THE EQUATIONS OF MOTION AND THE SOLUTION METHODS

→The Equations of Motion for the three basic types of element

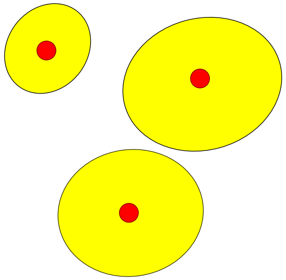
→ Numerical techniques for time integration (Repetition of previous studies)

THE EQUATIONS OF MOTION

Three main types of the elements:

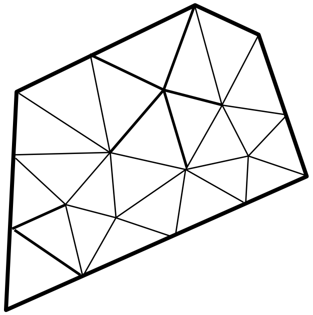
(1) perfectly rigid elements

→ reference point



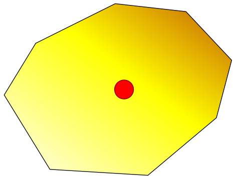
(2) elements being deformable because of an internal FEM mesh

→ nodes



(3) elements being deformable because of a simple strain field

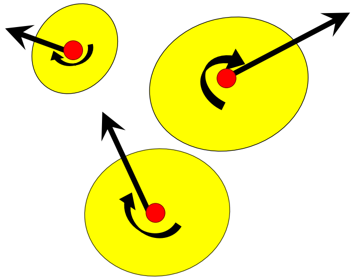
→ a reference point + a simple strain function



THE EQUATIONS OF MOTION

„f = ma”

a) Perfectly rigid elements



Reference point

to every element

the displacement vector of the p-th element:

$$\mathbf{u}^p(t) = \begin{bmatrix} u_x^p(t) \\ u_y^p(t) \\ u_z^p(t) \\ \varphi_x^p(t) \\ \varphi_y^p(t) \\ \varphi_z^p(t) \end{bmatrix}$$

total displacement vector:

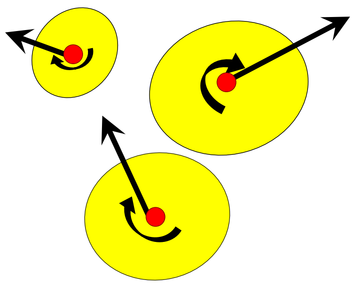
$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}^1(t) \\ \mathbf{u}^2(t) \\ \vdots \\ \mathbf{u}^N(t) \end{bmatrix}$$

summed up of small increments!

THE EQUATIONS OF MOTION

„f = ma”

a) Perfectly rigid elements



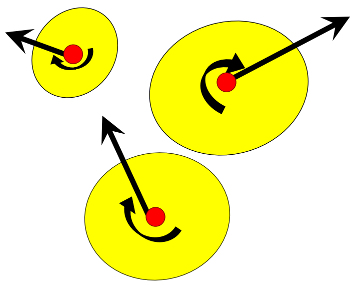
velocity vector: $\mathbf{v}(t) = \dfrac{d\mathbf{u}(t)}{dt}$

pl. $$\mathbf{v}^p(t) = \begin{bmatrix} v_x^p(t) \\ v_y^p(t) \\ v_z^p(t) \\ \omega_x^p(t) \\ \omega_y^p(t) \\ \omega_z^p(t) \end{bmatrix} = \begin{bmatrix} \dfrac{du_x^p(t)}{dt} \\ \dfrac{du_y^p(t)}{dt} \\ \dfrac{du_z^p(t)}{dt} \\ \dfrac{d\varphi_x^p(t)}{dt} \\ \dfrac{d\varphi_y^p(t)}{dt} \\ \dfrac{d\varphi_z^p(t)}{dt} \end{bmatrix}$$

THE EQUATIONS OF MOTION

„f = ma”

a) Perfectly rigid elements

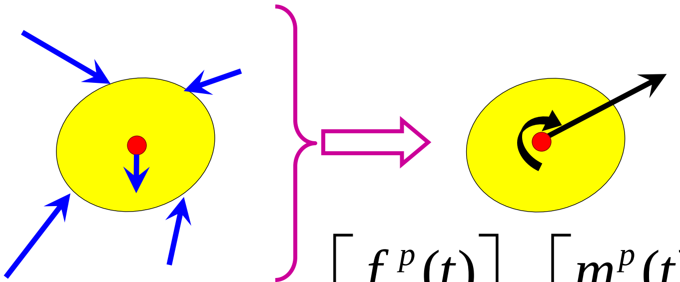


velocity vector: $\mathbf{v}(t) = \frac{d\mathbf{u}(t)}{dt}$

acceleration vector: $\mathbf{a}(t) = \frac{d^2\mathbf{u}(t)}{dt^2}$

pl. $$\mathbf{a}^p(t) = \begin{bmatrix} a_x^p(t) \\ a_y^p(t) \\ a_z^p(t) \\ \beta_x^p(t) \\ \beta_y^p(t) \\ \beta_z^p(t) \end{bmatrix} = \begin{bmatrix} \frac{d^2u_x^p(t)}{dt^2} \\ \frac{d^2u_y^p(t)}{dt^2} \\ \frac{d^2u_z^p(t)}{dt^2} \\ \frac{d^2\varphi_x^p(t)}{dt^2} \\ \frac{d^2\varphi_y^p(t)}{dt^2} \\ \frac{d^2\varphi_z^p(t)}{dt^2} \end{bmatrix}$$

