6. QUASI-STATIC METHODS

6.1 Theoretical background

6.1.1 The elements

All quasi-static models existing today apply perfectly rigid elements, similarly to the BALL-type models. The elements have smooth strictly convex shapes (mostly circular or spherical) so that the contacts are small point-like domains. The detailed derivations of this section are also restricted to circular/spherical elements only.

Consider a collection of *N* elements in the three-dimensional, right-handed (*x*, *y*, *z*) Euclidean coordinate frame. A reference point, *O*, is defined on each element, coinciding with the centre of gravity for simplicity. Each element has the same six degrees of freedom as the elements in the BALL-type models.

6.1.2 The contacts

The concept of ***contact point*** and the vector pointing from the reference point to the contact will have a very important role in the forthcoming derivations. The contact point is where contact force and moment is transmitted between the touching elements. (Since the elements are perfectly rigid, their deformability is carried by the contacts, similarly to the BALL-type models.)

An analysed element ('*p*') and its neighbour ('*q*') form a contact either if their surfaces touch in a single point, or if the two surfaces intersect. In the first case the position of the contact point is trivial. In the second case the extended domain of intersection has to be replaced by a single point; in addition, it is assumed that the overlapping domain is small in comparison to the size of the elements. For spherical elements the straight line connecting the centres of the two spheres intersect with the two surfaces; the middle point between these two points of intersection is defined to be the contact point, as shown in Figure 1. The point "*pc*" is attached to the analysed element "*p*" and moves together with it; similarly, "*qc*" moves together with the neighbour, *q*.

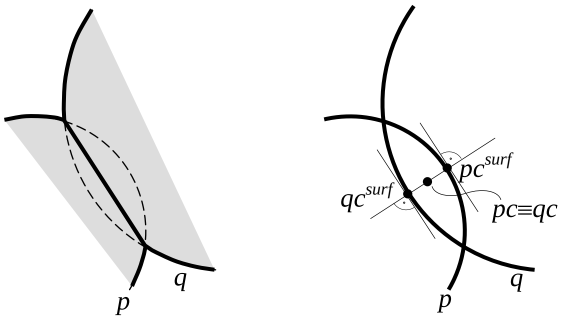

Figure 1.
Contact point between two spherical elements

In principle, the elements could have other non-spherical shapes; Figure 2. illustrates such a situation. In this case there exists a unique line which is normal to both surfaces in the points of intersections. The middle point of this straight section is the contact.)

In both cases, the relative displacements of *qc* with respect to *pc* give the deformations of the contact.

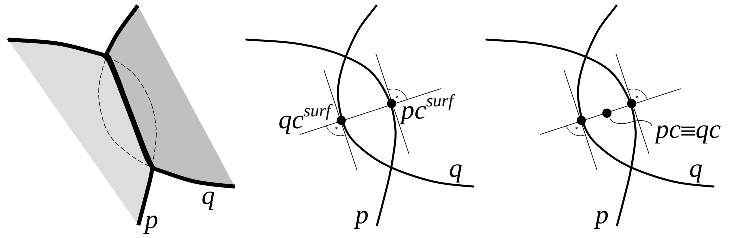

Figure 2.
Contact point between two elements with arbitrary strictly convex, smooth shape

Since the overlapping domain is small, the vector pointing from the rerefence point of p to the surface point pc^{surf} is approximately identical to the vector pointing from the reference point to pc. This vector will be denoted by $\mathbf{r}^{pc}$, and will have an important role in the forthcoming derivations.

A local coordinate system is assigned separately to each contact. This local frame is formed by the unit $\mathbf{n}^c$ contact normal (pointing outwards of p), and by and by an arbitrarily chosen pair of perpendicular unit vectors $\mathbf{t}^c$ and $\mathbf{w}^c$, lying in the plane of the contact (i.e. being perpendicular to $\mathbf{n}^c$ too). The ($\mathbf{n}^c$, $\mathbf{t}^c$, $\mathbf{w}^c$) vectors form a right-hand frame.

Remember that the local and global coordinates of an arbitrary vector $\mathbf{v}$ are related to each other through the transformation matrix $\mathbf{T}^c$:

$$\mathbf{v}_{glob} = \mathbf{T}^c \mathbf{v}_{loc} \quad ; \quad \mathbf{v}_{loc} = \mathbf{T}^{cT} \mathbf{v}_{glob}$$

where

$$\mathbf{T}^c = \begin{bmatrix} \cos(\mathbf{x},\mathbf{n}^c) & \cos(\mathbf{x},\mathbf{t}^c) & \cos(\mathbf{x},\mathbf{w}^c) \\ \cos(\mathbf{y},\mathbf{n}^c) & \cos(\mathbf{y},\mathbf{t}^c) & \cos(\mathbf{y},\mathbf{w}^c) \\ \cos(\mathbf{z},\mathbf{n}^c) & \cos(\mathbf{z},\mathbf{t}^c) & \cos(\mathbf{z},\mathbf{w}^c) \end{bmatrix} .$$

(here **x**, **y**, and **z** are the unit vectors pointing in the direction of the *x*, *y*, and *z* axes).

