment diagram is linear all over the inclined part, while it has two parallel linear segments separated by a jump on the horizontal part. There are two connected segments at the corner point. If the bending moment is calculated from the same direction at either side of the corner, then positive senses for both moments will be identical. In addition, both moment values are sums of moments of the same forces about the same point, making the moment value to be identical at both sides of the corner.

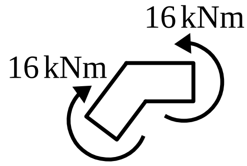
In accordance with this observation, characteristic values of the bending moment are as follows:

$M_1 = M_5 = 0\,\text{kNm}$, $M_2 = +6.4\cdot 4 - 3.2\cdot 3 = +16\,\text{kNm}$,
$M_3 = +8\cdot 1.5 - 8 = +4\,\text{kNm}$, $M_4 = +8\cdot 1.5 = +12\,\text{kNm}$.



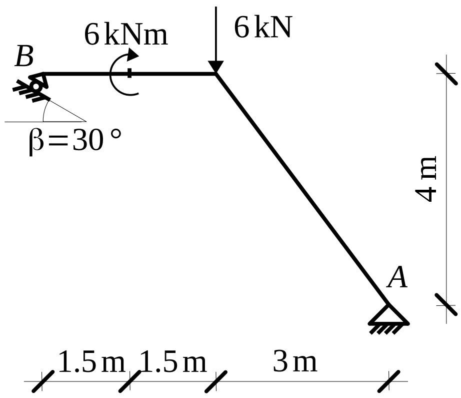
Balance of moments about a corner

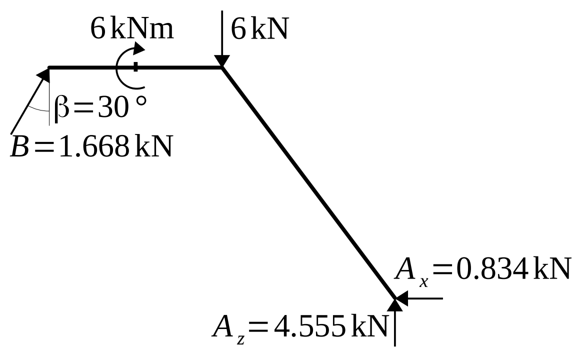
Let the corner be isolated from its neighbourhood and let the stub be drawn with bending moments exerted on it at both cuts. (Normal and shear forces are irrelevant now from the aspect of equilibrium, since they would all be associated with a zero moment arm; for that reason, they are not even drawn in the figure.) Bending moments are read from the diagram; the curved arrow of the moment is started with its tail at the side of the cut where the respective ordinate is drawn. In the current case, a clockwise arrow on the bottom left end of the stub is directed from the bottom right towards the top left side, while the arrow at the opposite end starts at the bottom and points towards the top side in a counter-clockwise sense. Both moments are of a magnitude of $16\,\text{kNm}$, their signs in a moment balance are decided just based on the graphic appearance of arrows: $16-16=0$.



Exercise 3

Draw internal force diagrams for the structure based on calculations (reactions are given).





Solution

As a preliminary step, decide upon positive sides of different parts and draw the positive arrows of internal force components accordingly. Find the angle that the inclined axis makes with the horizontal and draw it into the figure as well.

$\alpha =$

What kind of segments is the normal force diagram composed of? What are its characteristic values?

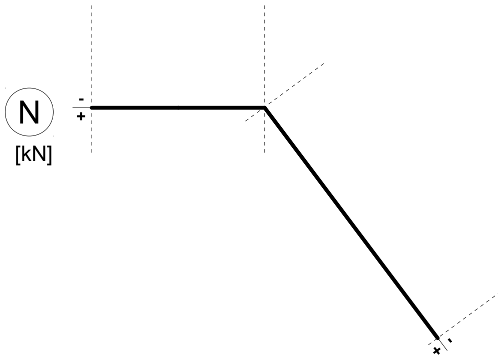
$N_1 =$

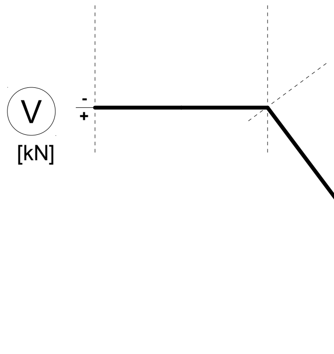
$N_2 =$

What kind of segments is the shear force diagram composed of? What are its characteristic values?

$V_1 =$

$V_2 =$





What kind of segments is the moment diagram composed of? What are its characteristic values?

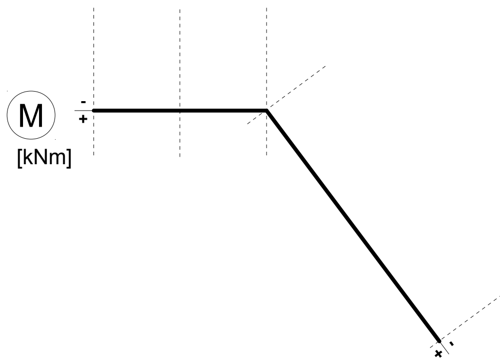
$M_1 =$

$M_2 =$

$M_3 =$

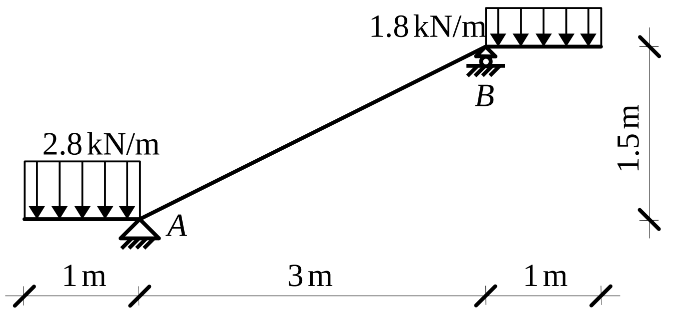
$M_4 =$

$M_5 =$



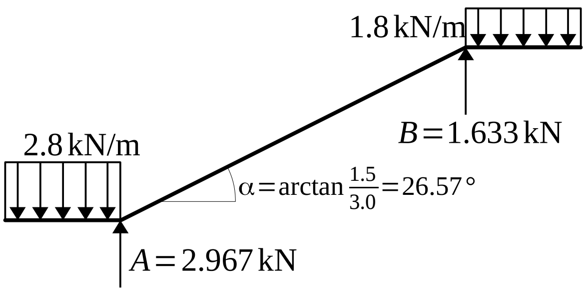
Example 4

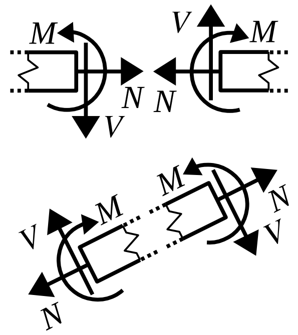
Draw internal force diagrams for the structure based on calculations.



Solution

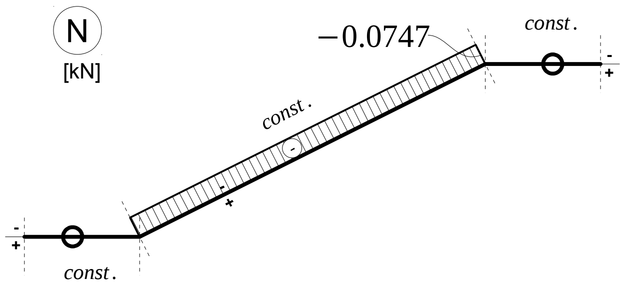
Reactions as well as positive senses of internal force components are shown in the figures below (the bottom side is taken positive for all three segments).

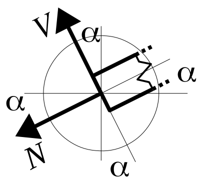




The normal force diagram is composed of three constant segments (two horizontal and an inclined one). On horizontal parts, any normal force is due to horizontal components. Since all external forces are vertical, those are all zero: $N_1 = N_3 = 0\,\text{kNm}$. If a value on the inclined part is calculated from (bottom) left, normal resolutions of reaction *A* and distributed load (of intensity 2.8 kN/m) should be considered:

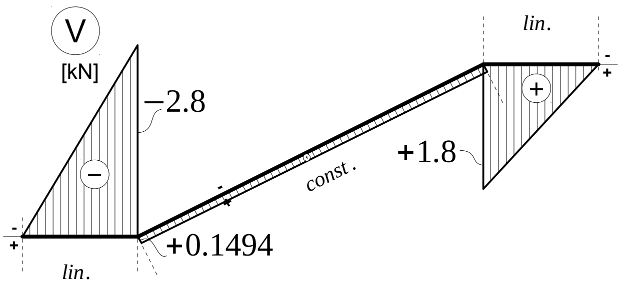
$N_2 = -2.967 \cdot \sin 26.57° + 2.8 \cdot 1.0 \cdot \sin 26.57° = -0.07470\,\text{kN}$



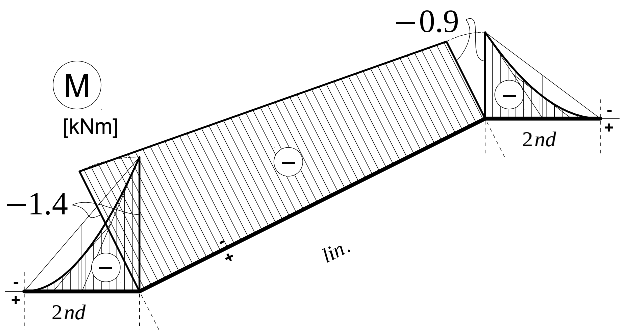


The shear force diagram is linear over horizontal segments and constant over the inclined part. Values at two ends of the bottom horizontal part are $V_1 = 0$ and $V_2 = -2.8 \cdot 1 = -2.8\,\text{kN}$. The same for the top horizontal part: $V_4 = +1.8 \cdot 1 = +1.8\,\text{kN}$ and $V_5 = 0\,\text{kN}$. The value on the inclined part is calculated from left by accounting for shearing components of reaction *A* and distributed load (of intensity 2.8 kN/m):

$V_3 = +2.967 \cdot \cos 26.57° - 2.0 \cdot 1.0 \cdot \cos 26.57° = +0.1494\,\text{kN}$

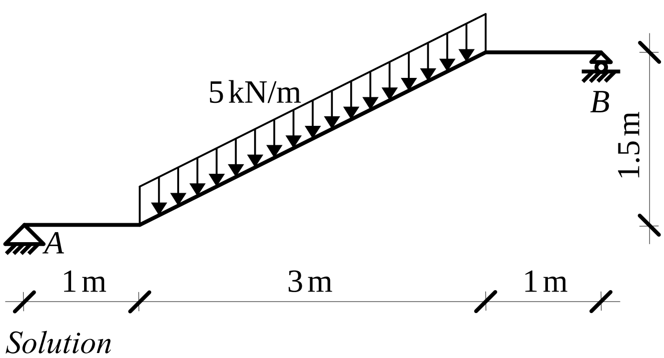


The bending moment diagram is parabolic over horizontal segments and linear over the inclined part. Values at two ends of the bottom horizontal part are $M_1=0$ and $M_2=-2.8\cdot1\cdot0.5=-1.4\,\text{kNm}$. The depth of the parabola is $2.8\cdot1.0^2/8=0.7\,\text{kNm}$. The same for the top horizontal part: $M_5=-1.8\cdot1.0\cdot0.5=-0.9\,\text{kNm}$ and $M_6=0\,\text{kNm}$. The depth of the parabola here is $1.8\cdot1.0^2/8=0.45\,\text{kNm}$. Two endpoints of the inclined part coincide with corners of the frame where moments can be calculated from the same equations as have already been used on the opposite side, so the values are unchanged: $M_3=-1.4\,\text{kNm}$ and $M_4=-0.9\,\text{kNm}$. These two ordinates should be connected by a straight line.



Exercise 4

Draw internal force diagrams for the structure based on calculations.