THE EQUATIONS OF MOTION



$$\begin{bmatrix} f_x^p(t) \\ f_y^p(t) \\ f_z^p(t) \end{bmatrix} ; \begin{bmatrix} m_x^p(t) \\ m_y^p(t) \\ m_z^p(t) \end{bmatrix}$$

a) Perfectly rigid elements

Equations of motion of the *p*-th element:

$$m^p a_x^p = f_x^p$$
$$m^p a_y^p = f_y^p$$
$$m^p a_z^p = f_z^p$$

$$I_{xx}^p \beta_x - I_{xy}^p \beta_y - I_{xz}^p \beta_z + \omega_y^p \left(\omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) - \omega_z^p \left(\omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) = m_x^p$$

$$I_{yy}^p \beta_y - I_{yx}^p \beta_x - I_{yz}^p \beta_z - \omega_x^p \left(\omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) + \omega_z^p \left(\omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right) = m_y^p$$

$$I_{zz}^p \beta_z - I_{zx}^p \beta_x - I_{zy}^p \beta_y + \omega_x^p \left(\omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) - \omega_y^p \left(\omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right) = m_z^p$$

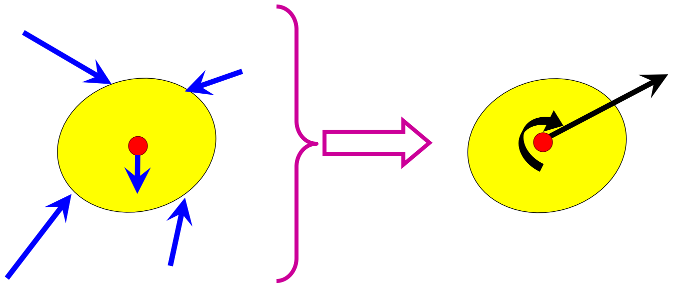
$$I_{xy}^p = \int_{V^p} (x - x^p) \cdot (y - y^p) \cdot \mu(x,y,z) \cdot dV \quad ;$$

$$I_{zy}^p = \int_{V^p} (z - z^p) \cdot (y - y^p) \cdot \mu(x,y,z) \cdot dV \quad \text{etc.}$$

THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Equations of motion of the *p*-th element:



the load vector: forces reduced to the reference point

$\rightarrow$ partly from the external forces

acting on the elements (e.g. weight)

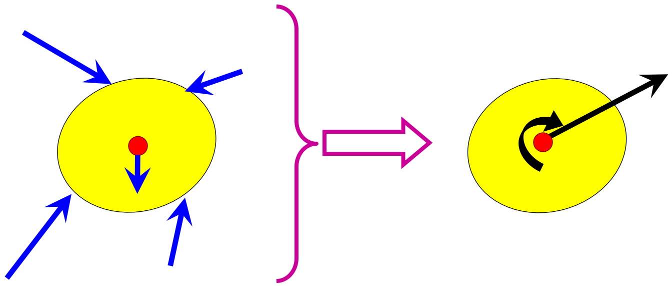
depend on position and velocity

$\rightarrow$ partly from the contact forces

expressed by the neighbouring elements

depend on position and velocity

THE EQUATIONS OF MOTION



a) Perfectly rigid elements

Equations of motion of the *p*-th element:

$$m^p a_x^p = f_x^p$$

$$m^p a_y^p = f_y^p$$

$$m^p a_z^p = f_z^p$$

$$I_{xx}^p \beta_x - I_{xy}^p \beta_y - I_{xz}^p \beta_z = m_x^p - \omega_y^p \left(\omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) + \omega_z^p \left(\omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right)$$

$$I_{yy}^p \beta_y - I_{yx}^p \beta_x - I_{yz}^p \beta_z = m_y^p + \omega_x^p \left(\omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) - \omega_z^p \left(\omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right)$$

$$I_{zz}^p \beta_z - I_{zx}^p \beta_x - I_{zy}^p \beta_y = m_z^p - \omega_x^p \left(\omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) + \omega_y^p \left(\omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right)$$

$$\mathbf{M}^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

THE EQUATIONS OF MOTION

$$
\mathbf{M}^p = \begin{bmatrix} m^p & & & & & \\ & m^p & & & & \\ & & m^p & & & \\ & & & I^p_{xx} & -I^p_{xy} & -I^p_{xz} \\ & & & -I^p_{yx} & I^p_{yy} & -I^p_{yz} \\ & & & -I^p_{zx} & -I^p_{zy} & I^p_{zz} \end{bmatrix}
$$

<u>a) Perfectly rigid elements</u>

Equations of motion of the *p*-th element:

$$
m^p a^p_x = f^p_x
$$

$$
m^p a^p_y = f^p_y
$$

$$
m^p a^p_z = f^p_z
$$

$$
I^p_{xx}\beta_x - I^p_{xy}\beta_y - I^p_{xz}\beta_z = m^p_x - \omega^p_y\left(\omega^p_z I^p_{zz} - \omega^p_x I^p_{zx} - \omega^p_y I^p_{zy}\right) + \omega^p_z\left(\omega^p_y I^p_{yy} - \omega^p_x I^p_{yx} - \omega^p_z I^p_{yz}\right)
$$

$$
I^p_{yy}\beta_y - I^p_{yx}\beta_x - I^p_{yz}\beta_z = m^p_y + \omega^p_x\left(\omega^p_z I^p_{zz} - \omega^p_x I^p_{zx} - \omega^p_y I^p_{zy}\right) - \omega^p_z\left(\omega^p_x I^p_{xx} - \omega^p_y I^p_{xy} - \omega^p_z I^p_{xz}\right)
$$

