IMPLICIT TIMESTEPPING METHODS

→ Discontinuous Deformation Analysis

→ Contact Dynamics

OVERVIEW OF DEM SOFTWARES

Quasi-static metods ← ***an equilibrium state is searched for***

From an initial approximation of the equilibrium state searched for, the displacements **u** are to be determined taking the system to the equilibrium (assumption: time-independent behaviour, zero accelerations!!!)

$$"\mathbf{K}\cdot\Delta\mathbf{u}+\mathbf{f}=\mathbf{0}"$$

→ Kishino, 1988
→ Bagi-Bojtár, 1991

circular, perfectly rigid elemets, deformable contacts

Time-stepping methods $"\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))"$ ← ***a process in time is searched for***

simulate the motion of the system along small, but finite Δt timesteps

Explicit timestepping methods:

→ UDEC *deformable polyhedral elements, deformable contacts*

→ BALL-type models, e.g. PFC *rigid elements, deformable contacts*

Implicit timestepping methods:

→ DDA („Discontinuous Deformation Analysis”) *deformable polyhedral elements*

→ contact dynamics models *rigid elements, non-deformable contacts*

DISCONTINUOUS DEFORMATION ANALYSIS

„DDA”: Gen-Hua Shi (1988), Berkeley
then many others applied or developed

research softwares!!!

1. The elements: The unknowns and the reduced loads
2. Contacts
3. The equations of motion: Potential energy minimization
4. Numerical solution of the equations of motion

Applications

DISCONTINUOUS DEFORMATION ANALYSIS

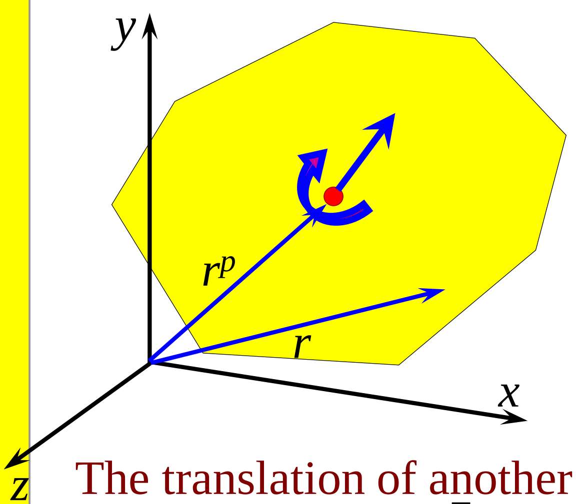
Repetition:

<u>1.The elements:</u> polyhedral [Deformable without subdivision]

displacement vector of the *p*-th element:

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

(reference point;
rigid-body translation and rotation;
the ***<u>uniform</u>*** strain of the element)



The translation of another point in the element in e.g. in 2D:

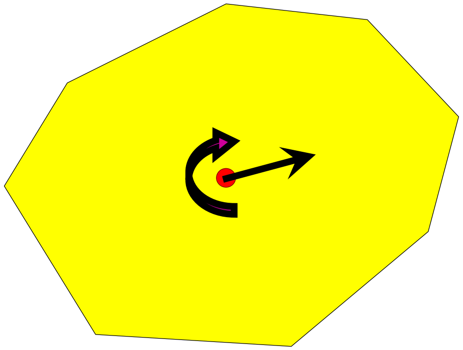
$$\begin{bmatrix} u_x(x,y) \\ u_y(x,y \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(y-y^p) & (x-x^p) & 0 & \dfrac{(y-y^p)}{2} \\ 0 & 1 & (x-x^p) & 0 & (y-y^p) & \dfrac{(x-x^p)}{2} \end{bmatrix} \begin{bmatrix} u_x^p \\ u_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \gamma_{xy}^p \end{bmatrix}$$

in 3D: $\mathbf{u}(x,y,z) = \mathbf{T}(x,y,z) \cdot \mathbf{u}^p$

DISCONTINUOUS DEFORMATION ANALYSIS

Repetition:

„reduced load” belonging to the *p*-th element:

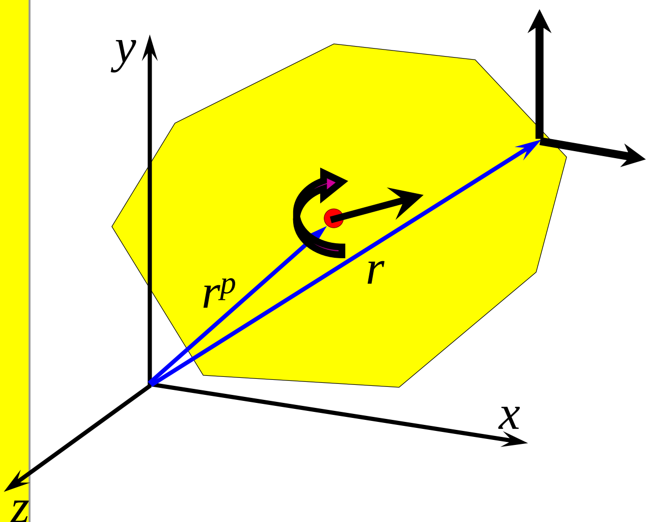


$$\mathbf{f}^{p}=\begin{bmatrix} f_x^p \\ f_y^p \\ f_z^p \\ m_x^p \\ m_y^p \\ m_z^p \\ V^p\sigma_x^p \\ V^p\sigma_y^p \\ V^p\sigma_z^p \\ V^p\tau_{yz}^p \\ V^p\tau_{zx}^p \\ V^p\tau_{xy}^p \end{bmatrix}$$