Solution

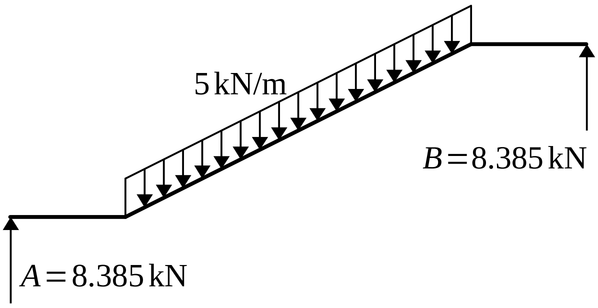
The length of inclined part:

$$l=$$

Angle between the inclined axis and the horizontal:

$$\alpha=$$

Reactions:



Where will positive arrows point to on different segments in calculations from different sides?

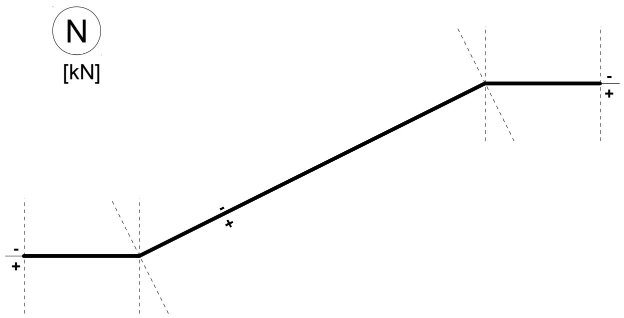
Decide upon the order of segments in the normal force diagram. Find characteristic values:

$N_1=$

$N_2=$

$N_3=$

$N_4=$



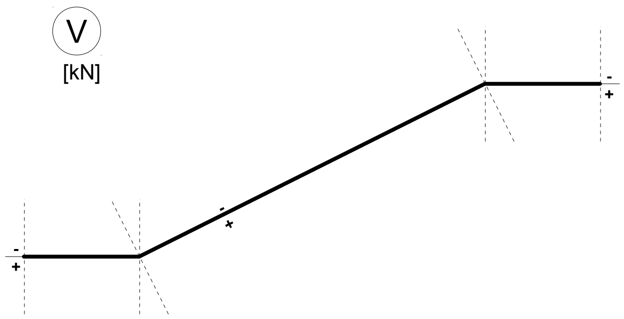
Decide upon the order of segments in the shear force diagram. Find characteristic values:

$V_1=$

$V_2=$

$V_3=$

$V_4=$



Decide upon the order of segments in the bending moment diagram. Find characteristic values:

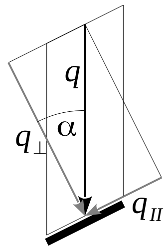
$M_1 =$

$M_2 =$

$M_3 =$

$M_4 =$

In order to find the depth of the parabola, the distributed load component perpendicular to the inclined axis must be known:



$q_\perp =$

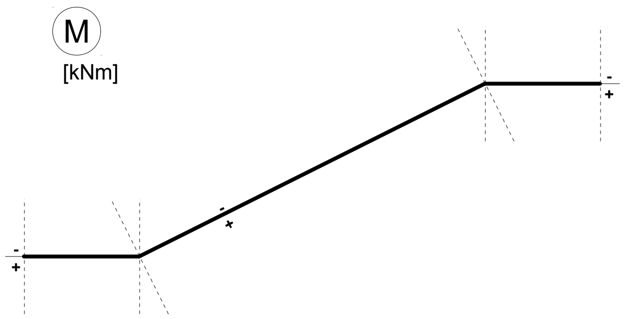
which yields the depth as:

$$\frac{q_\perp \cdot l^2}{8} =$$

Where is a maximum of bending moment? (The answer should be given based on the shear force diagram.)

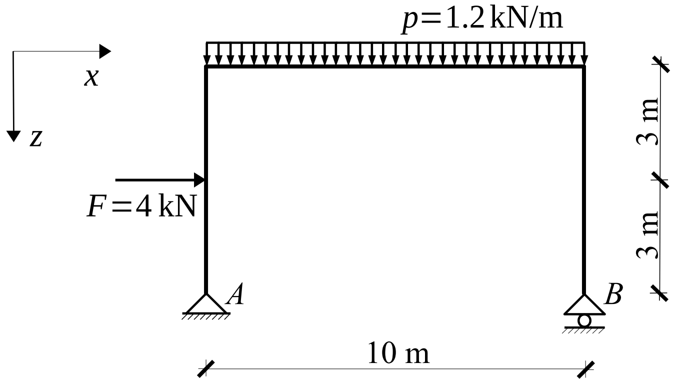
Find its value:

$M_{max} =$

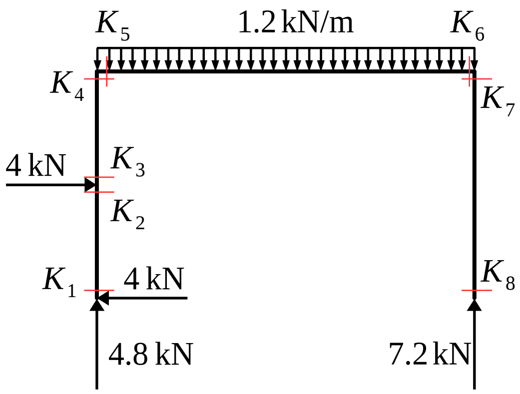


Internal force diagrams of oblique and bifurcating frames I.

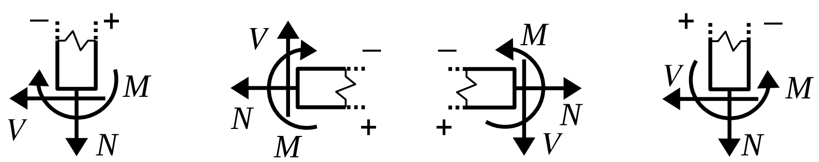
Example 1
Draw internal force diagrams of the structure shown.



Solution

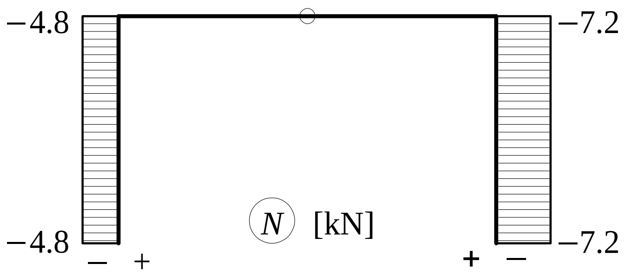


The solution starts by finding all reaction components; they are sketched in the figure to the left (the method of their calculation has already been demonstrated in Example 1 of Lecture 11). All cross sections where internal forces should be calculated are marked with numbers in the same figure. Internal forces must always be given at supports, at points of application of concentrated loads and at points where the frame axis changes its direction. Cross sections in all previous cases are set at an infinitely small distance (practically at 0 m) from reactions, concentrated active loads or corner points of the frame. Positive side of the frame is set to the bottom of horizontal segments; thus, for convenience, the inner side is considered positive in both vertical segments. Cross sections K_1-K_5 and K_6-K_8 are calculated from the side of support A and B, respectively. Sign conventions for the left hand side vertical segment (calculation from the bottom side), for the horizontal segment (from left and from right), as well as for the right hand side vertical segment (from the bottom) are as follows:



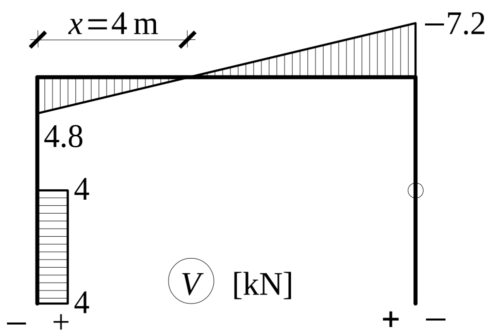
Let the normal force diagram be considered first. The frame axis is vertical at both supports, therefore vertical and horizontal reaction components are aligned with normal and shear force components, respectively. Any cross sections of the left hand side column are calculated from the bottom side: all forces found below the cross section in case are replaced by an equivalent

force-couple system at the centroid of the cross section. The normal force is constant all along the column, since force F is perpendicular to the axis of the column. The reaction component A_z is directed upwards and results therefore in compression (negative normal force):
$N_1 = N_2 = N_3 = N_4 = -4.8\,\text{kN}$
The normal force is still constant all along the horizontal segment on the top because it is not influenced by the perpendicular distributed load. The normal force in cross section K_5 can be obtained from the left (from A_x and F) as follows:



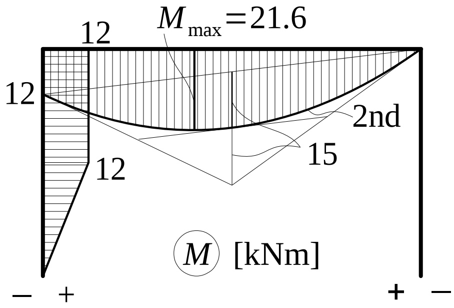
$N_5 = 4 - 4 = 0\,\text{kN}$
It can be verified that the normal force at K_6 is also obtained as $N_6 = 0\,\text{kN}$, since there is no horizontal force component acting on the left hand side of the cross section.
On the right hand side column, the normal force is constant: looking at the bottom part, reaction B causes compression:
$N_7 = N_8 = -7.2\,\text{kN}$

The shear force diagram is divided into two constant parts on the left column, connected to each other by a jump. In the region below the active load, there is only the horizontal component of A that causes shear (which is positive by the sign convention):
$V_1 = V_2 = 4\,\text{kN}$
In calculating shear in cross sections above the concentrated load, force F together with A_x should also be accounted for:
$V_3 = V_4 = 4 - 4 = 0\,\text{kN}$
Due to the constant intensity of load along the horizontal part, the shear force diagram is linear there. Considering forces to the left of K_5, the shear force component is positive since A_z is directed upwards:
$V_5 = 4.8\,\text{kN}$
Considering forces to the right of K_6, upwards reaction B yields a negative shear:
$V_6 = -7.2\,\text{kN}$
After the linear segment between K_5 and K_6 having been drawn, the cross section of zero shear (where the moment has a local extremum) is found. The shear force obtained from the left side is $V(x) = 4.8 - 1.2 \cdot x = 0$, whence $x = 4.8/1.2 = 4\,\text{m}$



On the right hand side column there is no shear at all: the only force B at the bottom part has a component just parallel to the current frame axis; thus,
$V_6 = V_7 = 0\,\text{kN}$.

The moment diagram on the left hand side column consists of two linear segments. The value of moment at the pin-jointed support is zero:



$M_1 = 0$ kNm

In the cross section of concentrated load, the moment diagram has a kink. In both cross sections immediately above and below F, a moment causing tension on the inner side arises from reaction A_x (directed to the left). This bending moment is therefore positive (in the calculation of moment in cross section K_3 from below, the moment arm of force F is zero):

$M_2 = M_3 = 4\cdot 3 = 12\,\text{kNm}$.

The bending moment does not change between K_3 and K_4:

$M_4 = 12\,\text{kNm}$,

since the value of shear and thus the slope of moment function is zero here. On the horizontal part; however, the bending moment function is quadratic.

Magnitudes of moments M_4 and M_5 must be equal due to the equilibrium of the top left corner of the frame; if M_4 represents a clockwise rotation, M_5 should rotate against the clock, that is, it causes tension at the bottom (positive) side.

The bending moment is zero at the rightmost cross section of the horizontal part, because the moment arm of the only force B on the bottom right part equals zero.

The depth of the parabolic segment is $\dfrac{p\cdot l^2}{8} = \dfrac{1.2\cdot 10^2}{8} = 15\,\text{kNm}$.

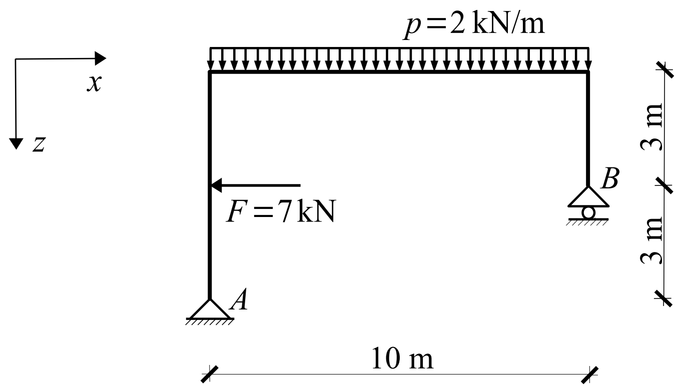
All cross sections of the right hand side column undergo zero bending (M_7=M_8=0), since the moment arm of force B (standing alone below such cross sections) is zero.