6.1.3 The equations of equilibrium

The basic unknowns of the quasi-static models are those incremental displacements which move the system from the actual to the equilibrated state. The equilibrium equations serve for the approximation of these displacements.

Similarly to the BALL-type models, a 6-scalar displacement vector describes the incremental motion of an element. Collect these vectors into a hypervector, $\Delta\mathbf{u}$, consisting of as many 6-scalar blocks as N, the number of elements forming the analysed system:

$$\Delta\mathbf{u} = \begin{bmatrix} \Delta\mathbf{u}^1 \\ \Delta\mathbf{u}^2 \\ \vdots \\ \Delta\mathbf{u}^N \end{bmatrix}$$

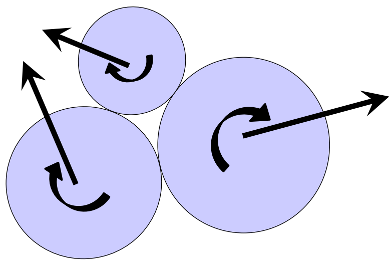

Figure 3.
Degrees of freedom of rigid elements

in which the *p*-th block contains the incremental displacements of the *p*-th element:

$$\Delta\mathbf{u}^p = \begin{bmatrix} \Delta u_x^p \\ \Delta u_y^p \\ \Delta u_z^p \\ \Delta\varphi_x^p \\ \Delta\varphi_y^p \\ \Delta\varphi_z^p \end{bmatrix}$$

Displacements occur in a quasi-static model only if the elements are not in equilibrium in the analysed state. The *p*-th element is not in equilibrium if the reduced force vector of the element $\Delta\mathbf{f}^p$ (i.e. the external and contact forces and moments acting on *p*, reduced to the reference point of *p*) is a non-zero vector. Collect these reduced force vectors into a hypervector $\Delta\mathbf{f}$:

$$\Delta\mathbf{f} = \begin{bmatrix} \Delta\mathbf{f}^1 \\ \Delta\mathbf{f}^2 \\ \vdots \\ \Delta\mathbf{f}^N \end{bmatrix} .$$

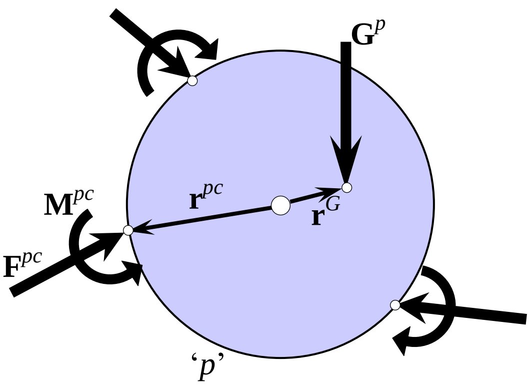

Figure 4.
Forces and moments acting on element p

Focus now on element p only. Let $\mathbf{F}^{pc}$ and $\mathbf{M}^{pc}$ denote the contact force and moment acting on p in the contact point pc. The vector $\mathbf{r}^{pc}$ points from the reference point to the contact. External forces (e.g. gravity) may also act on the element; let $\mathbf{G}^{p}$ denote their resultant, and $\mathbf{r}^{G}$ is the vector pointing from the reference point to the point of action of $\mathbf{G}^{p}$. The vector $\Delta\mathbf{f}^{p}$ can be calculated in the following way (summation for c runs along all contacts of p):

$$\Delta\mathbf{f}^{p} = \begin{bmatrix} \Delta f_x^p \\ \Delta f_y^p \\ \Delta f_z^p \\ \Delta m_x^p \\ \Delta m_y^p \\ \Delta m_z^p \end{bmatrix} = \begin{bmatrix} G_x^p + \sum_{(c)} F_x^{pc} \\ G_y^p + \sum_{(c)} F_y^{pc} \\ G_z^p + \sum_{(c)} F_z^{pc} \\ \left(r_y^G G_z^p - r_z^G G_y^p\right) + \sum_{(c)} \left(r_y^{pc} F_z^{pc} - r_z^{pc} F_y^{pc} + M_x^{pc}\right) \\ \left(r_z^G G_{x\,x}^p - r_x^G G_z^p\right) + \sum_{(c)} \left(r_z^{pc} F_x^{pc} - r_x^{pc} F_z^{pc} + M_y^{pc}\right) \\ \left(r_x^G G_y^p - r_y^G G_x^p\right) + \sum_{(c)} \left(r_x^{pc} F_y^{pc} - r_y^{pc} F_x^{pc} + M_z^{pc}\right) \end{bmatrix}$$

This vector shows the equilibrium error of element p.