DISCONTINUOUS DEFORMATION ANALYSIS

How to reduce a force acting at (x, y) to the reference point of the element:

e.g. in 2D:



$$\mathbf{f}^p = \begin{bmatrix} f_x^p \\ f_y^p \\ m_z^p \\ A^p\sigma_x^p \\ A^p\sigma_y^p \\ A^p\tau_{xy}^p \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y-y^p) & (x-x^p) \\ (x-x^p) & 0 \\ 0 & (y-y^p) \\ \dfrac{(y-y^p)}{2} & \dfrac{(x-x^p)}{2} \end{bmatrix} \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} =$$

$$= \mathbf{T}^T(x,y)\cdot\mathbf{f}(x,y)$$

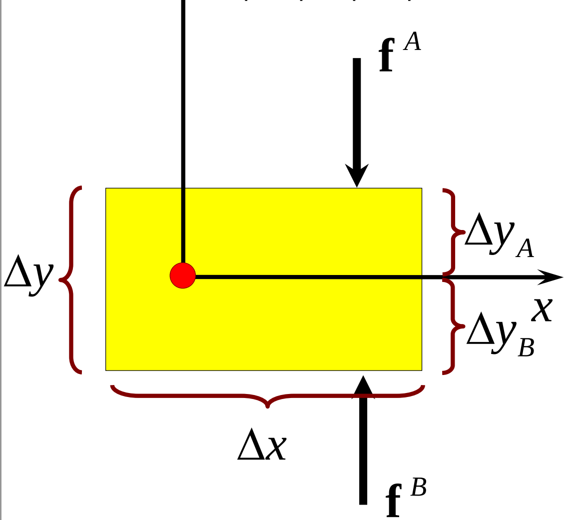
Illustration of its meaning:

DISCONTINUOUS DEFORMATION ANALYSIS

How to reduce a force acting at (x, y) to the reference point of the element:

e.g. in 2D:

$$\left|\mathbf{f}^A\right| = \left|\mathbf{f}^B\right| = Q$$



0

$$\mathbf{f}^p = \begin{bmatrix} f_x^p \\ f_y^p \\ m_z^p \\ A^p\sigma_x^p \\ A^p\sigma_y^p \\ A^p\tau_{xy}^p \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y-y^p) & (x-x^p) \\ (x-x^p) & 0 \\ 0 & (y-y^p) \\ \frac{(y-y^p)}{2} & \frac{(x-x^p)}{2} \end{bmatrix} \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} =$$

$$\mathbf{f}^A = \begin{bmatrix} 0 \\ -Q \end{bmatrix}; \quad \mathbf{f}^B = \begin{bmatrix} 0 \\ Q \end{bmatrix}$$

$$y^A - y^p = \Delta y_A; \; y^B - y^p = -\Delta y_B$$

$$A^p\sigma_x^p = 0 \quad A^p\tau_{xy}^p = (x^A - x^p)/2 \cdot f_y^A + (x^B - x^p)/2 \cdot f_y^B = 0$$

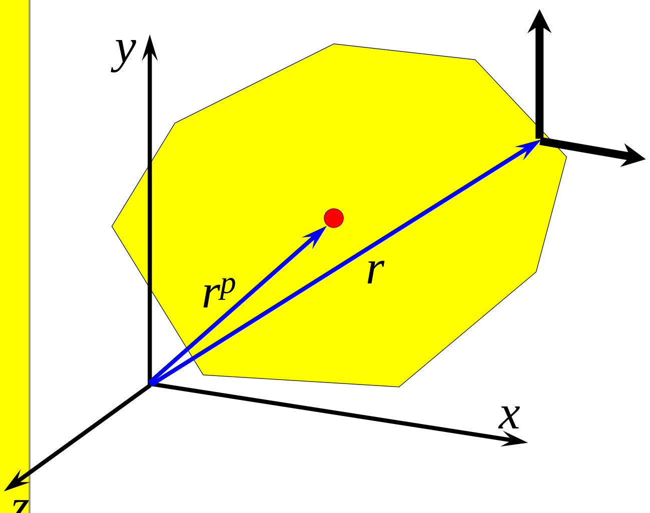
$$A^p\sigma_y^p = (y^A - y^p)\cdot f_y^A + (y^B - y^p)\cdot f_y^B =$$

$$= \Delta y_A \cdot (-Q) + (-\Delta y_B)\cdot Q = -(\Delta y_A + \Delta y_B)\cdot Q = -\Delta y \cdot Q$$

$$\Delta y \cdot \Delta x \cdot \sigma_y^p = -\Delta y \cdot Q \qquad \boxed{\sigma_y^p = -\frac{Q}{\Delta x}}$$

DISCONTINUOUS DEFORMATION ANALYSIS

How to reduce a force acting at (x, y) to the reference point of the element:



the reduced force in 3D: $\mathbf{T}^T(x,y,z)\cdot\mathbf{f}(x,y,z)$

Remark: Higher-order polynomials also possible! [strains/stresses]

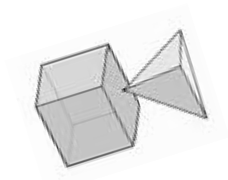
(e.g: M. MacLaughlin)

DISCONTINUOUS DEFORMATION ANALYSIS

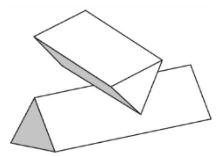
2.The contacts: (material point) with (material point)

in 2D: Node – to – Edge contacts

in 3D: Node – to – Face contacts:



Edge – to – Edge contacts:



→ „first entrance position"

⇒ contact deformation: $\Delta u_N ; \Delta u_T$

normal & tangential, perhaps sliding

→ direction of the contact:

the normal vector of the face

???? for edge-to-edge contact

Mechanical model:

→ originally: infinitely rigid contacts, Coulomb-friction

→ recent codes: deformable contacts included

+ other friction conditions, cohesion etc.