Tangents to the parabolic segment at both its endpoints and at the midpoint can be obtained as usual. Local maximum of the bending moment is obtained at zero shear; calculated from the right hand side as follows:

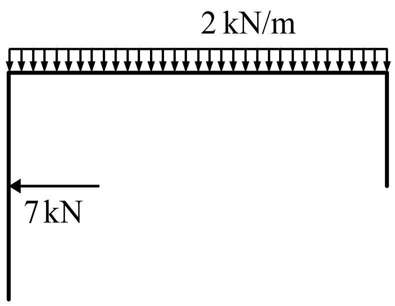
$$M_{max} = 7.2\cdot 6 - \frac{1.2\cdot 6^2}{2} = 21.6\,\text{kNm} .$$

Exercise 1

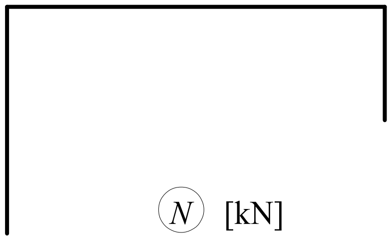
Draw internal force diagrams of the structure shown.

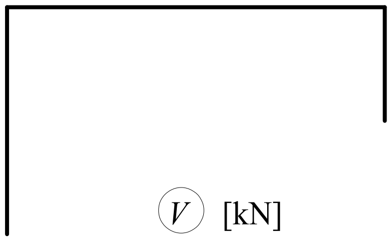


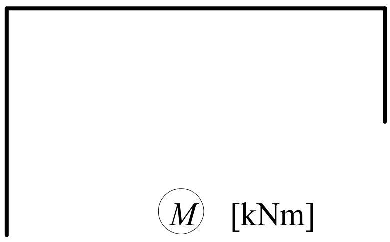
Solution
Reactions:



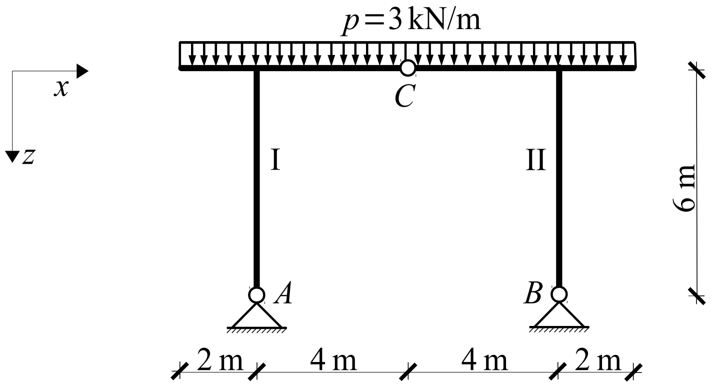
Internal force diagrams



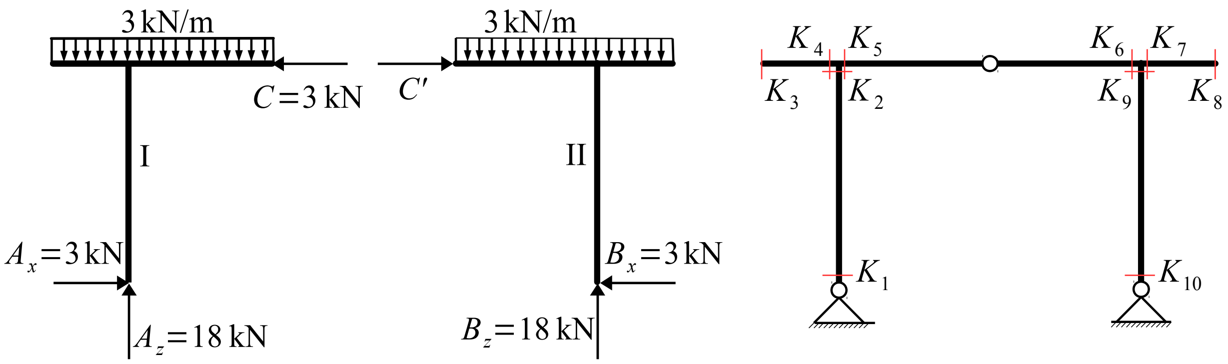




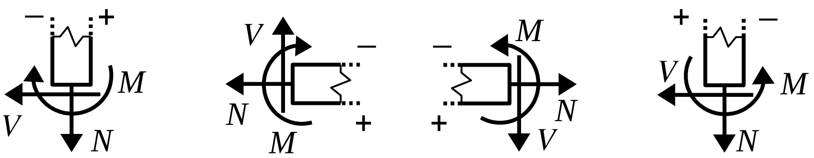
Example 2
Draw internal force diagrams of the structure shown.



Solution
Reactions are determined first: external and internal reaction components are shown in the next figure (the method for calculation of reactions of three hinged structures have already been shown in details at compound structures). All cross sections where internal forces should be calculated are marked with numbers in the figure at the right hand side.



Positive side of horizontal segments of the frame is set to the bottom; thus, for convenience, the inner side is considered positive in both vertical segments. Sign conventions for the left hand side vertical segment (calculation from the bottom side), for the horizontal segment (from left and from right), as well as for the right hand side vertical segment (from the bottom) are as follows:



Let the normal force diagram be considered first. The frame axis is vertical at both supports, therefore vertical and horizontal reaction components are aligned with normal and shear force components, respectively. Any cross sections of the left hand side column are calculated from the bottom side: all forces found below the cross section in case are replaced by an equivalent force-couple system at the centroid of the cross section. The normal force is constant all along the column, since only force A_z is not perpendicular to the axis of the column. Such an upwards force results in compression (negative normal force):

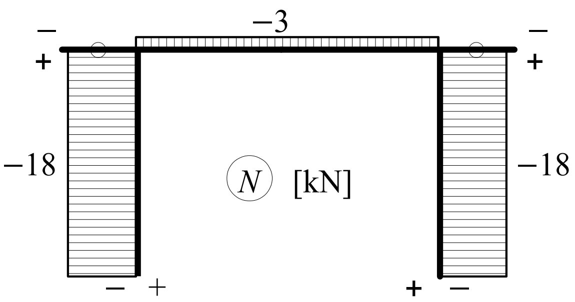
$N_1 = N_2 = -18\,\text{kN}$

The normal force, calculated from left and right on the left and right hand side overhangs, respectively, is constant zero, since distributed load components perpendicular to the current frame axis do not result in normal force:

$N_3 = N_4 = N_7 = N_8 = 0\,\text{kN}$

The normal force between two bifurcations (corners) of the frame axis is constant because the perpendicular distributed load makes still no change in the normal component. Its value is obtained in cross section K_5 from left, considering the component A_x causing compression:

$N_5 = -3\,\text{kN}$



Of course, the same value is obtained from right, based on the force component B_x, e.g., in cross section K_6:

$N_6 = -3\,\text{kN}$.

The normal force is constant along the right hand side column, it can be calculated easily from the bottom part, based on the upwards component B_z causing compression again:

$N_9 = N_{10} = -18\,\text{kN}$.

The shear force along the left hand column is constant: there is only the horizontal component of reaction A that generates shear if calculated from below. Its inwards sense means that the sher is negative:

$V_1 = V_2 = -3\,\text{kN}$.

The diagram is linear on the left overhang. There is no concentrated force at the extremity of that overhang; thus, a calculation from the left yields:

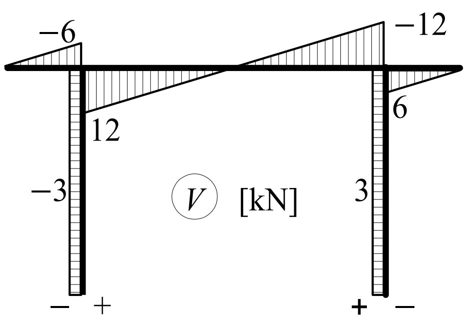
$V_3 = 0\,\text{kN}$.

Calculating at cross section K_4 from the left, the resultant of vertical distributed load on the overhang should be accounted for, which results in a negative shear:

$V_4 = -3 \cdot 2 = -6\,\text{kN}$.

The shear force diagram is linear along the middle horizontal segment due to the constant intensity of distributed load. Looking at all forces to the left of the cross section K_5, component A_z upwards and the resultant of load on the left overhang downwards has a positive and negative contribution to the shear, respectively:

$V_5 = 18 - 3 \cdot 2 = 12\,\text{kN}$.



Similarly, the shear force at cross section K_6 is calculated from the right by accounting for a negative shear component from B upwards and a positive one from the vertical distributed load on the overhang downwards:

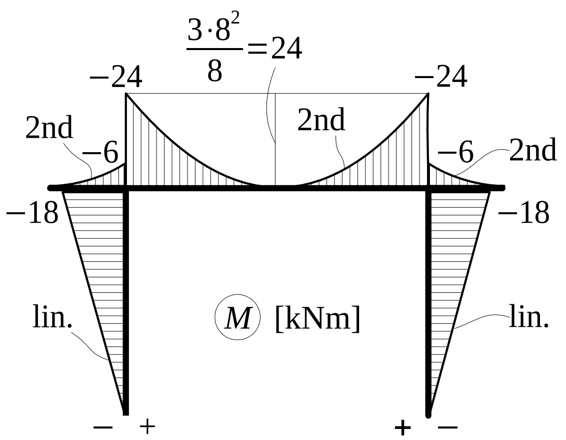
$V_6 = -18 + 3 \cdot 2 = -12\,\text{kN}$.

The zero value between cross sections K_5 and K_6 is exactly at the hinge due to symmetry.

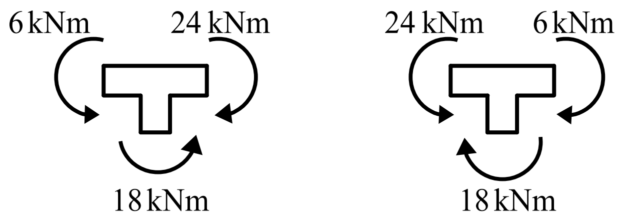
The shear force diagram is linear between K_7 and K_8. At

the end of the right overhang, the shear force is obtained as zero from the right, since there is no concentrated load at the end of overhang. The shear in K_7 is determined from the right merely by the resultant of the distributed load on the overhang which is equivalent to a positive shear force $V_7 = 3 \cdot 2 = 6\,\text{kN}$.
Important: slopes of three linear segments of the shear force diagram over horizontal parts of the frame are equal, since the intensity of distributed load is the same for all of them.

The bending moment diagram is linear along all vertical parts; moment values at pin-joints are zero. Calculating from the part below cross section K_2, the reaction component A_x (inwards) results in tension on the left hand side of the column and is therefore negative:
$M_2 = -3 \cdot 6 = -18\,\text{kNm}$

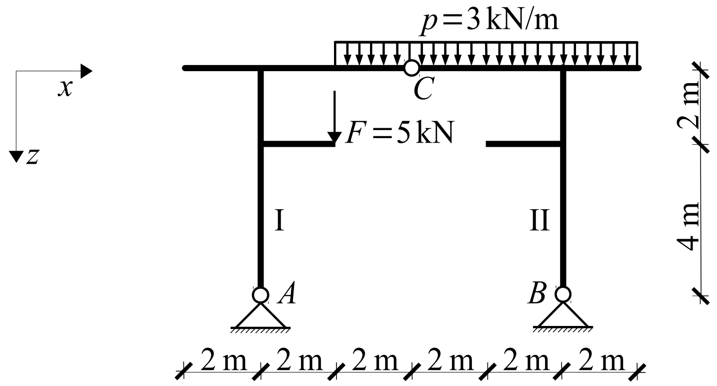


Similarly, if the moment in K_9 is calculated from below, the reaction component B_x (inwards) results in tension on the right hand side of the column and is therefore negative:
$M_9 = -3 \cdot 6 = -18\,\text{kNm}$.
The bending moment diagram on all horizontal parts of the frame is quadratic and is convex from below due to the downwards orientation of the distributed load. The moment is zero at the end of both overhangs.
The distributed load results in tension at the top of both overhangs; thus, the moment is negative:
$M_4 = M_7 = -3 \cdot 2 \cdot 1 = -6\,\text{kNm}$.
Because of the zero shear at each free end of overhangs, the slope of the moment diagram is also zero there. The quadratic curve can be drawn based on its initial and final value as well as an initial slope (these arguments also apply to a simple cantilever beam).
Consider the equilibrium of member I only, and calculate moment in K_5 from the right: it is only influenced by the distributed load between K_5 and C, since internal reaction at C is found to be horizontal and has therefore a zero moment arm. Similarly, forces on the left hand side of member II lead to the same moment in K_6. Both moments are drawn to the negative side because of the tension arising on the top: $M_5 = M_6 = -3 \cdot 4 \cdot 2 = -24\,\text{kNm}$.
The depth of the parabolic segment between corners is $p \cdot l^2/8 = 3 \cdot 8^2/8 = 24\,\text{kNm}$.
The result can be verified at the internal hinge where the moment should be zero (it is also seen from the FBD, since moment arm of either C or C' is zero about the centroid of adjacent cross sections). The tangent to the moment diagram at C is now horizontal due to zero shear.