$$
I^p_{zz}\beta_z - I^p_{zx}\beta_x - I^p_{zy}\beta_y = m^p_z - \omega^p_x\left(\omega^p_y I^p_{yy} - \omega^p_x I^p_{yx} - \omega^p_z I^p_{yz}\right) + \omega^p_y\left(\omega^p_x I^p_{xx} - \omega^p_y I^p_{xy} - \omega^p_z I^p_{xz}\right)
$$

$$
\mathbf{M}^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))
$$

THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Special case: e.g. Spheres:

$$I_{xy}^{p} = 0; \quad I_{zy}^{p} = 0; \quad \text{etc.}; \quad I_{xx}^{p} = I_{yy}^{p} = I_{zz}^{p} := I^{p}$$

$$m^{p} a_{x}^{p} = f_{x}^{p}$$

$$m^{p} a_{y}^{p} = f_{y}^{p}$$

$$m^{p} a_{z}^{p} = f_{z}^{p}$$

$$I^{p} \beta_{x} = m_{x}^{p}$$

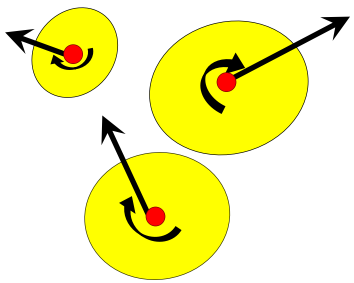
$$I^{p} \beta_{y} = m_{y}^{p}$$

$$I^{p} \beta_{z} = m_{z}^{p}$$

THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Equations of motion of the *p*-th element: (6 scalar equations)



$$\mathbf{M}^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

for the complete system (*N* elements):

$$\mathbf{M}(t)\mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))$$

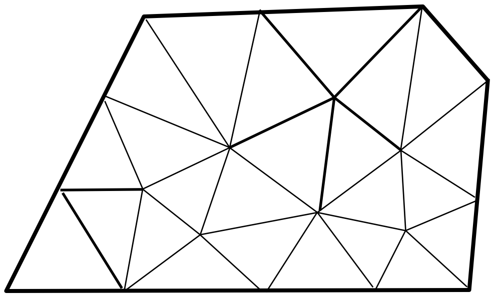
$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N \end{bmatrix} \qquad \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t)) = \begin{bmatrix} \mathbf{f}^1(t, \mathbf{u}(t), \mathbf{v}(t)) \\ \mathbf{f}^2(t, \mathbf{u}(t), \mathbf{v}(t)) \\ \vdots \\ \mathbf{f}^N(t, \mathbf{u}(t), \mathbf{v}(t)) \end{bmatrix}$$

THE EQUATIONS OF MOTION

„f = ma”

b) Elements made deformable by being subdivided

as an example: SIMPLEX subdivision



displacement vector of the *p*-th node:

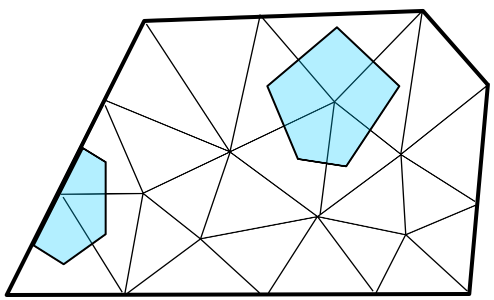
$$\mathbf{u}^p(t) = \begin{bmatrix} u_x^p(t) \\ u_y^p(t) \\ u_z^p(t) \end{bmatrix}$$

displacement vector of the whole system:

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}^1(t) \\ \mathbf{u}^2(t) \\ \vdots \\ \mathbf{u}^N(t) \end{bmatrix}$$

THE EQUATIONS OF MOTION

b) Elements made deformable by being subdivided

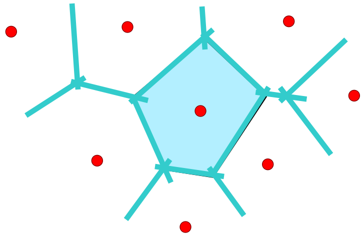


the equations of motion of the *p*-th node:

$$m^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

mass assigned to the *p*-th node: m^p

Voronoi tessellation:



in 2D:

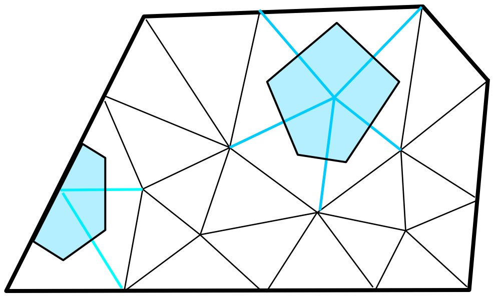
bisecting lines $\Rightarrow$ 2D domains assigned to the nodes

in 3D:

bisecting planes $\Rightarrow$ 3D domains assigned to the nodes

THE EQUATIONS OF MOTION

b) Elements made deformable by being subdivided



the equations of motion of the *p*-th node:

$$m^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

mass assigned to the *p*-th node: m^p

the force acting on the *p*-th node: $\mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$ (3 components)

$\leftarrow$ from the stresses inside the simplexes

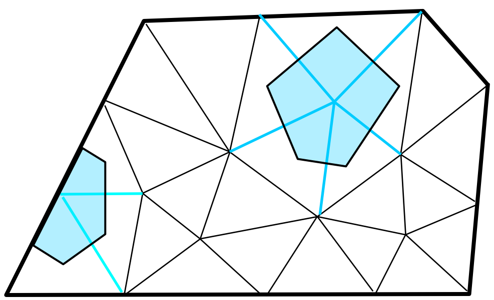
$\leftarrow$ from the neighbouring element

$\leftarrow$ from external forces (e.g. self weight, drag force)

ASSUMED TO ACT ON THE NODE !!!