Remark: infinitely rigid contact: „penalty function": $F_N = k_N \Delta u_N ;\ dF_T = k_T d(\Delta u_T)$

≡ linearly elastic in normal and in tangential directions

DISCONTINUOUS DEFORMATION ANALYSIS

↓ more exactly: „ Hamilton principle ”

3. The equations of motion: „Potential energy” stationarity principle

„Potential” of the system: $\Pi = \Pi^{blocks} + \Pi^{contacts}$

deformed springs

external pot.

strain energy

inertial forces

velocity-proportional damping

initial stress

prescribed displacement history

$$\boxed{\frac{\partial \Pi}{\partial u_i^p} = 0} \quad \text{for all } p, i$$

$$\mathbf{M} \cdot \mathbf{a}(t) + \mathbf{C} \cdot \mathbf{v}(t) + \mathbf{K} \cdot \mathbf{u}(t) = \mathbf{f}^{ext}(t, \mathbf{v}(t), \mathbf{u}(t))$$

generalized displacement increment

stiffness matrix + etc

damping + etc

inertia

or: $\boxed{\mathbf{M} \cdot \mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))}$

DISCONTINUOUS DEFORMATION ANALYSIS

4. Numerical solution of the equations of motion:

(t_i, t_{i+1}) time interval:

at t_i : known $\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{f}(t_i, \mathbf{u}_i, \mathbf{v}_i)$; satisfy the eqs. of motion

Find $\mathbf{u}_{i+1}$, $\mathbf{v}_{i+1}$, $\mathbf{a}_{i+1}$ so that the eqs of motion would be satisfied at t_{i+1}

$$\boxed{\mathbf{r}(t_{i+1},\mathbf{u}_{i+1},\mathbf{v}_{i+1}) = \mathbf{f}(t_{i+1},\mathbf{u}_{i+1},\mathbf{v}_{i+1}) - \mathbf{M}\cdot\mathbf{a}_{i+1} = 0}$$

Remember: Newmark's β–method:

$$\boxed{\begin{aligned} \mathbf{u}_{i+1} &= \mathbf{u}_i + \Delta t\cdot\mathbf{v}_i + \frac{\Delta t^2}{2}\left[(1-2\beta)\mathbf{a}_i + 2\beta\cdot\mathbf{a}_{i+1}\right] \\ \mathbf{v}_{i+1} &:= \mathbf{v}_i + (1-\gamma)\cdot\Delta t\cdot\mathbf{a}_i + \gamma\cdot\Delta t\cdot\mathbf{a}_{i+1} \end{aligned}}$$

[stability: $2\beta \geq \gamma \geq ½$]

DDA: Newmark's β–method, with $\beta = 1/2$; $\gamma = 1$:

$$\boxed{\begin{aligned} \mathbf{u}_{i+1} &= \mathbf{u}_i + \Delta t\cdot\mathbf{v}_i + \frac{\Delta t^2}{2}\mathbf{a}_{i+1} \\ \mathbf{v}_{i+1} &:= \mathbf{v}_i + \Delta t\cdot\mathbf{a}_{i+1} \end{aligned}}$$

$$\text{let}\quad \left\{\begin{aligned} \Delta\mathbf{u}_{i+1} &= \mathbf{u}_{i+1} - \mathbf{u}_i \\ \mathbf{a}_{i+1} &= \frac{1}{\Delta t^2/2}(\Delta\mathbf{u}_{i+1} - \Delta t\cdot\mathbf{v}_i) \\ \mathbf{v}_{i+1} &= \mathbf{v}_i + \Delta t\cdot\mathbf{a}_{i+1} = \mathbf{v}_i + \frac{2}{\Delta t}(\Delta\mathbf{u}_{i+1} - \Delta t\cdot\mathbf{v}_i) = \frac{2}{\Delta t}\Delta\mathbf{u}_{i+1} - \mathbf{v}_i \end{aligned}\right.$$

DISCONTINUOUS DEFORMATION ANALYSIS

4. Numerical solution of the equations of motion :

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t)) \quad\Rightarrow\quad \mathbf{0}=\mathbf{f}(t_{i+1},\mathbf{u}_{i+1}(\Delta\mathbf{u}_{i+1},\mathbf{v}_{i+1}(\Delta\mathbf{u}_{i+1}))-\mathbf{M}\cdot\mathbf{a}_{i+1}(\Delta\mathbf{u}_{i+1})$$

Determine $\Delta\mathbf{u}_{i+1}$, so that the residual

$$\mathbf{r}(t_{i+1},\Delta\mathbf{u}_{i+1})=\mathbf{f}(t_{i+1},\mathbf{u}_{i+1}(\Delta\mathbf{u}_{i+1}),\mathbf{v}_{i+1}(\Delta\mathbf{u}_{i+1}))-\mathbf{M}\cdot\mathbf{a}_{i+1}(\Delta\mathbf{u}_{i+1})$$

would be sufficiently close to zero!