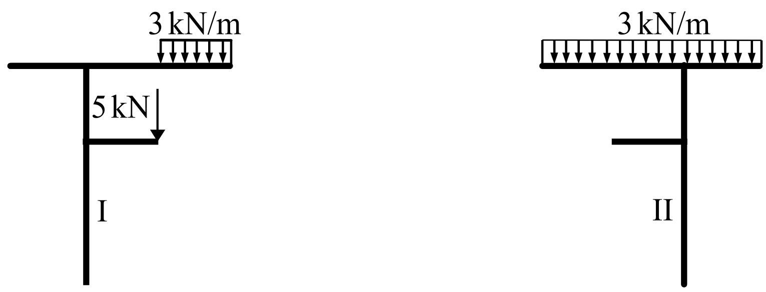


Finally, nodal moment equilibrium of each bifurcation should be checked. Accounting for tensile sides of all cross sections, arrows of bending moments are drawn to the nodes and their sums are calculated.
left node $\sum M_i : 6 - 24 + 18 = 0$
right node $\sum M_i : 24 - 6 - 18 = 0$

Exercise 2
Draw internal force diagrams of the structure shown.



Solution
Isolation:



Equilibrium statements:

e u new u

I:

II:

Str:
Analytic solution:

$\sum$

$\sum$

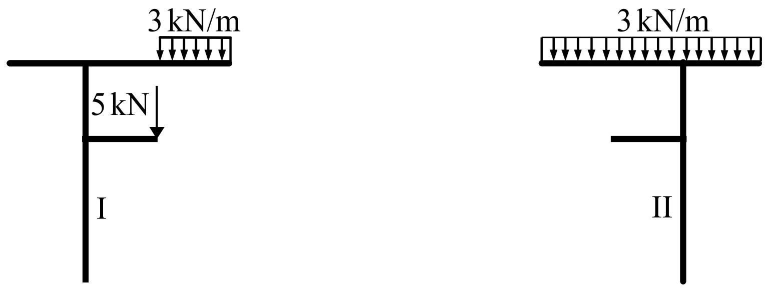
$\sum$

$\sum$

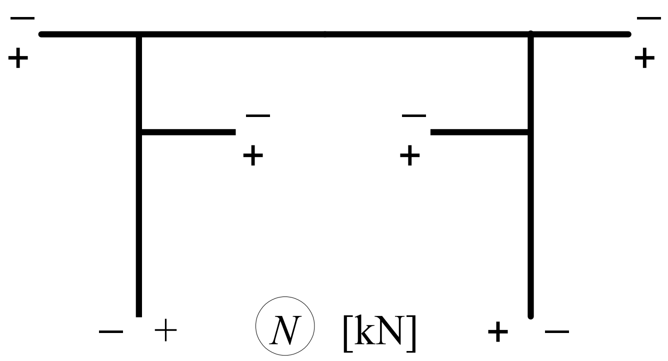
$\sum$

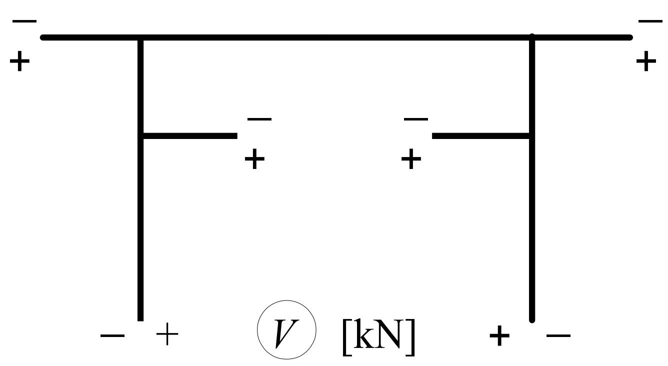
$\sum$

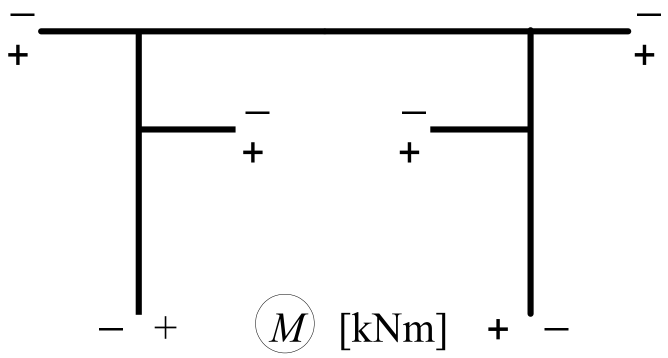
Final sketch of reactions:



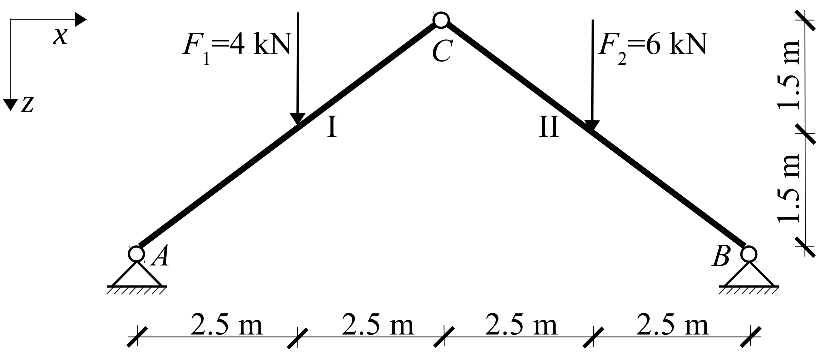
Internal force diagrams:



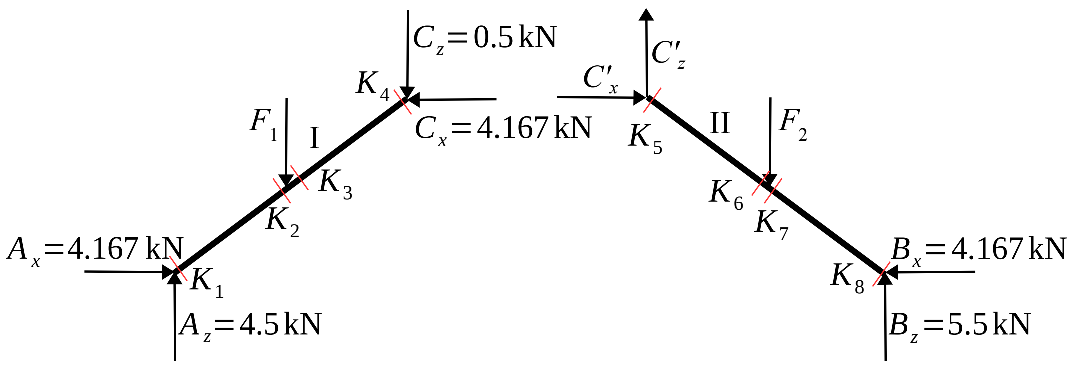




Example 3
Draw internal force diagrams of the structure shown.



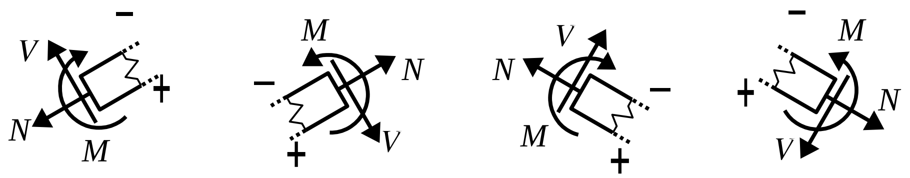
Solution
Reactions are determined first: external and internal reaction components are shown in the next figure (the method for calculation of reactions of similar structures have already been dealt with in Example 2 of Lecture 13). All cross sections where internal forces should be calculated are marked with numbers in the figure.



The inclination of both oblique axes to the horizontal is

$\alpha = \arctan\frac{3}{5} = 30.96°$.

Positive side of the frame is set to the bottom. Sign conventions for the left hand side (for calculation from either left or right), as well as for the right hand side (from left and from right) are as follows:



Let the normal force diagram be considered first: it consists of two constant segments on either side; both jumps are due to the concentrated load not perpendicular to the frame axis. Looking left from all cross sections between K_1 and K_2, reactions A_z and A_x result both in negative components of normal force:

$N_1 = N_2 = -4.5 \cdot \sin\alpha - 4.167 \cdot \cos\alpha = -5.888\,\text{kN}$.

Likewise, in any cross section between K_3 and K_4 , normal forces are obtained negative from left due to either C_z or C_x :

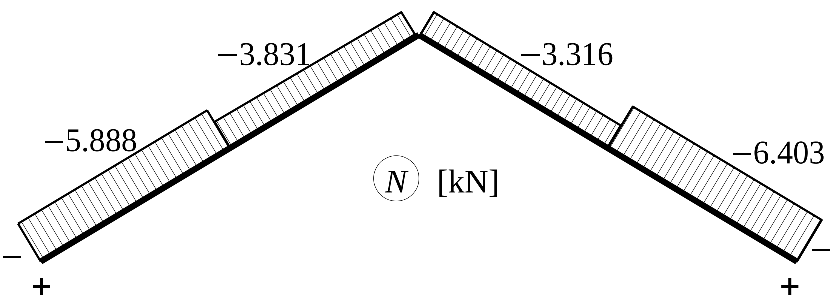
$N_3 = N_4 = -0.5 \cdot \sin\alpha - 4.167 \cdot \cos\alpha = -3.831\,\text{kN}$.

Between K_5 and K_6, the normal force obtained from the left hand side is positive from component C'_z but is negative from C'_x :

$N_5 = N_6 = 0.5 \cdot \sin\alpha - 4.167 \cdot \cos\alpha = -3.316\,\text{kN}$.

In cross sections between K_7 and K_8, both B_z and B_x causes a negative normal force (forces are now taken from the right hand side):

$N_7 = N_8 = -5.5 \cdot \sin\alpha - 4.167 \cdot \cos\alpha = -6.403\,\text{kN}$.



The shear force diagram still consists of two constant segments on both sides; two jumps are due to the concentrated forces not parallel to the frame axis. Considering all forces to the left of cross sections between K_1 and K_2, reactions A_z and A_x generate a positive and a negative shear, respectively:

$V_1 = V_2 = 4.5 \cdot \cos\alpha - 4.167 \cdot \sin\alpha = 1.715\,\text{kN}$.

Between K_3 and K_4, looking at forces on member I to the right of all cross sections, C_z results in a positive, C_x in a negative component of shear:

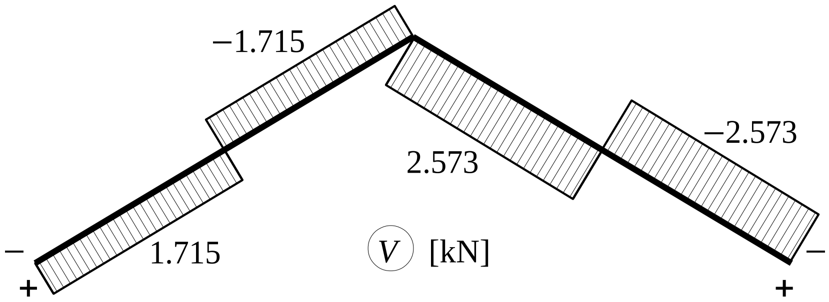
$V_3 = V_4 = 0.5 \cdot \cos\alpha - 4.167 \cdot \sin\alpha = -1.715\,\text{kN}$.