The relation between $\Delta\mathbf{u}$ and $\Delta\mathbf{f}$ is given by the ***stiffness matrix***, **K**. this matrix consists of $N\times N$ blocks each of them containing 6×6 scalars. To compile **K**, the $\mathbf{r}^{pc}$ vectors, the directions of the contact frame axes, and the contact stiffnesses have to be known, so **K** reflects the geometrical and stiffness characteristics of the system. Then the basic equation of the matrix displacement method can be written:

$$\mathbf{K}\,\Delta\mathbf{u} = \Delta\mathbf{f}$$

This equation expresses that the $\Delta\mathbf{u}$ incremental displacements of the system lead from the original unbalanced state to the equilibrium position corresponding to the (originally unbalanced) forces acting on the elements, both externally and through the contacts.

In the forthcoming paragraphs the stiffness matrix will be compiled, but before this could be done, a couple of static and kinematic relations have to be written.

6.1.4 Static relations

The force expressed on element p by its neighbour in contact c can be written in the local frame of the contact:

$$\mathbf{S}^c = \begin{bmatrix} N^c \\ T_t^c \\ T_w^c \\ M_n^c \\ M_t^c \\ M_w^c \end{bmatrix} ; \qquad \mathbf{S} = \begin{bmatrix} \mathbf{S}^1 \\ \mathbf{S}^2 \\ \vdots \\ \mathbf{S}^M \end{bmatrix} \quad .$$

Here N^c is the normal force, T_t^c and T_w^c are the two components of the shear force(lying in the tangent plane of the contact), M_n^c is the twisting moment (corresponding to the relative rotation about the common normal of the two contacting elements), finally, M_t^c and M_w^c are the two components of the bending moment (corresponding to the relative rotations about the *t* and *w* axes, respectively).

The 6-scalar $\mathbf{S}^c$ blocks are collected into the hypervector $\mathbf{S}$, the internal force vector of the system, which consists of as many blocks as the *M* number of contacts in the system. Typically, each block is written in a different coordinate frame.

An $\mathbf{S}^c$ vector can be transformed to the global frame with the help of the transformation matrix of the contact, $\hat{\mathbf{T}}^c$. Since the forces and the moments as well have to be transformed, this matrix differs from the usual transformation matrices:

$$\hat{\mathbf{T}}^c = \begin{bmatrix} \mathbf{T}^c & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^c \end{bmatrix} = \begin{bmatrix} \cos(\mathbf{x},\mathbf{n}^c) & \cos(\mathbf{x},\mathbf{t}^c) & \cos(\mathbf{x},\mathbf{w}^c) & 0 & 0 & 0 \\ \cos(\mathbf{y},\mathbf{n}^c) & \cos(\mathbf{y},\mathbf{t}^c) & \cos(\mathbf{y},\mathbf{w}^c) & 0 & 0 & 0 \\ \cos(\mathbf{z},\mathbf{n}^c) & \cos(\mathbf{z},\mathbf{t}^c) & \cos(\mathbf{z},\mathbf{w}^c) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\mathbf{x},\mathbf{n}^c) & \cos(\mathbf{x},\mathbf{t}^c) & \cos(\mathbf{x},\mathbf{w}^c) \\ 0 & 0 & 0 & \cos(\mathbf{y},\mathbf{n}^c) & \cos(\mathbf{y},\mathbf{t}^c) & \cos(\mathbf{y},\mathbf{w}^c) \\ 0 & 0 & 0 & \cos(\mathbf{z},\mathbf{n}^c) & \cos(\mathbf{z},\mathbf{t}^c) & \cos(\mathbf{z},\mathbf{w}^c) \end{bmatrix}.$$

Using this, the contact force vector belonging to *pc* can be expressed in the global frame:

$$\mathbf{f}^{pc} = \begin{bmatrix} F_x^{pc} \\ F_y^{pc} \\ F_z^{pc} \\ M_x^{pc} \\ M_y^{pc} \\ M_z^{pc} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^c & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^c \end{bmatrix} \begin{bmatrix} N^c \\ T_t^c \\ T_w^c \\ M_n^c \\ M_t^c \\ M_w^c \end{bmatrix} = \hat{\mathbf{T}}^c \mathbf{S}^c$$

The individual contact force can be reduced to the reference point of *p* with the help of the ***transition matrix***, $\mathbf{B}^{pc}$:

$$\mathbf{f}_{red}^{pc} = \begin{bmatrix} f_{red,x}^{pc} \\ f_{red,y}^{pc} \\ f_{red,z}^{pc} \\ m_{red,x}^{pc} \\ m_{red,y}^{pc} \\ m_{red,z}^{pc} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -r_z^{pc} & +r_y^{pc} & 1 & 0 & 0 \\ +r_z^{pc} & 0 & -r_x^{pc} & 0 & 1 & 0 \\ -r_y^{pc} & +r_x^{pc} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x^{pc} \\ F_y^{pc} \\ F_z^{pc} \\ M_x^{pc} \\ M_y^{pc} \\ M_z^{pc} \end{bmatrix} = \mathbf{B}^{pc}\mathbf{f}^{pc} = \mathbf{B}^{pc}\mathbf{T}^c\mathbf{S}^c$$

Note that $\mathbf{B}^{pc}$ is different from contact to contact.