Newton-Raphson:

the Jacobian of the residual: $\mathcal{K}(t,\Delta\mathbf{u})=\dfrac{d\mathbf{r}(t,\Delta\mathbf{u})}{d\Delta\mathbf{u}}$

this matrix can be compiled from elementary calculations at t_i:

$\leftarrow$ contains the stiffness matrix

$\leftarrow$ contains the inertia, contact forces, geometric characteristics etc.

the residual can also be compiled from elementary calculations at t_i:

$\leftarrow$ contains the external forces, inertia effects, prescribed displacements, damping etc.

DISCONTINUOUS DEFORMATION ANALYSIS

4. Numerical solution of the equations of motion :

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t)) \quad\Rightarrow\quad \mathbf{0}=\mathbf{f}(t_{i+1},\mathbf{u}_{i+1}(\Delta\mathbf{u}_{i+1},\mathbf{v}_{i+1}(\Delta\mathbf{u}_{i+1}))-\mathbf{M}\cdot\mathbf{a}_{i+1}(\Delta\mathbf{u}_{i+1})$$

$$\boxed{\mathbf{r}(t_{i+1},\Delta\mathbf{u}_{i+1})=\mathbf{f}(t_{i+1},\mathbf{u}_{i+1}(\Delta\mathbf{u}_{i+1}),\mathbf{v}_{i+1}(\Delta\mathbf{u}_{i+1}))-\mathbf{M}\cdot\mathbf{a}_{i+1}(\Delta\mathbf{u}_{i+1})}$$

$$\mathcal{K}(t,\Delta\mathbf{u})=\frac{d\mathbf{r}(t,\Delta\mathbf{u})}{d\Delta\mathbf{u}}$$

Analysis of a time interval:

initial estimation for $\Delta\mathbf{u}_{i+1}$: $\Delta\mathbf{u}_{i+1}^{(0)}:=\mathbf{0}$

k+1-th estimation for $\Delta\mathbf{u}_{i+1}$: $\Delta\mathbf{u}_{i+1}^{(k+1)}:=\Delta\mathbf{u}_{i+1}^{(k)}-\mathcal{K}(t_{i+1},\Delta\mathbf{u}_{i+1}^{(k)})^{-1}\cdot\mathbf{r}(t_{i+1},\Delta\mathbf{u}_{i+1}^{(k)})$

then continue until $\left|\mathbf{r}(t_{i+1},\Delta\mathbf{u}_{i+1}^{k+1})\right|$ becomes sufficiently small

„Open – close iterations”: at the end of Δt: **check** the topology and the forces;

→ **modify the topology** if necessary (e.g. new contacts, sliding, contact loss)

→ with the new topology, **repeat:** Newton-Raphson to find another $\Delta\mathbf{u}_{i+1}$

if acceptable topology not found: **decrease timestep** Δt to 1/3 of its previous length

CONVERGENCE WITHIN A TIME STEP ???

DISCONTINUOUS DEFORMATION ANALYSIS

Comparison to UDEC:

Main differences from UDEC:

$\rightarrow$ basic unknowns: also the components of $\boldsymbol{\varepsilon}$;

$\rightarrow$ uniform stress and strain field inside the elements;

$\rightarrow$ numerical integration: implicit

$\rightarrow$ stiffness matrix included $\Rightarrow$ artificial damping not necessary

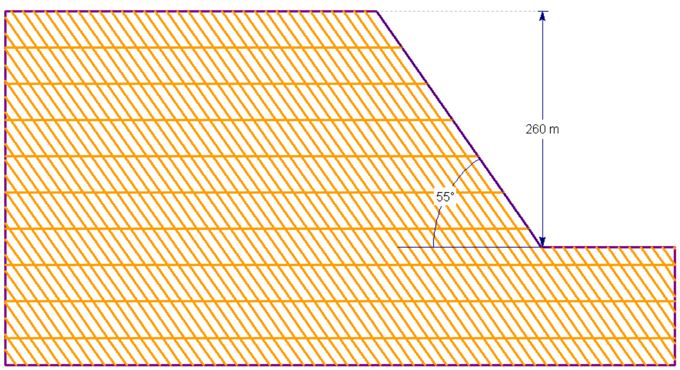
– advantages to UDEC: $\rightarrow$ implicit $\Rightarrow$ numerical stability;
fast convergence if topology does not change
no artificial damping required

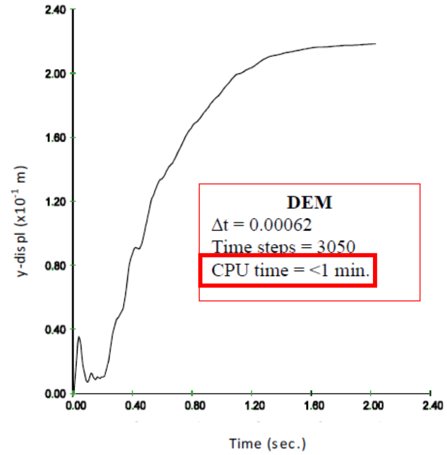
– disadvantages: no commercial software $\Rightarrow$ inconvenient
(several research codes; e.g. ask from Gen-Hua Shi)
too simple mechanics of the elements and of the contacts
large storage requirements & longer computations
open-close iterations: convergence is not ensured if topology changes