Looking left on member II, cross sections between K_5 and K_6 undergo positive shear due to both components C'_z and C'_x :

$V_5 = V_6 = 0.5 \cdot \cos\alpha + 4.167 \cdot \sin\alpha = 2.573\,\text{kN}$.

Forces to the right from all cross sections between K_7 and K_8 are subjected to a negative and a positive shear force component from reactions B_z and B_x, respectively:

$V_7 = V_8 = -5.5 \cdot \cos\alpha + 4.167 \cdot \sin\alpha = -2.573\,\text{kN}$.

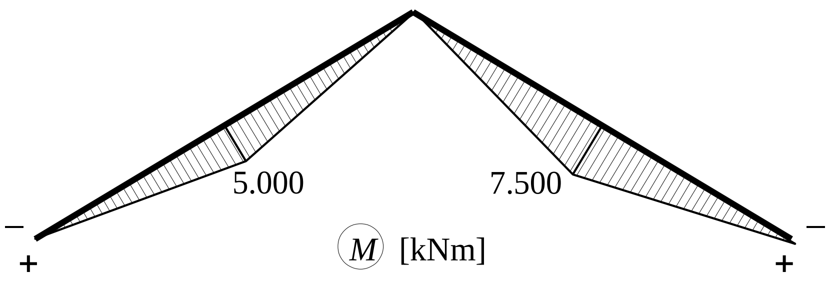


The bending moment diagram consists of two linear segments on both members; kinks are due to the concentrated loads not parallel to the frame axis. Moments in all external and internal hinges are strictly zero. A common value of moment for cross sections K_2 and K_3 is obtained from the left hand side as the sum of a positive moment of A_z and a negative moment of A_x :

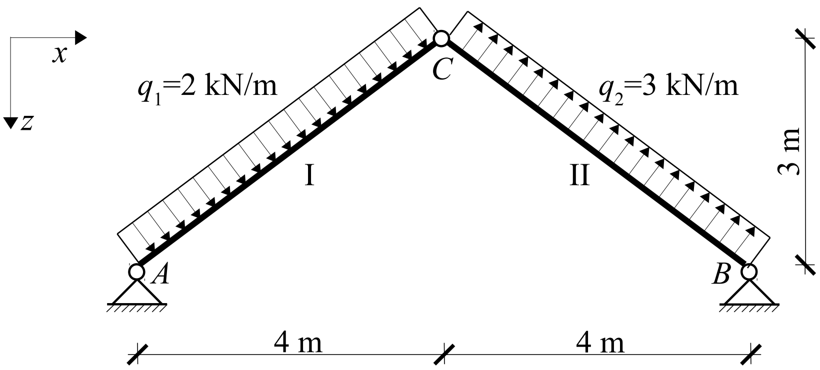
$M_2 = M_3 = 4.5 \cdot 2.5 - 4.167 \cdot 1.5 = 5.000\,\text{kN}$.

Similarly but looking at forces on the right hand side, equal moments of K_6 and K_7 can be obtained by adding the negative moment of B_z to that positive of B_x :

$M_6 = M_7 = 5.5 \cdot 2.5 - 4.167 \cdot 1.5 = 7.500\,\text{kN}$.



Exercise 3
Draw internal force diagrams of the structure shown.



Solution
Isolation:

Equilibrium statements:

e u new u

I:

II:

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

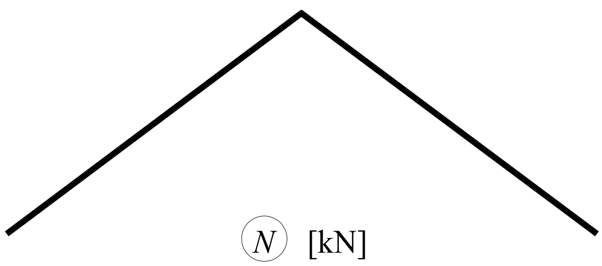
$\sum$

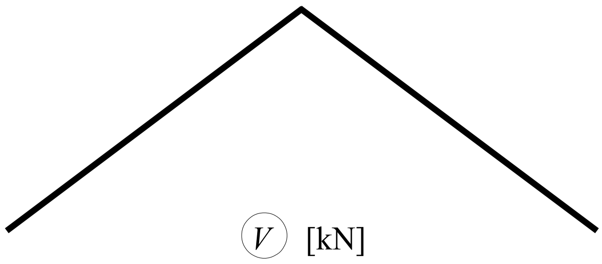
$\sum$

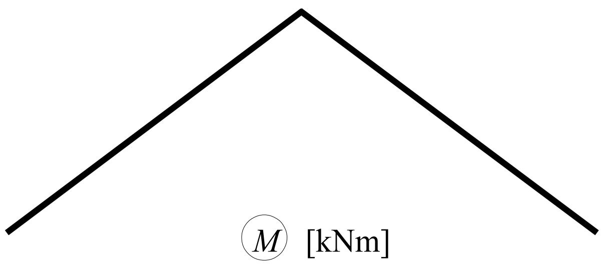
$\sum$

Final sketch of reactions:

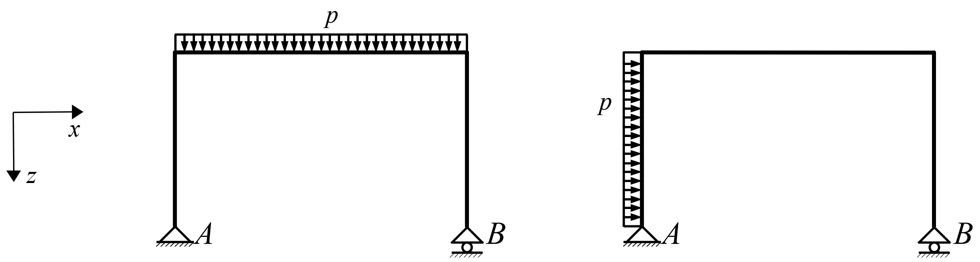
Internal force diagrams:



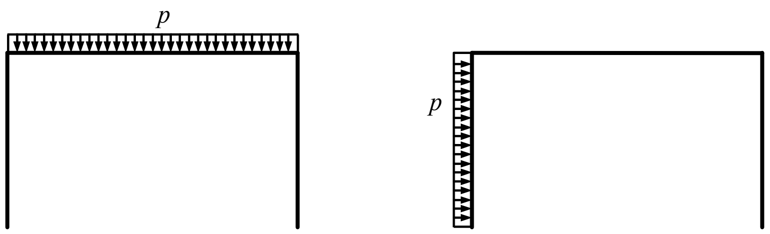

V [kN]


M [kNm]

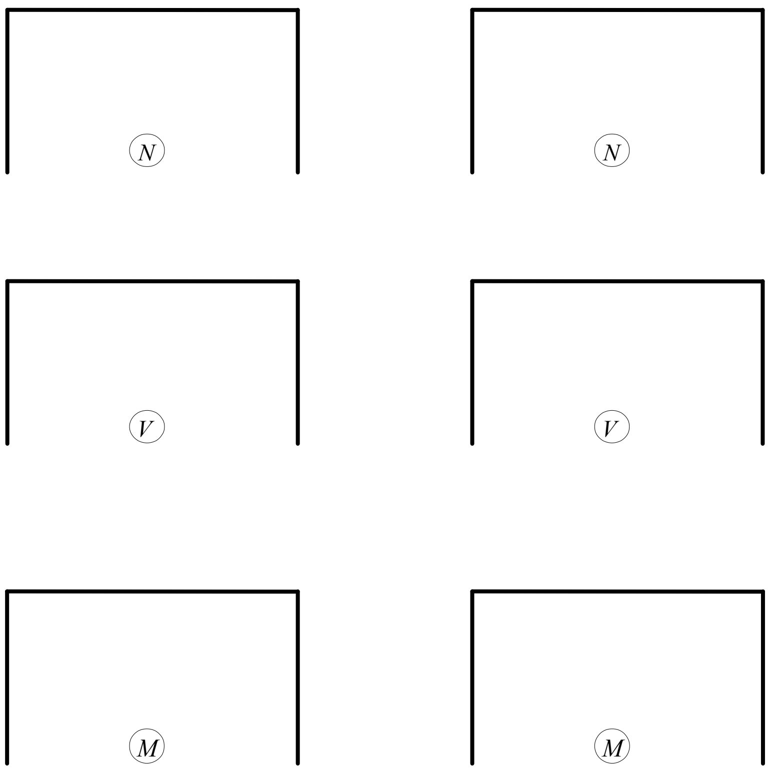
Exercise 4 (if needed)
Determine the correct shapes of internal force diagrams of the structures shown.



Solution
Draw lines of action and sense of reaction forces.

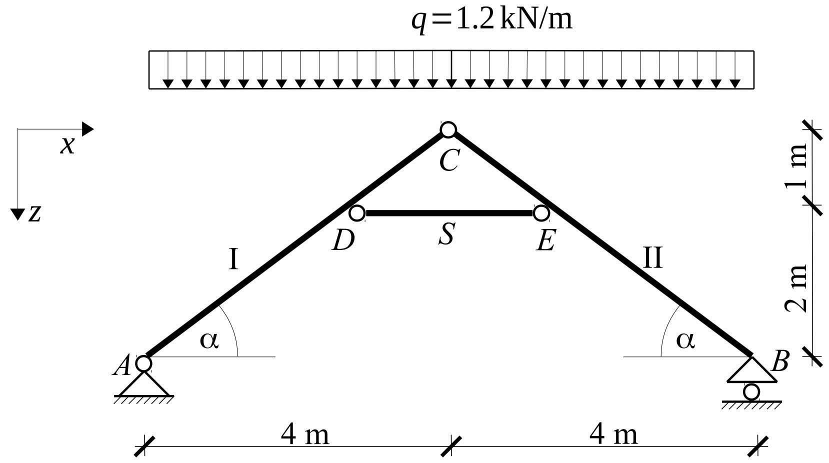


Internal force diagrams:



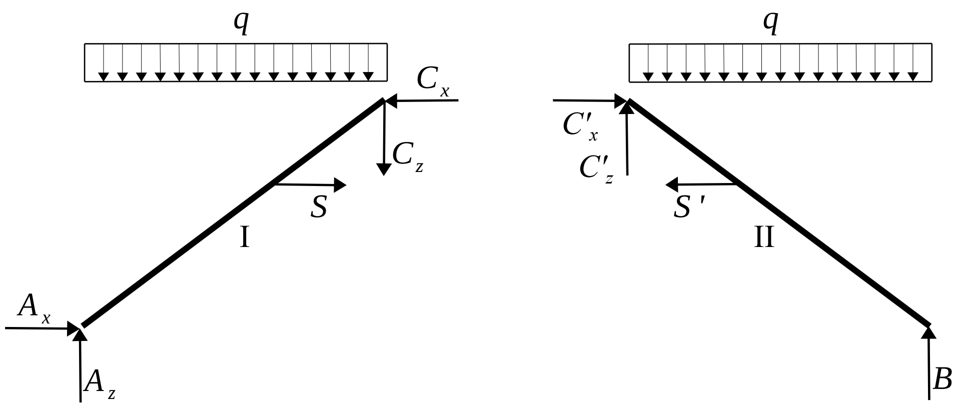
Internal force diagrams of oblique and bifurcating frames II.

Example 1
Draw internal force diagrams of the structure shown.