6.1.5 Kinematic relations

The contact forces become modified because of the displacement increments, if the incremental motions cause a change of the existing contact deformations. In order to express the deformation increments in terms of the displacement increments of the elements, consider a contact *c* which is formed by the first element *p* with the second element *q*; the two point-like material regions *pc* and *qc* occupy the same position before $\Delta\mathbf{u}$ begins. The well-known $\mathbf{r}^{pc}$ and $\mathbf{r}^{qc}$ vectors will also be needed in the calculations.

The incremental displacements of the two material points, $\Delta\mathbf{u}^{pc}$ és $\Delta\mathbf{u}^{qc}$, can be expressed with the help of the transpose of the matrices of transitions (already used at the static relations), $\mathbf{B}^{pc}$ and $\mathbf{B}^{qc}$:

$$\Delta\mathbf{u}^{pc} = \begin{bmatrix} 1 & 0 & 0 & 0 & +r_z^{pc} & -r_y^{pc} \\ 0 & 1 & 0 & -r_z^{pc} & 0 & +r_x^{pc} \\ 0 & 0 & 1 & +r_y^{pc} & -r_x^{pc} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta u_x^p \\ \Delta u_y^p \\ \Delta u_z^p \\ \Delta\varphi_x^p \\ \Delta\varphi_y^p \\ \Delta\varphi_z^p \end{bmatrix} = \mathbf{B}^{pcT}\Delta\mathbf{u}^p$$

and

$$\Delta\mathbf{u}^{qc} = \begin{bmatrix} 1 & 0 & 0 & 0 & +r_z^{qc} & -r_y^{qc} \\ 0 & 1 & 0 & -r_z^{qc} & 0 & +r_x^{qc} \\ 0 & 0 & 1 & +r_y^{qc} & -r_x^{qc} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta u_x^q \\ \Delta u_y^q \\ \Delta u_z^q \\ \Delta\varphi_x^q \\ \Delta\varphi_y^q \\ \Delta\varphi_z^q \end{bmatrix} = \mathbf{B}^{qcT}\Delta\mathbf{u}^q$$

Their difference, $\Delta\mathbf{u}^{qc} - \Delta\mathbf{u}^{pc}$, is by definition the incremental deformation of the contact, denoted by $\Delta\boldsymbol{\delta}^c$. Because of its physical meaning and being related to the contact force components, it has to be written in the local frame of the contact:

$$\Delta\boldsymbol{\delta}^c = \begin{bmatrix} \Delta\delta_n^c \\ \Delta\delta_t^c \\ \Delta\delta_w^c \\ \Delta\theta_n^c \\ \Delta\theta_t^c \\ \Delta\theta_w^c \end{bmatrix} = \begin{bmatrix} \Delta u_n^q \\ \Delta u_t^q \\ \Delta u_w^q \\ \Delta\varphi_n^q \\ \Delta\varphi_t^q \\ \Delta\varphi_w^q \end{bmatrix} - \begin{bmatrix} \Delta u_n^p \\ \Delta u_t^p \\ \Delta u_w^p \\ \Delta\varphi_n^p \\ \Delta\varphi_t^p \\ \Delta\varphi_w^p \end{bmatrix} = \hat{\mathbf{T}}^{cT}\Delta\mathbf{u}^{qc} - \hat{\mathbf{T}}^{cT}\Delta\mathbf{u}^{pc} = \hat{\mathbf{T}}^{cT}\left(\mathbf{B}^{qcT}\Delta\mathbf{u}^q - \mathbf{B}^{pcT}\Delta\mathbf{u}^p\right) \ .$$