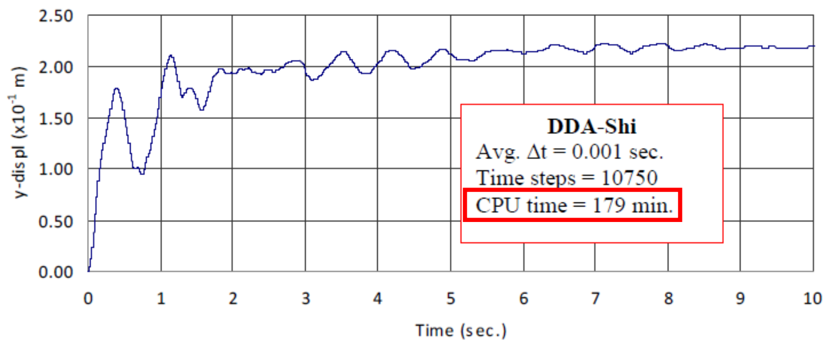
DISCONTINUOUS DEFORMATION ANALYSIS

Remark 1:

Comparison to UDEC:
M.S. Kahn (2010)





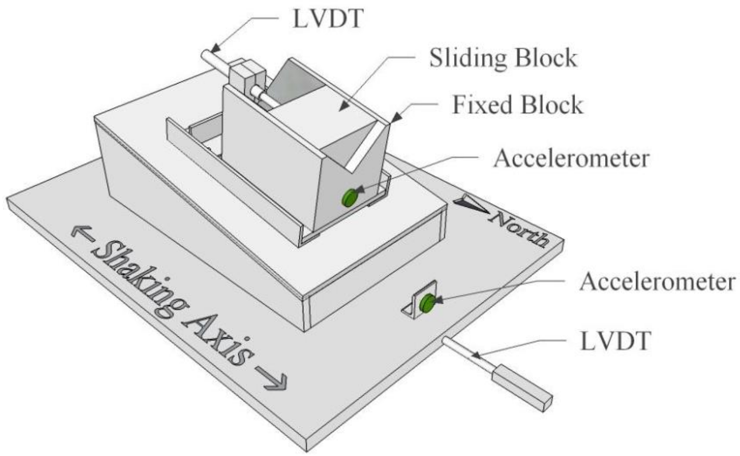


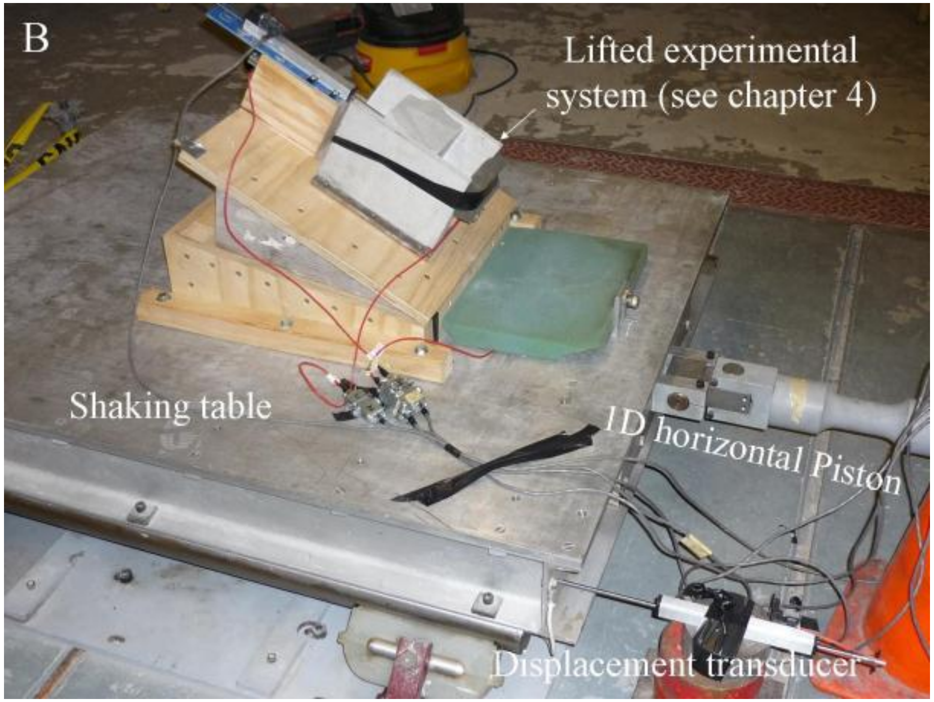
NOT EFFICIENT IN CASES IF
SIGNIFICANT TOPOLOGY MODIFICTIONS OCCUR !!!

DISCONTINUOUS DEFORMATION ANALYSIS

Remark 2: Importance of **experimentally verified** material parameters

e.g. D.B. Mazor (2011): in DDA-3D,
Aim: Simulate shaking table experiments
What is the proper friction angle?