Solution

The solution starts by finding all internal and external reactions.
FBD's are as follows:



Equilibrium statements:

	e	u	new u
I: $((q_{left}), A, S, C) \doteq 0$	3	5	5
II: $((q_{right}), C', S', B) \doteq 0$	3	4	1
Str: $((q), A, B) \doteq 0$	(3)	3	

Calculation of reactions:

Str $\sum M_i^{(A)}: -1.2 \cdot 8 \cdot 4 + B \cdot 8 = 0 \rightarrow B = 4.8\,\text{kN}(\uparrow)$

Str $\sum M_i^{(B)}: 1.2 \cdot 8 \cdot 4 - A_z \cdot 8 = 0 \rightarrow A_z = 4.8\,\text{kN}(\uparrow)$

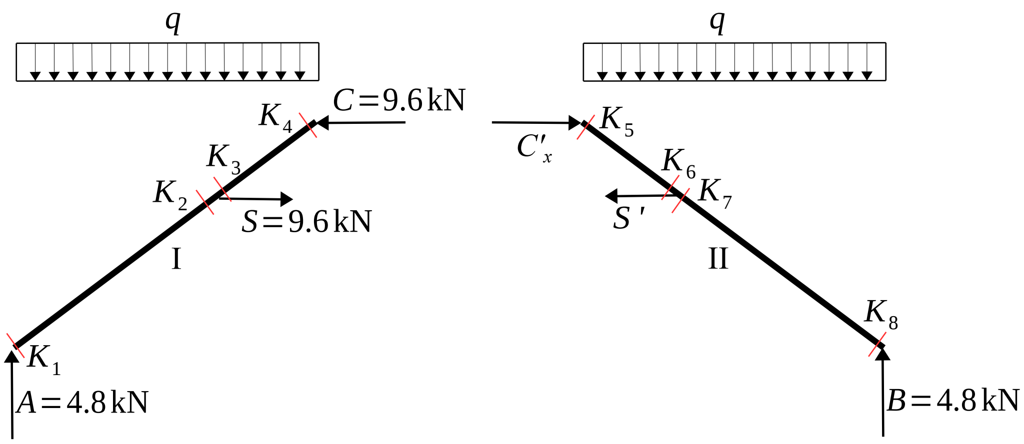
Str $\sum F_{ix}: A_x = 0$

$$\text{II} \sum M_i^{(C)}: -1.2 \cdot 4 \cdot 2 + 4.8 \cdot 4 - S' \cdot 1 = 0 \quad \rightarrow \quad S' = 9.6\,\text{kN}\,(t)$$

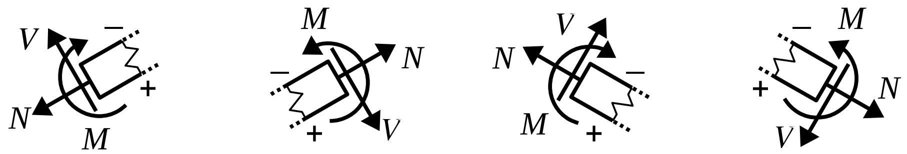
$$\text{II} \sum F_{ix}: C'_x - 9.6 = 0 \quad \rightarrow \quad C'_x = 9.6\,\text{kN}\,(\rightarrow)$$

$$\text{II} \sum F_{iz}: 1.2 \cdot 4 - C'_z - 4.8 = 0 \quad \rightarrow \quad C'_z = 0\,\text{kN}$$

The following figure shows all cross sections where internal force values should be calculated in order to draw their diagrams. All marked cross sections are infinitely close to points where internal or external effects take place, as well as to points where the frame axis change its direction.



Positive side is set to the bottom of each oblique member. Calculations from the left and right hand side on either member I or II are performed on the basis of sign conventions shown below:

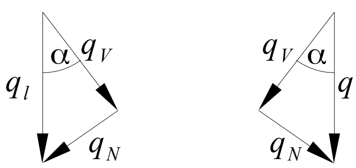


For the sake of simplicity, internal force diagrams for the horizontal member with two pinned connections are not drawn. In the lack of any direct external load, such a straight member undergoes a uniform normal force *N* (being equal to the member force *S*) and has zero shear or bending in each of its cross sections.

Internal forces of any cross section are calculated from equilibrium conditions of the member (I or II) it belongs to. Let normal forces be considered first. Instead of the angle itself, let the cosine and sine of angle α of inclination of oblique members to the horizontal be determined:

$$\cos\alpha = \frac{4}{5} = 0.8 \qquad \sin\alpha = \frac{3}{5} = 0.6\,.$$

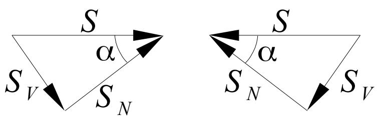
The *normal force diagram* consists of two linear segments on both members. The slope of each diagram is proportional to the axial component of the intensity of the distributed load. Meanings of load components parallel and perpendicular to the current axis (q_N and q_V, causing the change in *N* and *V* diagrams, respectively) are shown in the figure on the left for both members. Intensity q_l is understood as that of a load distributed over the axis of an inclined member.



Since $q_l \cdot 5 = q \cdot 4$ both equals the resultant of distributed loads over a member,
$q_l = q \cdot \frac{4}{5} = 0.96\,\text{kN/m}$.

Components directed along *N* and *V* can be expressed by trigonometry as follows:
$q_N = q_l \cdot \sin\alpha = 0.576\,\text{kN/m}$
$q_V = q_l \cdot \cos\alpha = 0.768\,\text{kN/m}$

As a consequence of member force *S*, both sides of the diagram contain a jump in the normal force diagram; its magnitude equals the component of *S* along the axis of the frame:



$S_N = S \cdot \cos\alpha = 9.6 \cdot 0.8 = 7.68\,\text{kN}$.

(The component perpendicular to the axis,
$S_V = S \cdot \sin\alpha = 9.6 \cdot 0.6 = 5.76\,\text{kN}$
modifies the value of shear.)

As will be shown, the normal force diagram can be prepared without the determination of the magnitude of jump, but it is worth checking as soon as the solution is complete.

Internal forces in cross section K_1 are calculated for convenience from the left. Axial component of reaction *A* upwards results in compression, making the normal force to be negative:
$N_1 = -4.8 \cdot \sin\alpha = -2.88\,\text{kN}$.

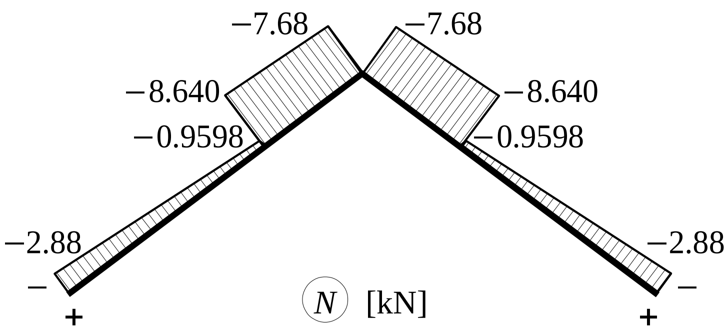
Cross section K_2 is also calculated from the left. The horizontal projection of the strucural part on the left of K_2 is 2.667 m by similar triangles. Axial component of reaction *A* upwards and distributed load downwards result in compression and tension, respectively:
$N_2 = (-4.8 + 1.2 \cdot 2.667) \cdot \sin\alpha = -0.9598\,\text{kN}$.

Internal forces in K_3 are better to get from the right hand side. The horizontal projection of the strucural part on the right of K_3 is 1.333 m by similar triangles. Both axial component of internal reaction *C* leftwards and distributed load downwards (now on the right of the cross section) result in compression:
$N_3 = -9.6 \cdot \cos\alpha - 1.2 \cdot 1.333 \cdot \sin\alpha = -8.640\,\text{kN}$.

K_4 is calculated from the right; reaction *C* directed to the left causes compression:
$N_4 = -9.6 \cdot \cos\alpha = -7.68\,\text{kN}$.

K_5 is calculated from the left; reaction *C′* directed to the right causes compression:
$N_5 = -9.6 \cdot \cos\alpha = -7.68\,\text{kN}$.

K_6 is calculated from the left; reaction *C′* directed to the right and distributed load on the left of K_6 causes compression as well:
$N_6 = -9.6 \cdot \cos\alpha - 1.2 \cdot 1.333 \cdot \sin\alpha = -8.640\,\text{kN}$



Calculation in K_7 is done from the right. Reaction *B* upwards and distributed load down wards result in compression and tension, respectively:

$N_7 = (-4.8 + 1.2 \cdot 2.667) \cdot \sin\alpha = -0.9598\ \text{kN}$.

Finally, the normal force in cross section K_8 is obtained from the right; reaction *B* upwards gives rise to compression:

$N_8 = -4.8 \cdot \sin\alpha = -2.88\ \text{kN}$.

The magnitude of jump betweeen linear segments must coincide with the component of member force *S* parallel to the axis of the frame; it provides a check after the diagram having been drawn:

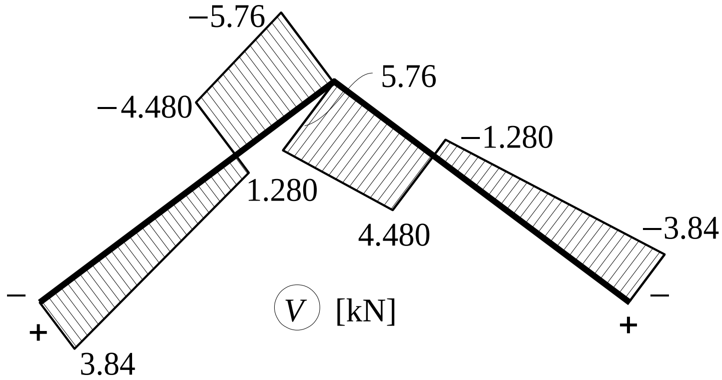
$-0.9598 - (-8.640) = 7.680\ \text{kN} = S_N$.

The *shear force diagram* consists of two linear segments on both members. The slope of each diagram is proportional to the component q_V of the intensity of distributed load in a perpendicular direction to the axis. Due to *S* again, the diagram contains a jump on both members of a magnitude equal to the component of *S* perpendicular to the axis. Calculation of cross section K_1 is done from the left. According to the sign convention taken, transversal component (i.e., perpendicular to the axis) of reaction *A* upwards result in a positive shear:

$V_1 = 4.8 \cdot \cos\alpha = 3.84\ \text{kN}$.

K_2 is calculated from the left. Reaction *A* upwards and distributed load downwards result in a positive and negative shear, respectively:

$V_2 = (4.8 - 1.2 \cdot 2.667) \cdot \cos\alpha = 1.280\ \text{kN}$.



K_3 is calculated from the right. Reaction *C* leftwards and distributed load on the right of the cross section result in a negative and positive shear, respectively:

$V_3 = -9.6 \cdot \sin\alpha + 1.2 \cdot 1.333 \cdot \cos\alpha = -4.480\ \text{kN}$.

K_4 is calculated from the right. Reaction *C* leftwards causes a negative shear:

$V_4 = -9.6 \cdot \sin\alpha = -5.76\ \text{kN}$.

K_5 is calculated from the left. Reaction *C′* rightwards causes a positive shear:

$V_5 = 9.6 \cdot \sin\alpha = 5.76\ \text{kN}$.

K_6 is calculated from the left. Reaction *C′* rightwards and distributed load on the left of the cross section result in a positive and negative shear, respectively:

$V_6 = 9.6 \cdot \sin\alpha - 1.2 \cdot 1.333 \cdot \cos\alpha = 4.480\ \text{kN}$.

K_7 is calculated from the right. Reaction *B* upwards and distributed load on the left of the cross section result in a negative and positive shear, respectively:

$V_7=(-4.8+1.2\cdot 2.667)\cdot\cos\alpha=-1.280\,\text{kN}$.

Finally, K_8 is calculated from the right. Reaction *B* upwards causes a negative shear:

$V_8=-4.8\cdot\cos\alpha=-3.84\,\text{kN}$.

The magnitude of jump betweeen linear segments must coincide with the component of *S* perpendicular to the axis of the frame; it provides a check after the diagram having been drawn:

$1.280-(-4.480)=5.76\,\text{kN}=S_V$.

The diagram of *bending moment* consists of two parabolic segments on each structural member. Force *S* results in a kink in the diagram. Moment values at internal hinges and pin-jointed supports are zero:

$M_1=M_4=M_5=M_8=0\,\text{kNm}$.