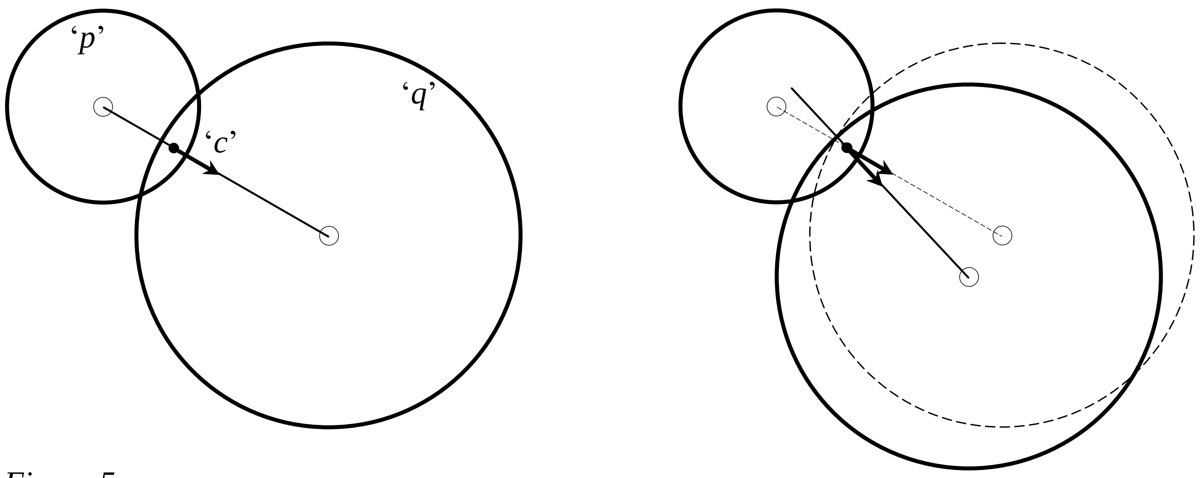

Figure 5.
A displacement leading to pure relative bending rotation in the contact: element ’p’ is fixed, element ’q’ moves in such a way that the material points pc and a qc remain at the same position while q rotates about the point qc

6.1.6 The contact stiffness matrix

The contact stiffness matrix expresses – in the local coordinate system of the contact – the increments of the contact force vector caused by the deformation increment $\Delta\boldsymbol{\delta}^c$:

$$\Delta\mathbf{S}^c = \begin{bmatrix} \Delta N^c \\ \Delta T_t^c \\ \Delta T_w^c \\ \Delta M_n^c \\ \Delta M_t^c \\ \Delta M_w^c \end{bmatrix} = \mathbf{k}^c \Delta\boldsymbol{\delta}^c = \begin{bmatrix} k_N^c & & & & & \\ & k_T^c & & & & \\ & & k_T^c & & & \\ & & & k_{twist}^c & & \\ & & & & k_{bend}^c & \\ & & & & & k_{bend}^c \end{bmatrix} \begin{bmatrix} \Delta\delta_n^c \\ \Delta\delta_t^c \\ \Delta\delta_w^c \\ \Delta\theta_n^c \\ \Delta\theta_t^c \\ \Delta\theta_w^c \end{bmatrix}.$$

The elements of the diagonal matrix are the actual normal, tangentional, twisting and bending stiffnesses of the contact. (Theoretically this matrix might have a more complex structure. Cross effects between the different types of contact force components may be expressed, for instance, with the help of out-of-diagonal terms in the matrix. The shear stiffness may be different in the different directions in the (t, w) plane; in this case the relations between tangential forces and deformations would become more complicated. The elements in the matrix could depend on the forces already acting in the contact etc. The quasi-static DEM models existing today apply only the simple version shown by the above expression.)

It is important to emphasize that the elements in the matrix may perhaps be zero: e.g. in a simple frictional contact which resists no bending the bending stiffnesses are zero, or if the contact is already sliding, the tangential stiffness k_T is zero etc.

6.1.7 Compilation of the equilibrium equations

The incremental displacements $\Delta\mathbf{u}$ cause increments of the contact deformations, and because of these, the contact forces will be modified. The incremental displacements have to be such that the modified contact force system would already be in equilibrium with the external forces. According to the usual models, the changes of the contact forces are purely due to the increments of the contact deformations. (The other reason could be the large displacements of the elements due to which the position and direction of the contacts could also change but these higher-order effects are usually neglected. An exception is, for instance, Bagi (2007).)

The increment of the contact forces in a contact *c* is:

$$\Delta\mathbf{S}^c = \mathbf{k}^c\Delta\boldsymbol{\delta}^c = \mathbf{k}^c\left(\hat{\mathbf{T}}^{cT}\Delta\mathbf{u}^{qc} - \hat{\mathbf{T}}^{cT}\Delta\mathbf{u}^{pc}\right) = \mathbf{k}^c\hat{\mathbf{T}}^{cT}\mathbf{B}^{qcT}\Delta\mathbf{u}^q - \mathbf{k}^c\hat{\mathbf{T}}^{cT}\mathbf{B}^{pcT}\Delta\mathbf{u}^p \ .$$

Assuming that the original state (still before $\Delta\mathbf{f}$ and $\Delta\mathbf{u}$) was an equilibrium state, and also assuming that the displacement increments are small so the geometry of the original state remains valid after $\Delta\mathbf{u}$, the equilibrium equation of an element *p* reduces to:

$$\Delta\mathbf{f}^p + \sum_{(c)}\mathbf{B}^{pc}\hat{\mathbf{T}}^c\Delta\mathbf{S}^c = 0$$

or

$$\Delta\mathbf{f}^p + \sum_{(c)}\mathbf{B}^{pc}\hat{\mathbf{T}}^c\mathbf{k}^c\left(\hat{\mathbf{T}}^{cT}\Delta\mathbf{u}^{qc} - \hat{\mathbf{T}}^{cT}\Delta\mathbf{u}^{pc}\right) = 0 \ .$$