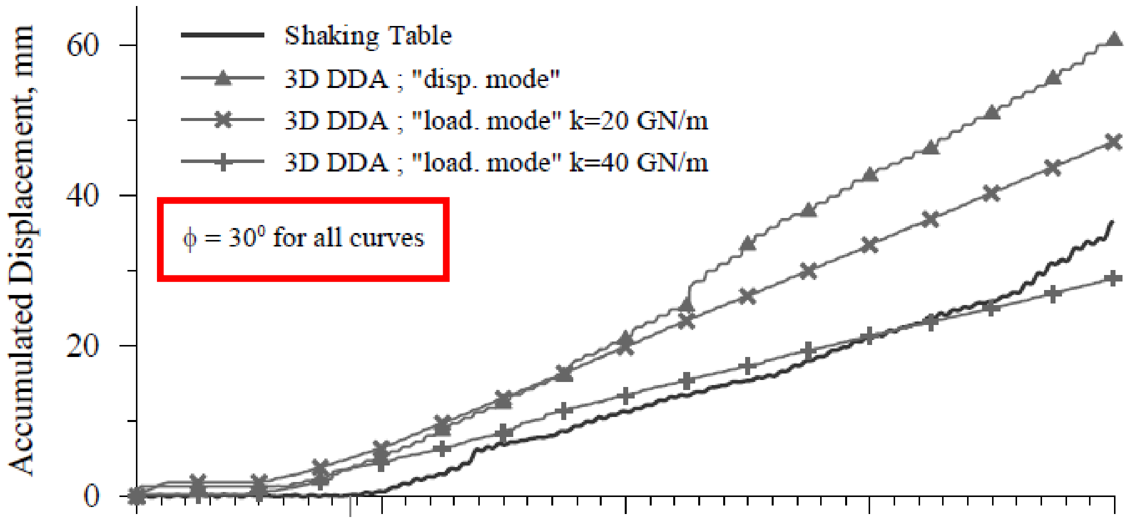


measurement of friction angle:
$\varphi = 36°$

DISCONTINUOUS DEFORMATION ANALYSIS

Remark 2: <u>Importance of **experimentally verified** material parameters</u>

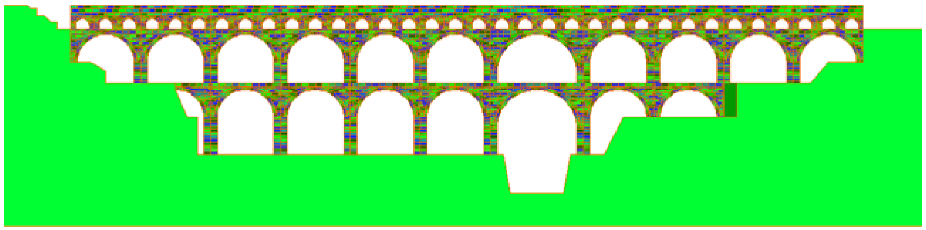
e.g. D.B. Mazor (2011): in DDA-3D,
Aim: Simulate shaking table experiments
What is the proper friction angle?



⇒ experience: „friction degradation" during dynamic sliding
⇒ importance of contact penalty stiffness

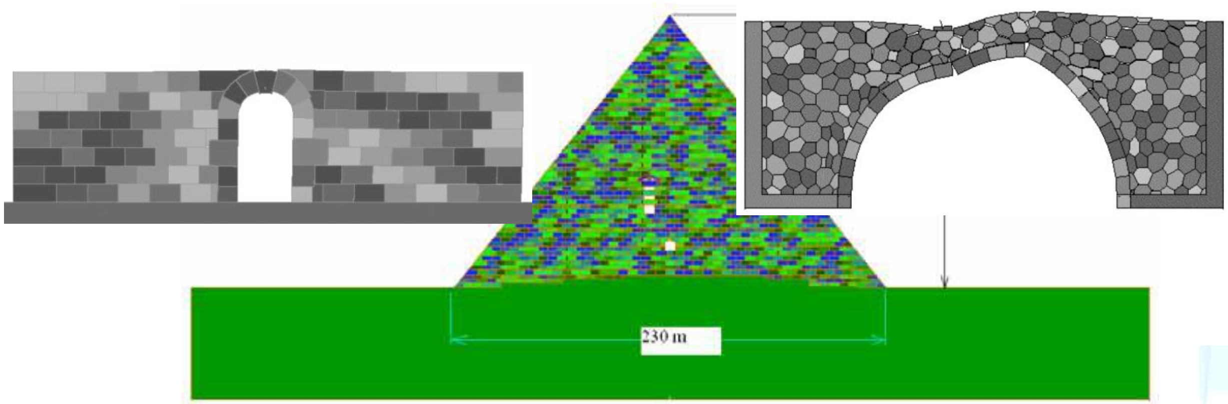
DISCONTINUOUS DEFORMATION ANALYSIS

<u>Applications</u>



arches, arch bridges

underground
excavations



masonry structures

e.g. Fractured rock for earthquake and thermal effects

DISCONTINUOUS DEFORMATION

Y.H.Hatzor, 1999-2004:
Masada-hill, Israel (King Herod’s Palace)

Herod the Great built his fortress a hundred kilometers away from Jerusalem, partly as a retreat, but more as a place of refuge for himself.