If cross sections next to the point of application of *S* (K_2 and K_3) are calculated from the left, moment of reaction *A* and of the distributed load causes tension on the positive and the negative side of the frame, respectively (in K_3, the moment arm of *S* is 0 m):

$$M_2=M_3=4.8\cdot 2.667-\frac{1.2\cdot 2.667^2}{2}=8.534\,\text{kNm} .$$

Similarly, calculating K_6 and K_7 from the right, reaction *B* and the distributed load generates tension on the positive and negative side of the frame, respectively:

$$M_6=M_7=4.8\cdot 2.667-\frac{1.2\cdot 2.667^2}{2}=8.534\,\text{kNm} .$$

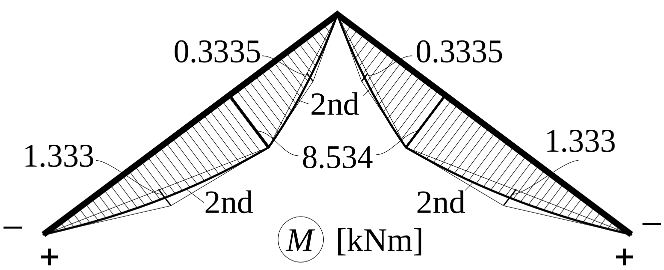
In order to find depths of parabolas, lengths of oblique segments above and below the short link are needed:

$$l_{12}=l_{78}=\frac{2}{3}\cdot 5=3.333\,\text{m},\quad l_{34}=l_{56}=\frac{1}{3}\cdot 5=1.667\,\text{m}$$

In calculating depths, only components of the distributed load in a direction perpendicular to the axis should be accounted for. The depths below and above the short link are as follows:

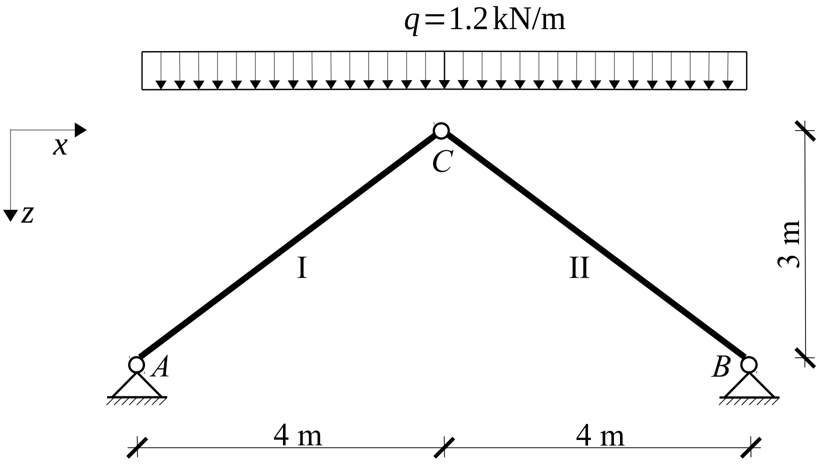
$$\frac{q_V\cdot l_{12}^2}{8}=\frac{0.768\cdot 3.333^2}{8}=1.066\,\text{kNm} ,$$

$$\frac{q_V\cdot l_{34}^2}{8}=\frac{0.768\cdot 1.667^2}{8}=0.2668\,\text{kNm} .$$

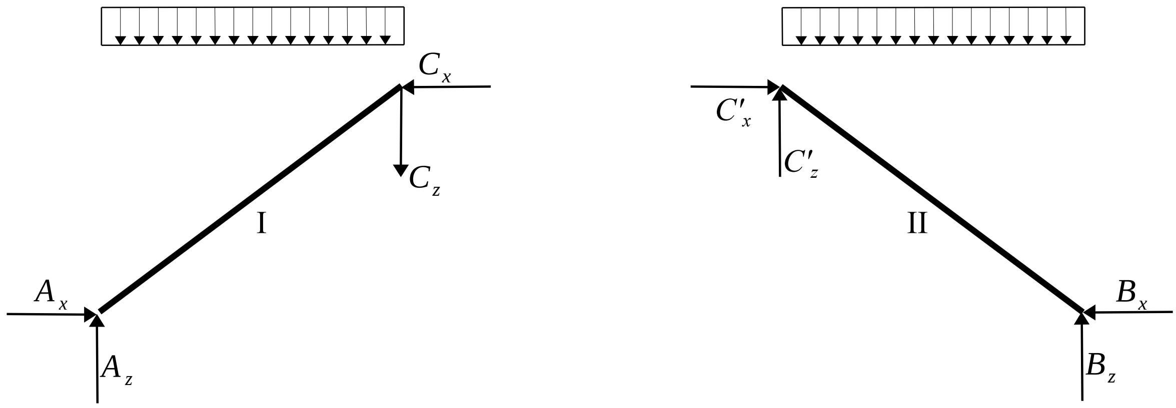


Note that symmetric loading (including active and passive forces exerted on the structure) on symmetric structures yield always symmetric normal and bending moment diagrams and an anti-symmetric shear force diagram. This last term refers to the property that ordinates in symmetric position are of equal magnitude but opposite sign.

Exercise 1
Draw internal force diagrams of the structure shown.



Solution
Isolation:



Equilibrium statements:

e u new u

I :

II :

Str:

Analytic solution:

$\sum$

$\sum$

$\sum$

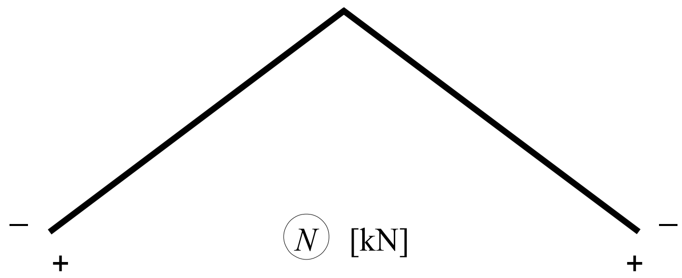
$\sum$

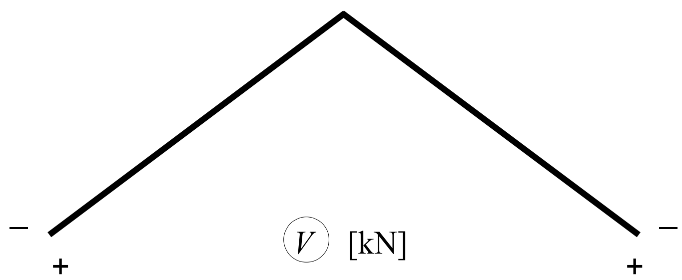
$\sum$

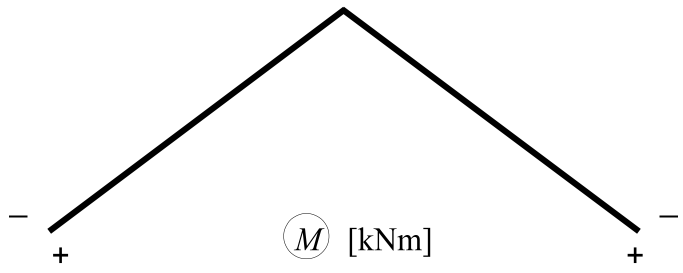
$\sum$

Final sketch of reactions:

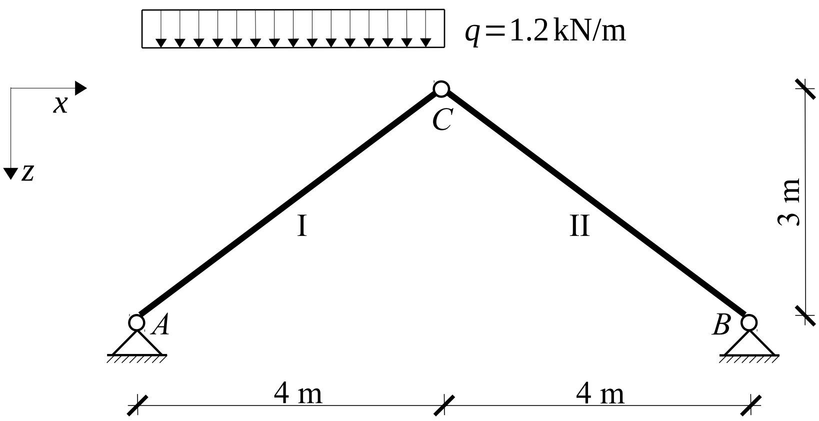
Internal force diagrams:





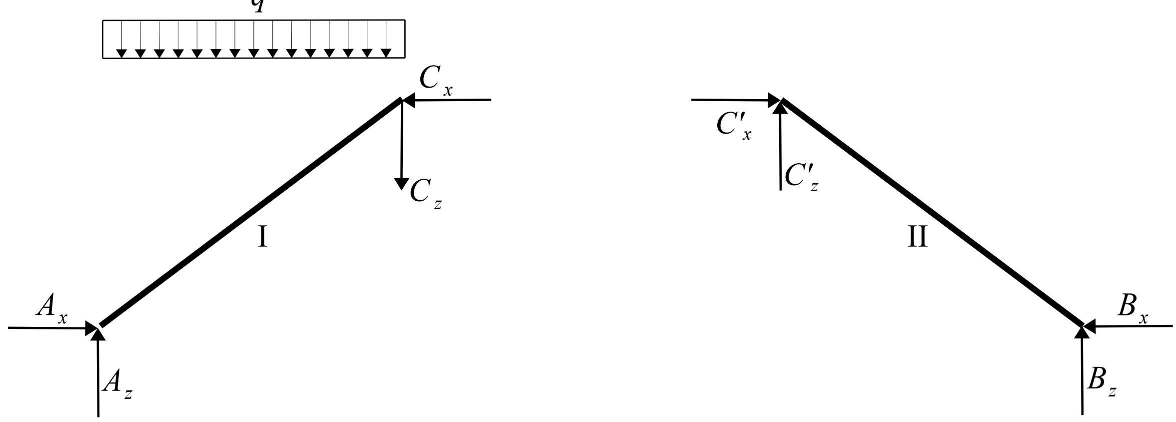


Example 2
Draw internal force diagrams of the structure shown.



Solution
Internal and external reactions are found first. The sight hand side of the structure is now isolated even if it is acted upon by two forces only. The reason for doing so is that the current problem is about a statical model typical for roof structures and, typically again, these structures undergo meteorological loads (wind and snow) responsible for bending in both structural members (see, e.g., Exercise 1 above or Exercise 3 of Lecture 25). The current scheme of loading corresponds to the partial snow load as it appears in design standards as well. The solution, of course, would be correct if the unloaded right hand side has not been isolated but treated as a short link (bar).

Isolation:



Equilibrium statements:

	e	u	new u
I $:((q), A, C) \doteq 0$	3	4	4
II $:(C', B) \doteq 0$	3	4	2
Str$:((q), A, B) \doteq 0$	(3)	3	

Calculation of reactions:

Str $\sum M_i^{(A)}: -1.2 \cdot 4 \cdot 2 + B \cdot 8 = 0 \;\rightarrow\; B_z = 1.2\,\text{kN}(\uparrow)$

Str $\sum M_i^{(B)}: 1.2 \cdot 4 \cdot 6 - A_z \cdot 8 = 0 \;\rightarrow\; A_z = 3.6\,\text{kN}(\uparrow)$

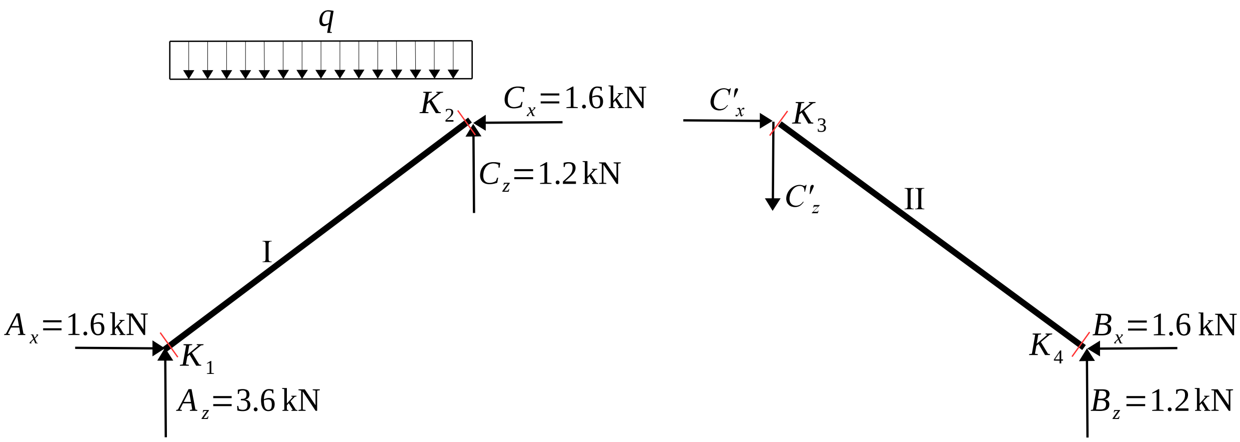
II $\sum M_i^{(C)}: 1.2 \cdot 4 - B_x \cdot 3 = 0 \;\rightarrow\; B_x = 1.6\,\text{kN}(\leftarrow)$

Str $\sum F_{ix}: A_x - 1.6 = 0 \;\rightarrow\; A_x = 1.6\,\text{kN}(\rightarrow)$

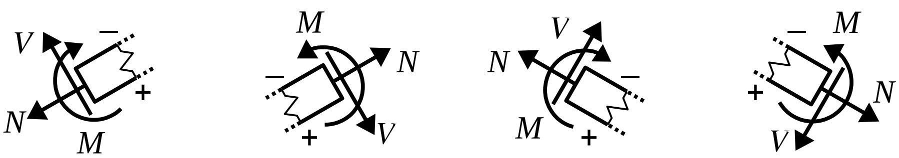
II $\sum F_{ix}: C'_x - 1.6 = 0 \;\rightarrow\; C'_x = 1.6\,\text{kN}(\rightarrow), \quad C_x = 1.6\,\text{kN}(\leftarrow)$

II $\sum F_{iz}: -C'_z - 1.2 = 0 \;\rightarrow\; C'_z = -1.2\,\text{kN}(\downarrow), \quad C_z = -1.2\,\text{kN}(\uparrow)$

In the next figure, all cross sections where internal forces should numerically be given in diagrams are marked.