THE EQUATIONS OF MOTION

b) Elements made deformable by being subdivided



the equations of motion of the whole system:

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$

($N \times 3$ scalar equations)

the complete inertial matrix consists of :

$$\mathbf{M}^p=\begin{bmatrix} m^p & & \\ & m^p & \\ & & m^p \end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix} \mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N \end{bmatrix}$$

the load vector: nodal forces

$$\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))=\begin{bmatrix} \mathbf{f}^1(t,\mathbf{u}(t),\mathbf{v}(t)) \\ \mathbf{f}^2(t,\mathbf{u}(t),\mathbf{v}(t)) \\ \vdots \\ \mathbf{f}^N(t,\mathbf{u}(t),\mathbf{v}(t)) \end{bmatrix}$$

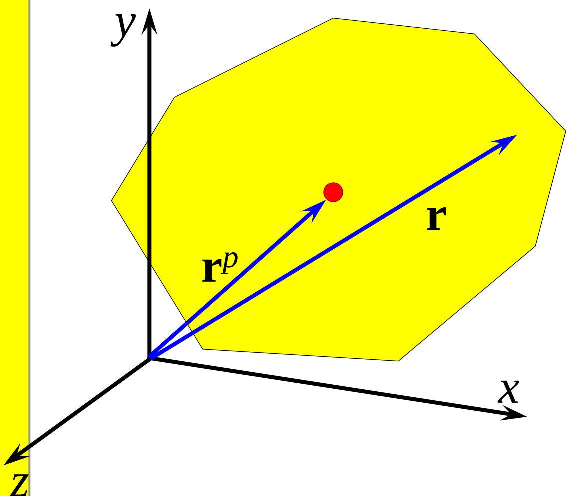
THE EQUATIONS OF MOTION

„f = ma"

c) Uniform-strain deformable elements without subdivision

displacement vector of the *p*-th element:

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$



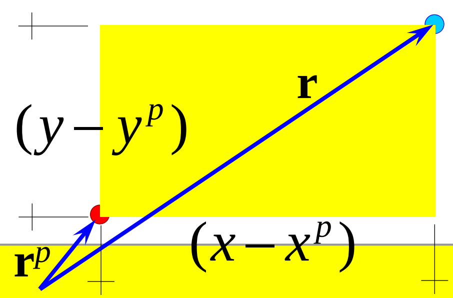
(reference point:
rigid-body translation and rotation;
the ***uniform*** strain of the element)

e.g. in 2D :

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \gamma_{xy}^p \end{bmatrix}$$

translation of another point in the element:

$$\mathbf{u} = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}$$



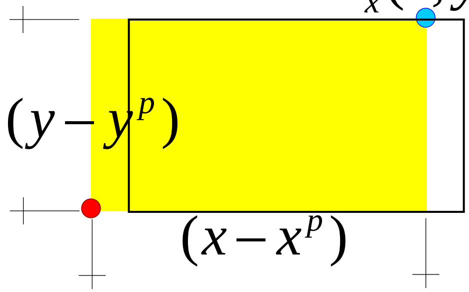
THE EQUATIONS OF MOTION

„f = ma”

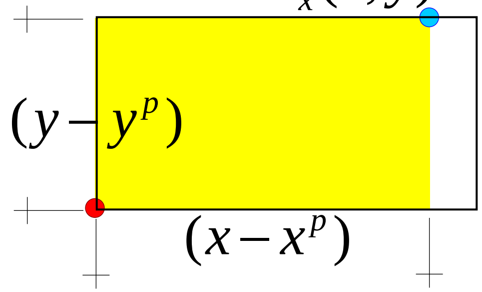
c) Uniform-strain deformable elements without subdivision HOME:

translation of another point in the element: with the help of superposition

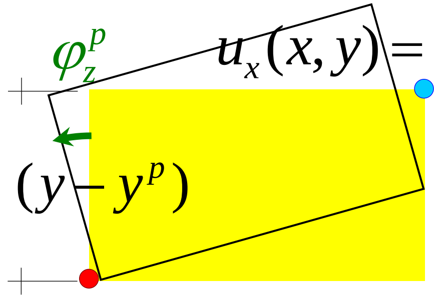
$$u_x(x,y) = u_x^p$$



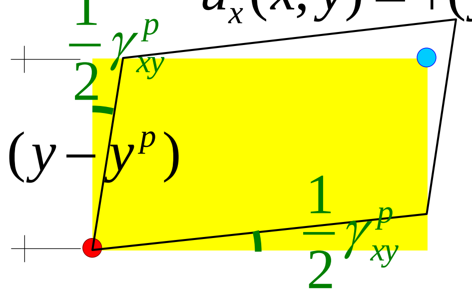
$$u_x(x,y) = (x-x^p)\varepsilon_x^p$$



$$u_x(x,y) = -(y-y^p)\varphi_z^p$$



$$u_x(x,y) = +(y-y^p)\frac{\gamma_{xy}^p}{2}$$



$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \gamma_{xy}^p \end{bmatrix}$$