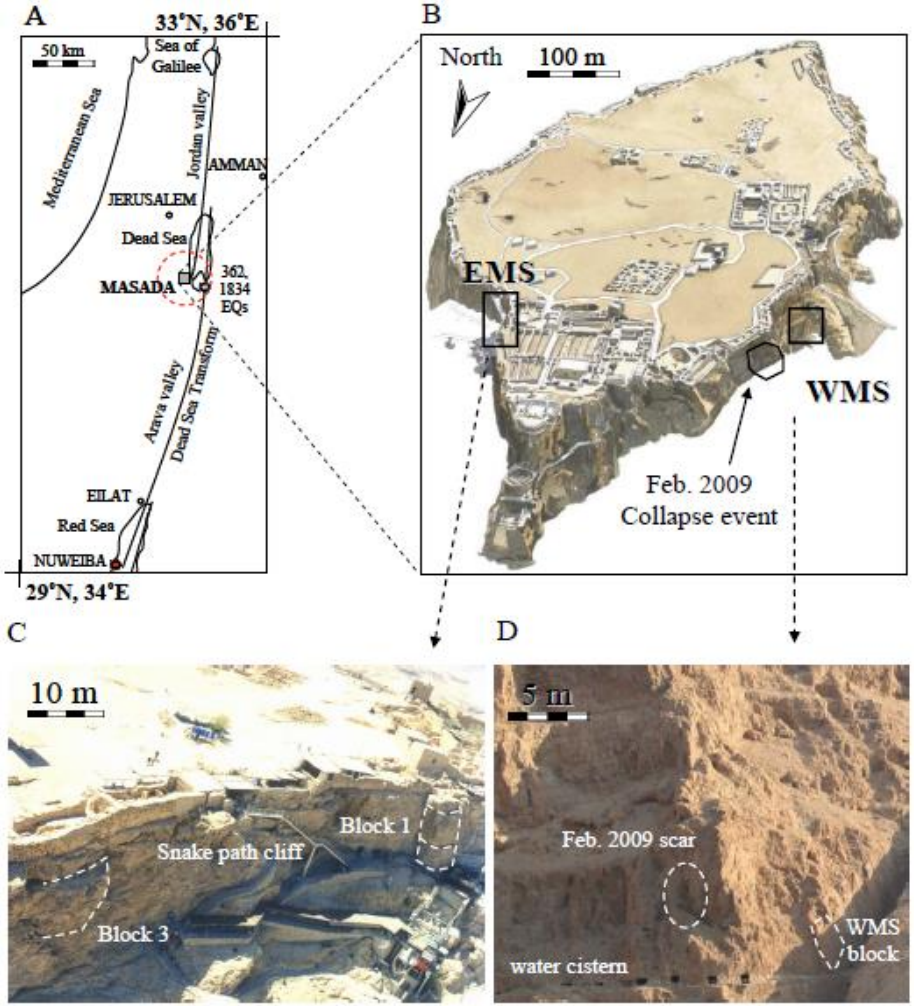
It happened at Masada seventy years after Herod's death that the Jewish Revolt against the Roman occupation was well underway. As the story goes, a last brave group of Jewish rebels fled Jerusalem to Masada, and withstood a two-year long siege by a 15,000-strong Roman Army. The crafty Romans finally built a ramp up the mountain slope.

On their last night in Masada before the Romans broke through, the 967 rebels opted for mass suicide rather than surrender.

„**Masada must not fall again**”

DISCONTINUOUS DEFORMATION ANALYSIS

Y.H.Hatzor, 1999-2004:
Masada-hill, Israel (King Herod's Palace)



DISCONTINUOUS DEFORMATION ANALYSIS

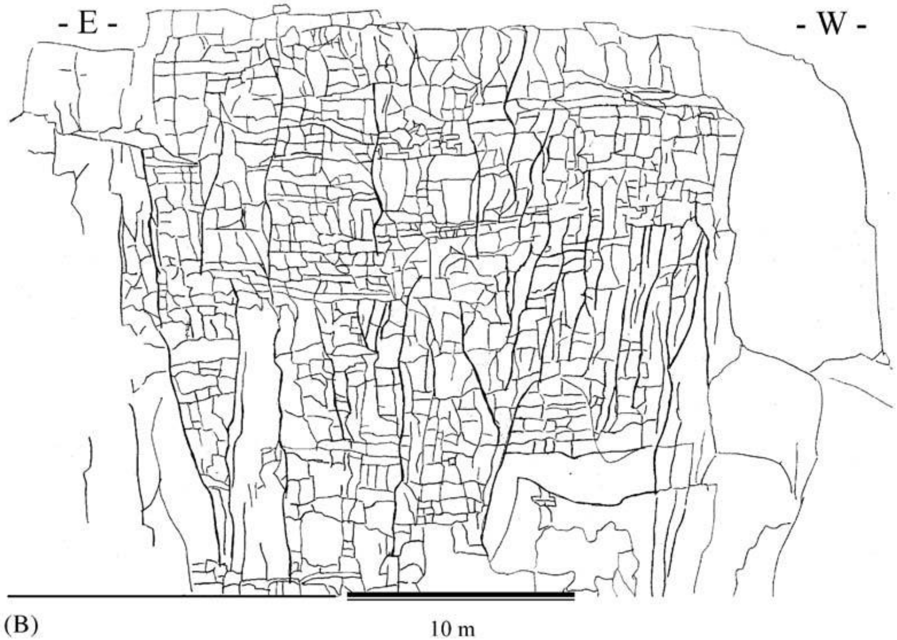
Y.H.Hatzor, 1999-2004:
Masada-hill, Israel (King Herod's Palace)

<u>experience:</u> huge blocks released, or already fallen out

<u>the aim of the analysis:</u> is it stable for an earthquake? „Masada must not fall again"

– DDA model, earthquake simulation;

(A)



(B)

DISCONTINUOUS DEFORMATION ANALYSIS

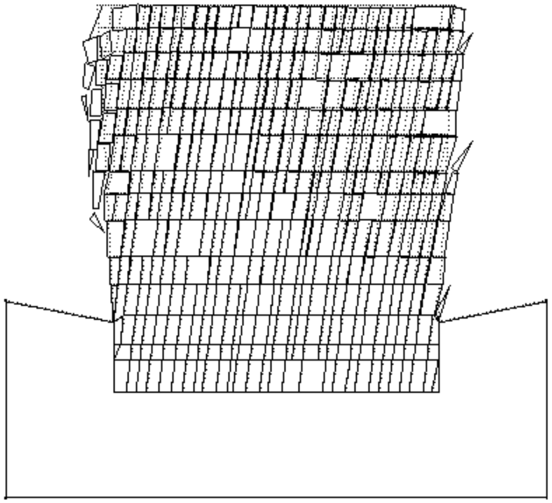
Y.H.Hatzor, 1999-2004:
Masada-hill, Israel (King Herod's Palace)