Here the summation for *c* goes along all contacts (with different neighbours having the running index *q*) of element *p*. After some rearrangements the equation of *p* is:

$$\Delta\mathbf{f}^p + \left(\sum_{(c)}\mathbf{B}^{pc}\hat{\mathbf{T}}^c\mathbf{k}^c\hat{\mathbf{T}}^{cT}\mathbf{B}^{qcT}\Delta\mathbf{u}^q\right) - \left(\sum_{(c)}\mathbf{B}^{pc}\hat{\mathbf{T}}^c\mathbf{k}^c\hat{\mathbf{T}}^{cT}\mathbf{B}^{pcT}\right)\Delta\mathbf{u}^p = 0 \ .$$

Here the first bracket contains those changes of the contact forces which are caused by the displacement increments of the neighbours, and the second bracket contains those changes which are due to the displacements of element *p* itself.

Introduce the following notations:

$$\mathbf{k}^{pq} = -\mathbf{B}^{pc}\hat{\mathbf{T}}^c\mathbf{k}^c\hat{\mathbf{T}}^{cT}\mathbf{B}^{qcT}$$

$$\mathbf{k}^{pp} = \sum_{(c)}\mathbf{B}^{pc}\hat{\mathbf{T}}^c\mathbf{k}^c\hat{\mathbf{T}}^{cT}\mathbf{B}^{pcT}$$

so the equilibrium equation of *p* is:

$$\mathbf{k}^{pp}\Delta\mathbf{u}^p + \left(\sum_{(q)}\mathbf{k}^{pq}\Delta\mathbf{u}^q\right) = \Delta\mathbf{f}^p$$

The summation in the bracket runs along those elements *q* which have a contact with *p*; other elements have no contribution to the resultant acting on *p*.

The same can be done for every element. These equations can be summarized into the huge matrix equation:

$$\widehat{\mathbf{K}}\,\Delta\mathbf{u} = \Delta\mathbf{f} \quad ,$$

in which the p-th block in the main diagonal of $\widehat{\mathbf{K}}$ is the $\mathbf{k}^{pp}$ matrix itself, the out-of-diagonal (p,q) block is $\mathbf{k}^{pq}$.

There are different possibilities to model the supports of the system. One of them is to imagine supporting springs whose one end is fixed and the other end is attached to the reference point of an element. In this case the main diagonal of the stiffness matrix has to be modified by adding the spring stiffness to the corresponding value in the main diagonal. (In principle, elements supported eccentrically could also be modelled this way, but the treatment of that case is more complicated and not applied in the existing models.) Another possibility is to consider the supporting walls of the system as elements having prescribed zero displacements. For these elements the displacements are not searched for among the unknowns, and the equilibrium equations are not taken into consideration when compiling global equations of equilibrium. Consequently, in the $\Delta\mathbf{u}$ and $\Delta\mathbf{f}$ vectors there are no blocks belonging to these elements. Their effects can be taken into consideration in the following way:

Consider element p which is supported by a wall element. The contact with the wall element is denoted by C. This contact also contributes do the $\mathbf{k}^{pp}$ block, since the displacements of p cause deformations in C:

$$\mathbf{k}^{pp} = \left(\sum_{(c)} \mathbf{B}^{pc} \hat{\mathbf{T}}^{c} \mathbf{k}^{c} \hat{\mathbf{T}}^{cT} \mathbf{B}^{pcT} \right) + \mathbf{B}^{pC} \hat{\mathbf{T}}^{C} \mathbf{k}^{C} \hat{\mathbf{T}}^{CT} \mathbf{B}^{pCT}$$

Since there are no unknowns belonging to the wall element, there is no out-of-diagonal block in the p-th block row of $\widehat{\mathbf{K}}$.

The global stiffness matrix is ready now. In the case of symmetrical contact stiffness matrices the global stiffness matrix is also symmetric.

We can recognize that the global stiffness matrix is just the same (apart from the negative sign) as the Jacobian matrix already seen in the description of the Newton-Raphson-method:

$$\widehat{\mathbf{K}}(\mathbf{u}) = -\mathbf{K}(\mathbf{u}) = -\frac{d\mathbf{f}(\mathbf{u})}{d\mathbf{u}}$$

or in details:

$$\widehat{K}_{pq} = -K_{pq} = -\frac{df_p(u_1, u_2, ..., u_n)}{du_q} \quad .$$

The mechanical meaning of a column of the stiffness matrix can easily be recognized from this also. The values in the first column, for instance, show the opposite value of what would be the increments in the reduced force vector if the first element is forced to make a unit translation in the x direction, while keeping all other degrees of freedom of this first element fixed, and all other elements also fixed with zero displacements. The last column contains the increments of the reduced force vector in the case of a unit rotation of the last element about z.