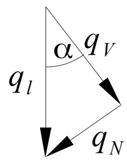


Positive side is set to the bottom of each oblique member. Calculations from the left and right hand side on either member I or II are performed on the basis of sign conventions shown below:



Internal forces of any cross section are calculated from equilibrium conditions of the member (I or II) it belongs to. Let normal forces be considered first. Instead of the angle itself, let the cosine and sine of angle α of inclination of oblique members to the horizontal be determined:

$\cos\alpha = \frac{4}{5} = 0.8 \quad \sin\alpha = \frac{3}{5} = 0.6$



The *normal force diagram* consists of a linear segment on the right and a constant segment on the left hand side member, respectively. both members. The slope of the ldiagram on the left is proportional to the axial component of the intensity of the distributed load. Meanings of load components parallel and perpendicular to the axis of the left member (q_N and q_V, causing the change in N and V diagrams, respectively) are shown in the figure on the left, the method of finding them has already been shown in Example 1.

Calculations are done from the left hand side in cross section K_1; both reaction components A_z (upwards) and A_x (rightwards) gives rise to a negative normal force:

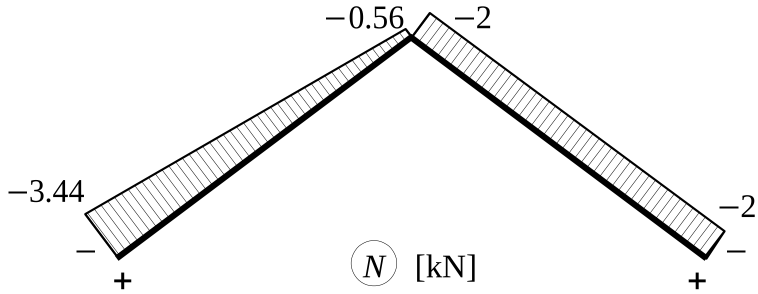
$N_1 = -3.6 \cdot \sin\alpha - 1.6 \cdot \cos\alpha = -3.44\,\mathrm{kN}$.

K_2 is calculated from the right; the upwards reaction C_z and leftwards reaction C_x together cause compression:

$N_2 = 1.2 \cdot \sin\alpha - 1{,}6 \cdot \cos\alpha = -0.56\,\mathrm{kN}$.

Calculation in K_3 and K_4 is done from the left. Both reactions C'_z downwards and C'_x rightwards result in compression:

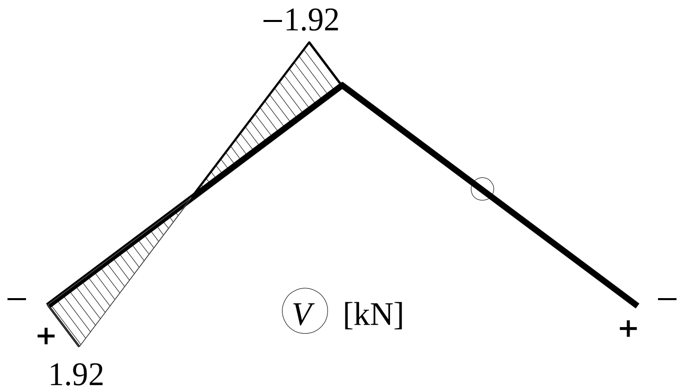
$N_3 = N_4 = -1.2 \cdot \sin\alpha - 1.6 \cdot \cos\alpha = -2\,\mathrm{kN}$.



The *shear force diagram* is linear on the left, whereas constant on the right hand side member.

K_1 is calculated from the left. Reaction A_z upwards and A_x rightwards result in a positive and negative shear, respectively:

$V_1 = 3.6 \cdot \cos\alpha - 1.6 \cdot \sin\alpha = 1.92\,\mathrm{kN}$.



K_2 is easier to be calculated from the right. Both reactions C_z upwards and C_x leftwards cause a negative shear:

$V_2 = -1.2 \cdot \cos\alpha - 1.6 \cdot \sin\alpha = -1.92\,\mathrm{kN}$.

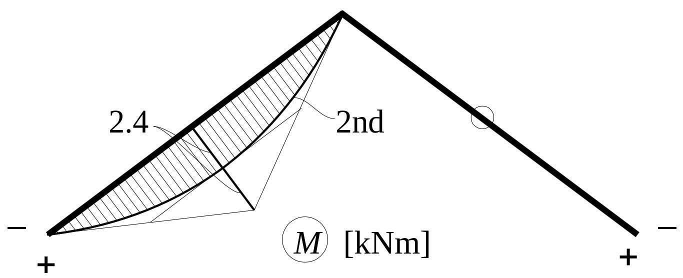
Calculation in cross sections K_3 and K_4 is done from the left. Both reactions C'_z downwards and C'_x rightwards cause a positive shear:

$$V_3 = V_4 = -1.2 \cdot \cos\alpha + 1.6 \cdot \sin\alpha = 0\,\text{kN} .$$

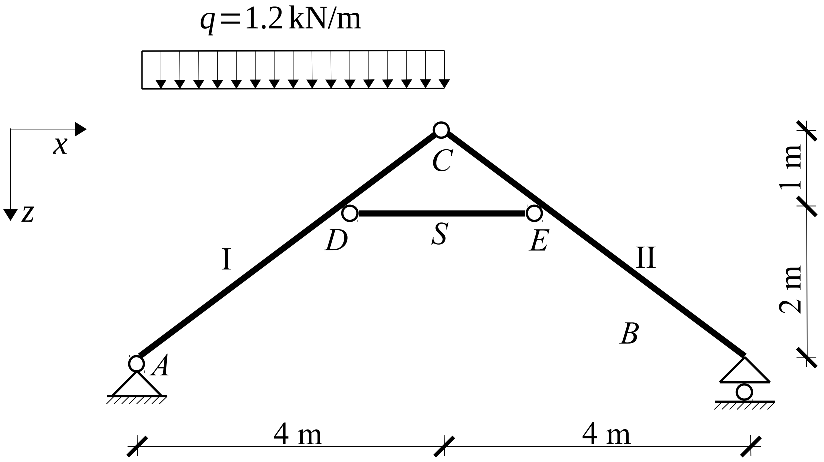
The bending moment diagram is parabolic on its left and linear on its right hand side. Moment values at internal hinges and pin-jointed supports are zero.
The depth of parabola on the right hand side is obtained from the transversal component of intensity of the distributed load as follows:

$$\frac{q_V \cdot l_{12}^2}{8} = \frac{0.768 \cdot 5^2}{8} = 2.4\,\text{kNm} .$$

On the right hand side member, the value of shear is zero, that is, the slope of the moment diagram is also zero. Because of the zero values at both ends, all values in between are still zero.

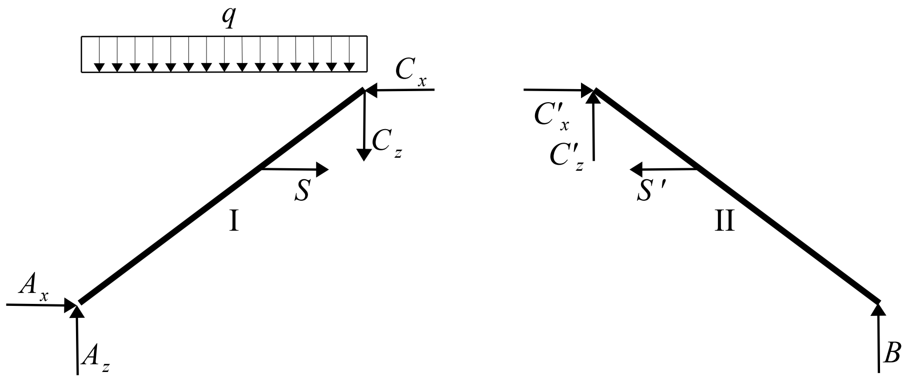


Exercise 2
Draw internal force diagrams of the structure shown.



Solution

Isolation:



Equilibrium statements:

e u new u

I :

II :

Str:

Analytic solution:

$\sum$

$\sum$

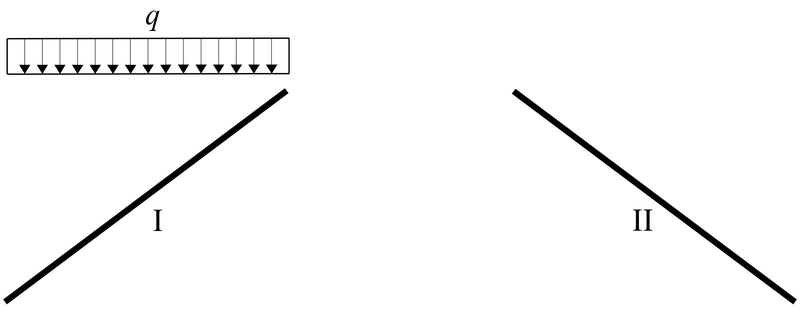
$\sum$

$\sum$

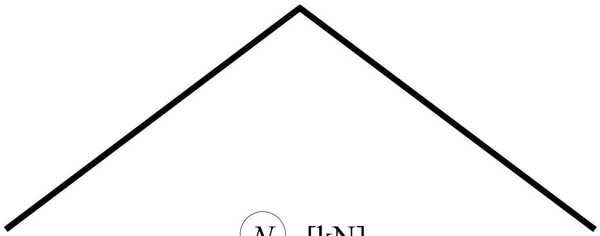
$\sum$

$\sum$

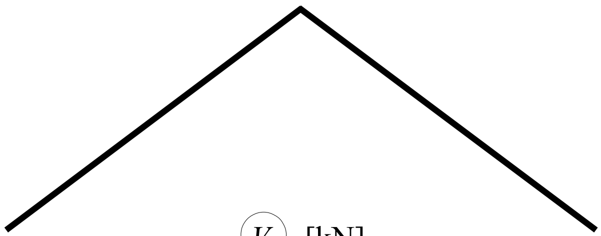
Final sketch of reactions:



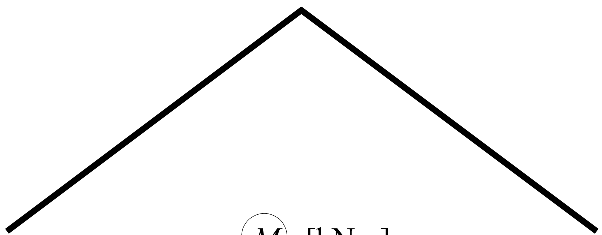
Internal force diagrams:



N [kN]



V [kN]



M [kNm]