<u>experience:</u> huge blocks released, or already fallen out

<u>the aim of the analysis:</u> is it stable for an earthquake? „Masada must not fall again"

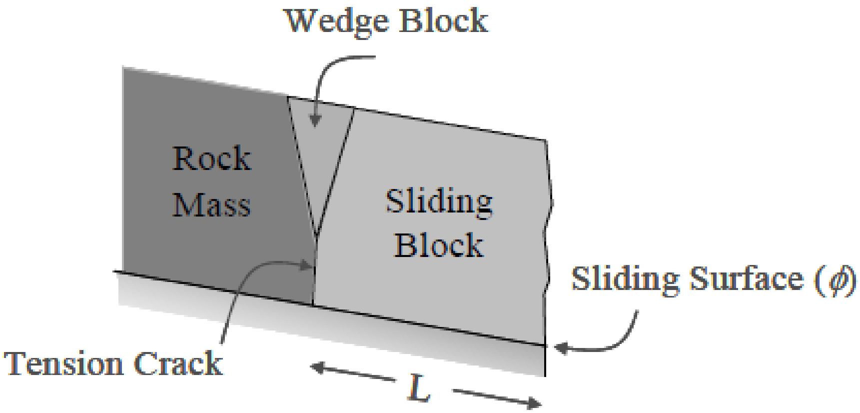
- DDA model, earthquake simulation;
- minor earthquake: considerable damages can be expected;
- 7,1 earthquake (1995, Sinai-desert): huge blocks would fall out

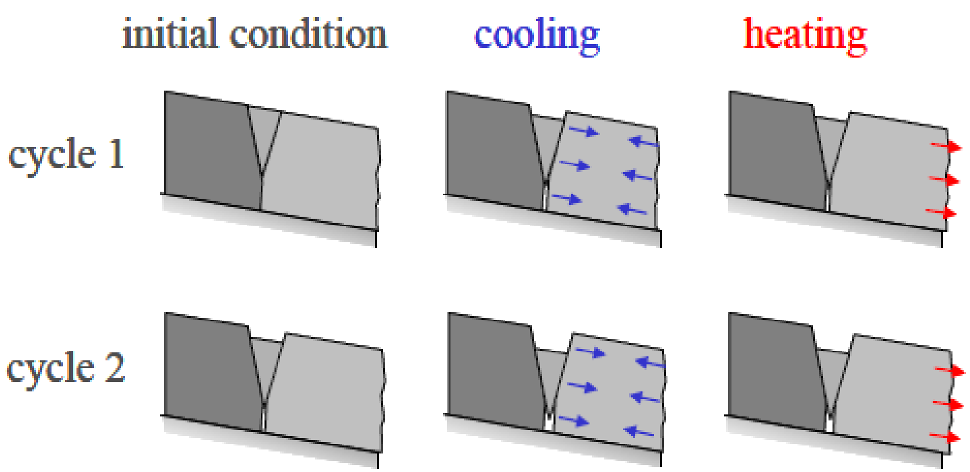
⇒ <u>proposals for strengthening</u>

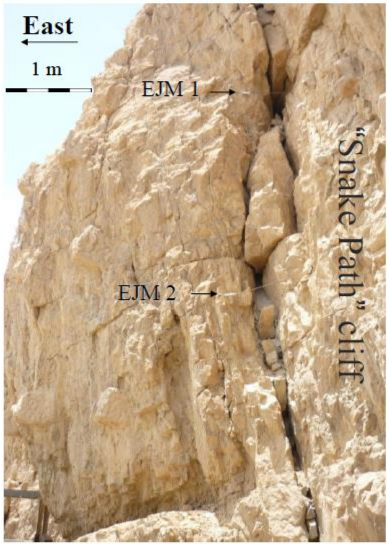


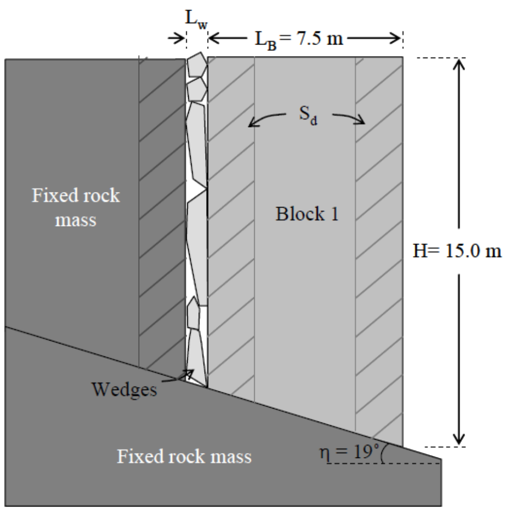
DISCONTINUOUS DEFORMATION ANALYSIS

Mazor (2011): 3D investigations









QUESTIONS

1. How the displacement vector of an element looks like in DDA? (i.e., what are the degrees of freedom of an element in DDA?)

2. Introduce the mechanical model of the contacts in DDA.

3. How does the equation of motion looks like in DDA? Write it, and tell the name of the terms in it.

4. What kind of time integration method is used in DDA? What values of the numerical control parameters is chosen in it, and what is the mechanical meaning of this choice?

5. Explain the meaning of the expression "open-close iteration" and how it is used in DDA.

6. Tell five differences between DDA and 3DEC.