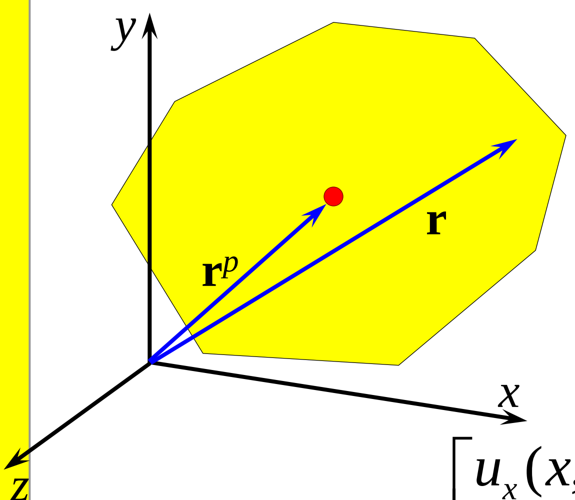
$$u_x(x,y) = u_x^p - (y-y^p)\varphi_z^p + (x-x^p)\varepsilon_x^p + (y-y^p)\frac{\gamma_{xy}^p}{2}$$

THE EQUATIONS OF MOTION

„f = ma”

c) Uniform-strain deformable elements without subdivision

translation of another point in the element:



$$u_x(x,y) = u_x^p - (y-y^p)\varphi_z^p + (x-x^p)\varepsilon_x^p + \frac{(y-y^p)}{2}\gamma_{xy}^p$$

$$u_y(x,y) = u_y^p + (x-x^p)\varphi_z^p + (y-y^p)\varepsilon_y^p + \frac{(x-x^p)}{2}\gamma_{xy}^p$$

$$\begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(y-y^p) & (x-x^p) & 0 & \frac{(y-y^p)}{2} \\ 0 & 1 & (x-x^p) & 0 & (y-y^p) & \frac{(x-x^p)}{2} \end{bmatrix} \begin{bmatrix} u_x^p \\ u_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \gamma_{xy}^p \end{bmatrix}$$

similarly in 3D!

$\Rightarrow$ relative translations in the contacts:

can be expressed from $\mathbf{u}^p$

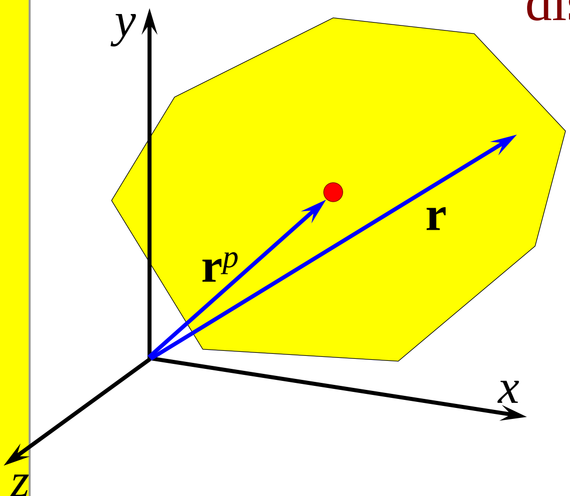
THE EQUATIONS OF MOTION

„f = ma”

c) Uniform-strain deformable elements without subdivision

remember:

displacement vector of the *p*-th element:



(reference point:

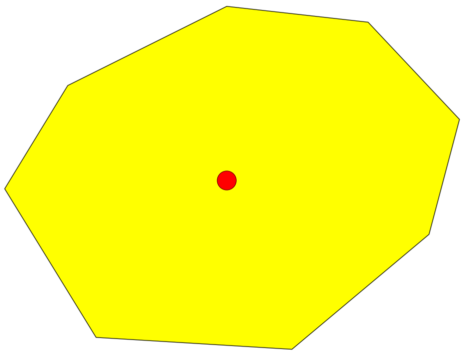
rigid-body translation and rotation;

the ***uniform*** strain of the element)

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

THE EQUATIONS OF MOTION

c) Uniform-strain deformable elements without subdivision



load vector beloning to element *p*:

– from the contacts with neighbouring elements

– from the external forces directly acting on the element

$$\mathbf{f}^p = \begin{bmatrix} f_x^p \\ f_y^p \\ f_z^p \\ m_x^p \\ m_y^p \\ m_z^p \\ V^p\sigma_x^p \\ V^p\sigma_y^p \\ V^p\sigma_z^p \\ V^p\tau_{yz}^p \\ V^p\tau_{zx}^p \\ V^p\tau_{xy}^p \end{bmatrix} \leftarrow \mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

the equations of motion of the *p*-th element:

$$\mathbf{M}^p \cdot \mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

the equations of motion of the whole system:

$$\boxed{\mathbf{M} \cdot \mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))}$$

SOLUTION OF THE EQUATIONS OF MOTION

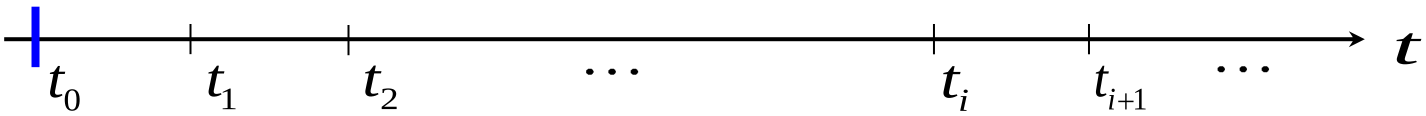
Numerical solutions only!

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$

The aim:

starting from a known $\mathbf{u}(t_0)=\mathbf{u}_0$ and $\mathbf{v}(t_0)=\mathbf{v}_0$ state at a t_0 time instant,

the aim is to determine the approximative solutions $(\mathbf{u}_1, \mathbf{v}_1)$, $(\mathbf{u}_2, \mathbf{v}_2)$, …, $(\mathbf{u}_i, \mathbf{v}_i)$, $(\mathbf{u}_{i+1}, \mathbf{v}_{i+1})$, … belonging to the t_1, t_2, …, t_i, t_{i+1}, … time instants.