Now try to solve the equation $\widehat{\mathbf{K}}\,\Delta\mathbf{u} = \Delta\mathbf{f}$. For an invertible stiffness matrix the $\Delta\mathbf{u}$ solution is unique. This is the case if the system is statically either determinate, or indeterminate. However, for statically overdeterminate (i.e. kinematically indeterminate) systems the stiffness matrix will be singular, which means that unrestricted displacements can be found in the system.

From practical point of view such situations often occur. The simulation of a rock sample just breaking in a compression test, the neighbourhood of a collapsing underground tunnel, or an arch just falling into voussoirs require the suitable mathematical treatment of this situation. The different quasi-static models use different approaches for this problem; these will be explained below at the introduction of the different models.

6.2 The most important quasi-static models

6.2.1 The model of Serrano and Rodriguez-Ortiz

The model published by Serrano and Rodriguez-Ortiz (1973) is the ancestor of the quasi-static discrete element techniques. It was a two-dimensional model using circular, perfectly rigid elements whose point-like contacts could resist tension, compression and shear forces. In the initial, unloaded state the circles just touch each other; the contact forces are zero at this stage. The mechanical behaviour of the contacts is characterised by two constants: the normal and the tangential stiffness. The global stiffness matrix can be compiled using these data. For given external forces acting on the elements the displacements can be calculated by inverting the stiffness matrix.

This model cannot be considered a discrete element model, for several reasons. Large displacements can not be treated: the loads can be applied in a single step, and the calculation of the corresponding displacements assumes geometrical linearity. New contacts cannot be recognized. The stiffness matrix must be invertible: unstable, collapsing parts cannot be present in the system. However, since it was the first computational method to avoid time integration, its historic value cannot be overestimated.

6.2.2 The model of Kishino

The model of Kishino (1987) also applies rigid circular 2D elements. The contacts are frictional: they resist infinite compression, and tangential force if the friction limit is not exceeded. The reference points are the centres of the circles. The elements have the usual three degrees of freedom (two translation components and a rotation).

Assume that all the elements are perfectly fixed, except from the *p*-th element whose three displacement components are restricted because of having contacts with the fixed neighbours. The equilibrium equations of this element can be written in the following form:

$$\begin{bmatrix} & & \\ & \mathbf{k}^{pp} & \\ & (3\times 3) & \end{bmatrix} \begin{bmatrix} \Delta u_x^p \\ \Delta u_y^p \\ \Delta \varphi_z^p \end{bmatrix} = \begin{bmatrix} \Delta f_x^{\,p} \\ \Delta f_y^{\,p} \\ \Delta m_z^p \end{bmatrix}$$

Here $\Delta\mathbf{f}^p$ is the reduced force vector of element *p*: all external and contact forces acting on *p* are reduced to the reference point. (If it is a zero vector, the element is in equilibrium.) The equations express that if all neighbours are fixed, the unknown $\Delta\mathbf{u}^p$ displacement increments will move the element into the position where the new contact forces equilibrate the external forces. The compilation of $\mathbf{k}^{pp}$ will be shown below.

The contacts transfer compression and frictional shear, so the incremental deformation vector of contact *c*, as well as the contact force vector, consist of two scalars:

$$\Delta\boldsymbol{\delta}^c = \begin{bmatrix} \Delta\delta_n^c \\ \Delta\delta_t^c \end{bmatrix}; \quad \mathbf{S}^c = \begin{bmatrix} N^c \\ T^c \end{bmatrix}$$

The constitutive relations describe the stiffnesses against normal and tangential displacement, include the condition that the normal force can only be compressional, and also contains the friction limit to the tangential force:

$$\mathbf{k}^c = \begin{bmatrix} k_N^c & 0 \\ 0 & k_T^c \end{bmatrix} \quad ; \qquad \begin{matrix} N < 0 \\ T \leq \boldsymbol{\nu} N \end{matrix}$$

The model of Kishino does not compile the global stiffness matrix; instead, it is based on solving the equilibrium equations of the individual elements assuming that when a certain element is considered, all its neighbours are fixed. In mathematical sense it corresponds to the ***Gauss-Seidel iteration technique*** (already seen in Section 3 as a numerical method to solve systems of linear equations). An iteration step is done in the following way:

1. At the beginning of the step the position and geometry of the elements, the forces acting in the contacts, and the contact stiffnesses are known. For each element separately, determine the reduced force vector $\Delta\mathbf{f}^p$. This is the equilibrium error of the element. If this vector is sufficiently close to a zero vector, e.g. its norm is below a pre-defined threshold, the element is approximately in equilibrium. If the equilibrium error of every element is below the threshold, the iteration can be stopped: the equilibrium state is found. Otherwise, continue with Step 2.
2. Consider the element having the largest equilibrium. Say this is element *p*. Compile its stiffness matrix:
$$\mathbf{k}^{pp} = \sum_{(c)} \mathbf{B}^{pc} \hat{\mathbf{T}}^c \mathbf{k}^c \hat{\mathbf{T}}^{cT} \mathbf{B}^{pcT}$$
Index *c* runs along the contacts of *p*. Check whether this matrix is invertible. Singular matrices have to be modified by assigning fictitious additional stiffnesses so that it would become invertible (the interested reader can find the details of the modification in Kishino (1987)).
3. Assume that all other elements are fixed, except from *p*. Its displacement caused by $\Delta\mathbf{f}^p$ is easy to calculate:
$$\Delta\mathbf{u}^p = (\mathbf{k}^{pp})^{-1} \cdot \Delta\mathbf{f}^p \quad .$$
4. Modify the position of *p* according to this displacement: assuming linearity, the element is in equilibrium now. Calculate the new contact forces by taking into account the relative displacement increments in the contacts, and check whether the normal force is compression, and the tangential force magnitude is below the friction limit. Modify the contact forces if necessary: for a disappearing contact both force components are to be deleted and the contact does not exist any more; for a sliding contact the tangential force is set to be just equal to the friction limit, and the tangential stiffness is set to zero. Check whether any new contacts are born: if a new

contact is found, it is added to the list of contacts, and a compression force is to be calculated from the overlap of the two elements. These modifications may destroy the balance of the analysed element.

5. The iteration cycle is finished and Step 1. can come again.

This model is already a real discrete element model: contacts can disappear or be born, and as an accumulation of small displacement increments, large displacements can be followed.

6.2.3 The model of Bagi and Bojtár

This model also applies circular (spherical in 3D) elements. The contacts transmit forces and moments; the contacts may be frictional or bonded.

Unlike the model of Kishino, this model compiles the ***whole stiffness matrix*** of the system. The mathematical background is given by the Newton-Raphson-method. An iteration step is as follows:

1. Start from an equilibrated, compatible state satisfying the contact conditions. (This can be, for instance, an unloaded, force-free arrangement where the elements just touch each other without any overlap.) The aim is to determine how the system gets into equilibrium if a small force increment is added on it.
2. Consider all elements separately, and for each of them, reduce the forces (all external and contact forces) to the reference point. This gives the vector $\Delta\mathbf{f}$. Compile the global stiffness matrix ($\widehat{\mathbf{K}}$) of the system, in the way explained above in 6.1.7. This matrix is often singular: local collapses, unstable parts just falling out, broken pieces not supported yet by their neighbours etc. can often be found in a system in an intermediate stage of a state-changing process. So, add a diagonal matrix $<\boldsymbol{\rho}>$, whose elements are all positive numbers, to $\widehat{\mathbf{K}}$. (The scalars in the main diagonal can be chosen arbitrarily, e.g. as a certain fraction of the average contact stiffness.) The modified matrix is:

$$\widehat{\mathbf{K}}^{\rho} := \widehat{\mathbf{K}} + <\boldsymbol{\rho}>$$

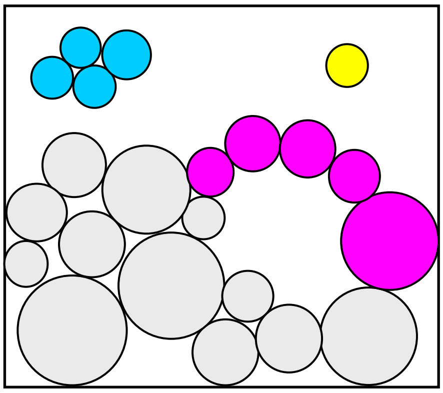

Figure 6.
A system having a singular global stiffness matrix

3. The modified stiffness matrix is invertible, so the displacement increments caused by $\Delta\mathbf{f}$ can be determined, assuming linear behaviour, in the following manner:

$$\Delta\mathbf{u} = (\widehat{\mathbf{K}}^{\rho})^{-1} \cdot \Delta\mathbf{f}$$

4. From the incremental displacements the increments of the contact deformations can be determined, based on them the contact forces can be upgraded. These forces and moments may have to be modified depending on the constitutive conditions (e.g. the shear force must not exceed the friction limit in a Coulomb-type contact; the moment resistance disappears if the contact breaks either for shear or for bending etc.).
5. Steps 2.-4. are repeated until reaching a state where the static, kinematic and constitutive relations are all satisfied. Then a new load increment can follow.

The Bagi-Bojtár-model is numerically more stable than the model of Kishino, and the number of iteration steps is definitely lower. However, the memory requirement is much higher and the calculations are usually much more time-consuming since the whole stiffness matrix has to be compiled and inverted in every iteration step.

6.3 Applications

These models (and other similar versions existing in the literature) have no commercial software versions. Their usage is restricted to researches related to granular and rock mechanics.

Questions

6.1. Explain the meaning of the individual elements of the stiffness matrix of a system!
6.2. The stiffness matrix of a system turns out to be singular. What is the physical meaning of this mathematical characteristic?
6.3. Introduce the model of Kishino!
6.4. Introduce the model of Bagi and Bojtár!