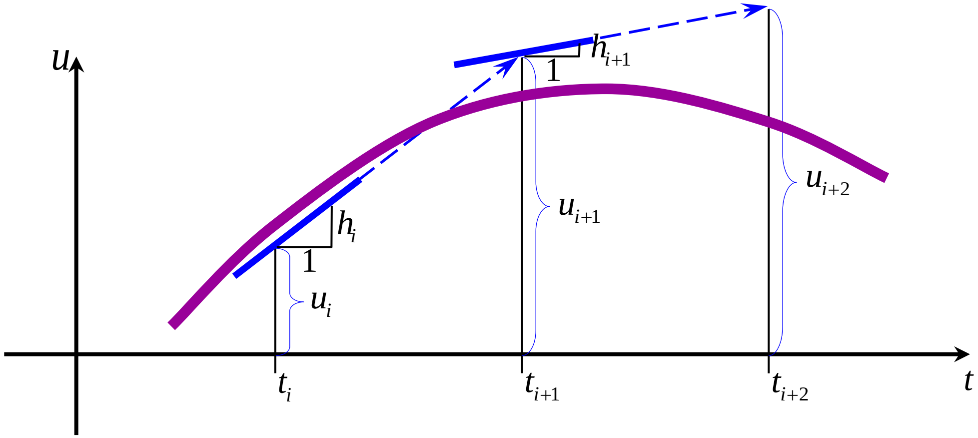


Initial remarks:

1. Explicit vs. implicit time integration methods
2. How to transform the equations of motion into first-order differential equations

SOLUTION OF THE EQUATIONS OF MOTION

1. Explicit vs. implicit methods:



→ explicit methods:

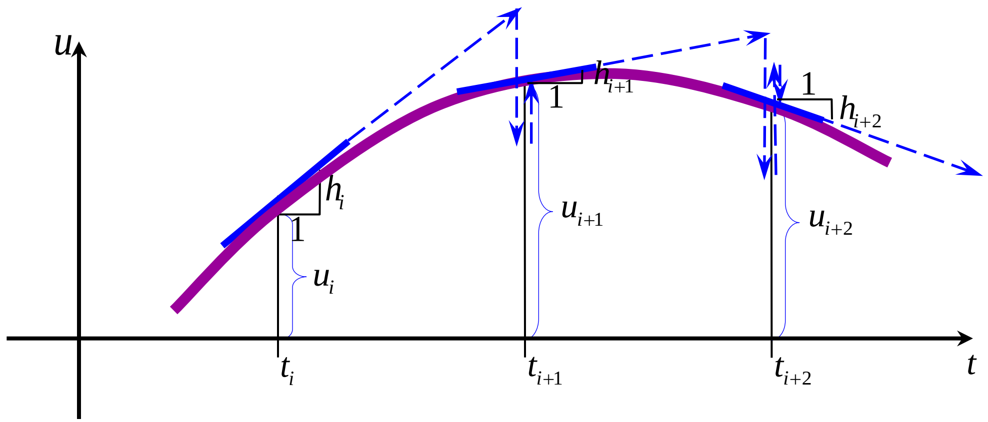
in the state at t_i: $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{f}_i)$ ⇒ equations of motion ⇒ approximate $(\mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{f}_{i+1})$ belonging to the state at t_{i+1}

NO checking of whether $(\mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{f}_{i+1})$ satisfy the eqs of motion, accept them and use them for the calculations of the next timestep

⇒ fast, but less reliable; numerical stability problems!

SOLUTION OF THE EQUATIONS OF MOTION

1. Explicit vs. implicit methods:



→ implicit methods:

in the state at t_i: $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{f}_i) \Rightarrow$ equations of motion $\Rightarrow$

approximate $(\mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{f}_{i+1})$ belonging to the state at t_{i+1};

then iterations, to improve this approximation belonging to t_{i+1},

so that the eqs of motion be satisfied at t_{i+1}

$\Rightarrow$ slow, but longer timesteps,

more reliable, better numerical stability

SOLUTION OF THE EQUATIONS OF MOTION

2. How to transform the equations of motion into first-order DE

The DE: $\mathbf{M}\cdot\frac{d^2\mathbf{u}(t)}{dt^2}=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$ where $\mathbf{v}(t)=\frac{d\mathbf{u}(t)}{dt}$

Notation:

new unknowns: $\mathbf{y}(t):=\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\end{bmatrix}$

new right-hand side:

$$\bar{\mathbf{a}}(t,\mathbf{u}(t),\mathbf{v}(t)):=\mathbf{M}^{-1}\cdot\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$ or: $$\bar{\mathbf{a}}(t,\mathbf{y}(t)):=\mathbf{M}^{-1}\cdot\mathbf{f}(t,\mathbf{y}(t))$$

$$\hat{\mathbf{a}}(t,\mathbf{u}(t),\mathbf{v}(t)):=\begin{bmatrix}\mathbf{v}(t)\\ \bar{\mathbf{a}}(t,\mathbf{u}(t),\mathbf{v}(t))\end{bmatrix}$$

so the equations become:

$$\frac{d\mathbf{y}(t)}{dt}=\hat{\mathbf{a}}(t,\mathbf{y}(t))$$

REPETITON: NUMERICAL METHODS

Numerical time integration of initial value problems:

→ Euler-method

→ Method of central differences

→ Newmark's β-method

The problem to solve, in mathematical sense:

to find that $\mathbf{y}(t)$ function for which the eqs. $\dfrac{d\mathbf{y}(t)}{dt} = \hat{\mathbf{a}}(t, \mathbf{y}(t))$ are satisfied at every *t*, and whose initial value is known: $\mathbf{y}(t_0) = \mathbf{y}_0$

Numerical solution:

Instead of trying to determine the explicit form of the function $\mathbf{y}(t)$,
the values $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_i, \mathbf{y}_{i+1}, \ldots$, belonging to $t_1, t_2, \ldots, t_i, t_{i+1}, \ldots$ are to be approximated.

EULER-METHOD

For the DEM eqs of motion:

The problem:

$$\begin{bmatrix} \dfrac{d\mathbf{u}(t)}{dt} \\ \dfrac{d\mathbf{v}(t)}{dt} \end{bmatrix} = \begin{bmatrix} \mathbf{v}(t) \\ \bar{\mathbf{a}}(t,\mathbf{u}(t),\mathbf{v}(t)) \end{bmatrix} \quad ; \quad \begin{bmatrix} \mathbf{u}(t_0) \\ \mathbf{v}(t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{bmatrix}$$

$$\bar{\mathbf{a}}(t,\mathbf{u}(t),\mathbf{v}(t)) := \mathbf{M}^{-1} \cdot \mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$

at t_i: known $\mathbf{v}_i$ and $\mathbf{f}$;

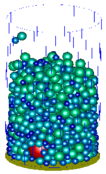
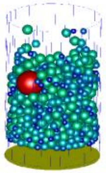
Let

$$\mathbf{h}_i = \begin{bmatrix} \mathbf{v}_i \\ \bar{\mathbf{a}}(t_i,\mathbf{u}_i,\mathbf{v}_i) \end{bmatrix}$$

meaning: the velocity and the acceleration keep their starting value along the time interval

from this, the new position and velocity:

$$\begin{bmatrix} \mathbf{u}_{i+1} \\ \mathbf{v}_{i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} + \Delta t \cdot \mathbf{h}_i = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} + \Delta t \cdot \begin{bmatrix} \mathbf{v}_i \\ \bar{\mathbf{a}}(t_i,\mathbf{u}_i,\mathbf{v}_i) \end{bmatrix}$$

or:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \cdot \mathbf{v}_i$$
$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta t \cdot \bar{\mathbf{a}}(t_i,\mathbf{u}_i,\mathbf{v}_i)$$

DEM: contact dynamics methods

disadvantage: oscillations

REPETITON: NUMERICAL METHODS

Numerical time integration of initial value problems:

→ Euler-method

→ Method of central differences

→ Newmark's β-method

The problem to solve:

to find that $\mathbf{y}(t)$ function for which the eqs. $\frac{d\mathbf{y}(t)}{dt} = \hat{\mathbf{a}}(t, \mathbf{y}(t))$ are satisfied at every t, and whose initial value is known: $\mathbf{y}(t_0) = \mathbf{y}_0$

Numerical solution:

Instead of trying to determine the explicit form of the function $\mathbf{y}(t)$,
the values $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_i, \mathbf{y}_{i+1}, \ldots$, belonging to $t_1, t_2, \ldots, t_i, t_{i+1}, \ldots$ are to be approximated.

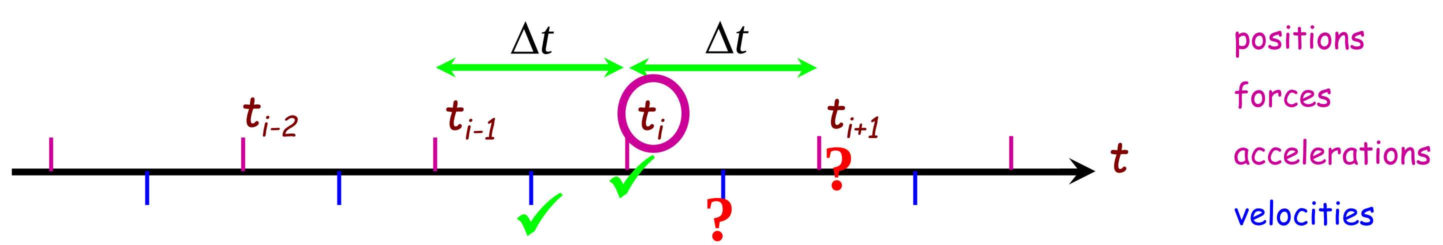
METHOD OF CENTRAL DIFFERENCES

For the DEM eqs of motion:

The problem:

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{v}(t); \qquad \mathbf{u}(t_0) = \mathbf{u}_0;$$

$$\frac{d\mathbf{v}(t)}{dt} = \bar{\mathbf{a}}(t, \mathbf{u}(t), \mathbf{v}(t)) \qquad \mathbf{v}(t_0) = \mathbf{v}_0$$



known: $\mathbf{v}_{i-1/2}$; $\bar{\mathbf{a}}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$ (initially: e.g. $\mathbf{v}_{1-1/2} := \mathbf{v}_0$)

Let $\mathbf{v}_{i+1/2} := \mathbf{v}_{i-1/2} + \Delta t \cdot \bar{\mathbf{a}}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$;

then from this: $\mathbf{u}_{i+1} := \mathbf{u}_i + \Delta t \cdot \mathbf{v}_{i+1/2}$

METHOD OF CENTRAL DIFFERENCES

For the DEM eqs of motion:

The problem:

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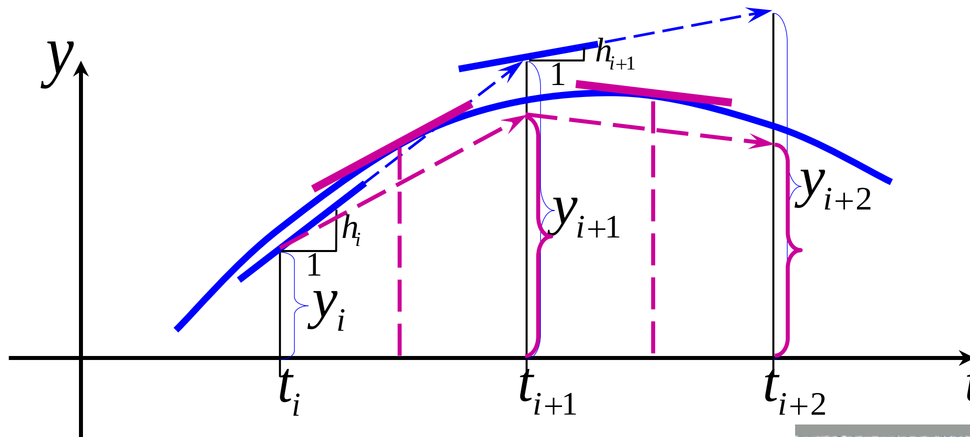

known: $\mathbf{v}_{i-1/2}$; $\bar{\mathbf{a}}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$

Let $\mathbf{v}_{i+1/2} := \mathbf{v}_{i-1/2} + \Delta t \cdot \bar{\mathbf{a}}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$;

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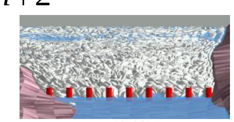

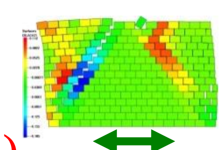

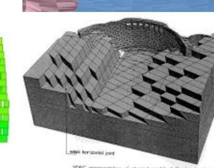

DEM: e.g. UDEC, PFC (most of the explicit timestepping methods)

REPETITON: NUMERICAL METHODS

Numerical time integration of initial value problems:

$\rightarrow$ Euler-method

$\rightarrow$ Method of central differences

$\rightarrow$ Newmark's β-method

The problem to solve:

to find that $\mathbf{y}(t)$ function for which the eqs. $\frac{d\mathbf{y}(t)}{dt} = \hat{\mathbf{a}}(t, \mathbf{y}(t))$ are satisfied at every t, and whose initial value is known: $\mathbf{y}(t_0) = \mathbf{y}_0$

Numerical solution:

Instead of trying to determine the explicit form of the function $\mathbf{y}(t)$,
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NEWMARK'S β-METHOD

For the DEM eqs of motion :

The problem: Find the $\mathbf{u}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ functions which satisfy the eqs.

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$

in which $\mathbf{v}(t)=\dfrac{d\mathbf{u}(t)}{dt}$, $\quad \mathbf{a}(t)=\dfrac{d^2\mathbf{u}(t)}{dt^2}$.

Notation: „residual”: $\mathbf{r}(t,\mathbf{u}(t),\mathbf{v}(t),\mathbf{a}(t))=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))-\mathbf{M}\cdot\mathbf{a}(t)$

The $\mathbf{u}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ functions are the solutions of the differential eqs if and only if: $\mathbf{r}(t,\mathbf{u}(t),\mathbf{v}(t),\mathbf{a}(t))=\mathbf{0}$

→ Assume that the $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ numerical solutions belonging to t_i satisfied this.

→ We would like to find $\mathbf{u}_{i+1}$, $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ belonging to t_{i+1} so that:

$$\mathbf{r}(t_{i+1},\mathbf{u}_{i+1},\mathbf{v}_{i+1},\mathbf{a}_{i+1})=0$$

NEWMARK'S β-METHOD

For the DEM eqs of motion:

Approximation of the position and velocity at the end of the timestep:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}\left[(1-2\beta)\mathbf{a}_i + 2\beta \cdot \mathbf{a}_{i+1}\right]$$

$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

Expression for the unknown values $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ in terms of the unknown $\mathbf{u}_{i+1}$:

$$\mathbf{a}_{i+1} := \frac{1}{\beta \cdot \Delta t^2}\left[\mathbf{u}_{i+1} - \left(\mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}(1-2\beta)\mathbf{a}_i\right)\right]$$

$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

here β and γ are constants controlling the behaviour of the method

The core of the method: Determine that $\mathbf{u}_{i+1}$, for which: $\mathbf{r}(t_{i+1}, \mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{a}_{i+1}) = 0$

→ e.g. Newton-Raphson iteration to find $\mathbf{u}_{i+1}$, then express $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ ✓

DEM: e.g. DDA models

NEWMARK'S β-METHOD

For the DEM eqs of motion:

Approximation of the position and velocity at the end of the timestep:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}\left[(1-2\beta)\mathbf{a}_i + 2\beta \cdot \mathbf{a}_{i+1}\right]$$

$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

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$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

here β and γ are constants controlling the behaviour of the method

specific β and γ values $\rightarrow$ several other methods

UNCONDITIONALLY STABLE IF: $2\beta \geq \gamma \geq ½$

e.g. $\gamma = ½$, $\beta = 0$: *method of central differences*, which is

ONLY CONDITIONALLY STABLE

QUESTIONS

1. Explain the meaning of the quantities in the equations of motion in the case of perfectly rigid elements!

2. Explain the meaning of the quantities in the equations of motion in the case of elements subdivided into uniform-strain simplexes!

3. Explain the meaning of the quantities in the equations of motion in the case of uniform-strain deformable elements without subdivision!

4. What is the difference between time-stepping and quasi-static methods?

5. What is the difference between explicit and implicit methods?

6. Introduce the Euler-method!

7. Introduce the method of central differences!

8. Introduce Newmark's β-method!