SHELL THEORIES

Citation:

K. Bagi (2024): Mechanics of Masonry Structures. Course handouts, Department of Structural Mechanics, Budapest University of Technology and Economics

In case of any question or problem, do not hesitate to contact Prof. K. Bagi, kbagi.bme@gmail.com .

This Lecture: Shell Theories

Definition of shells and the shell element

The most important shell theories

- Membranes; soap films
- Kirchhoff-Love shells („classical shells”, „shell bending”)
- Reissner-Mindlin shells („linear shear deformation theory”)
- Third-order shear deformation theory

Membrane solution: Examples

- Repetition from Maths: Principal curvatures at a surface point
- Membrane solution for spherical domes
- Membrane solution for fan vaults

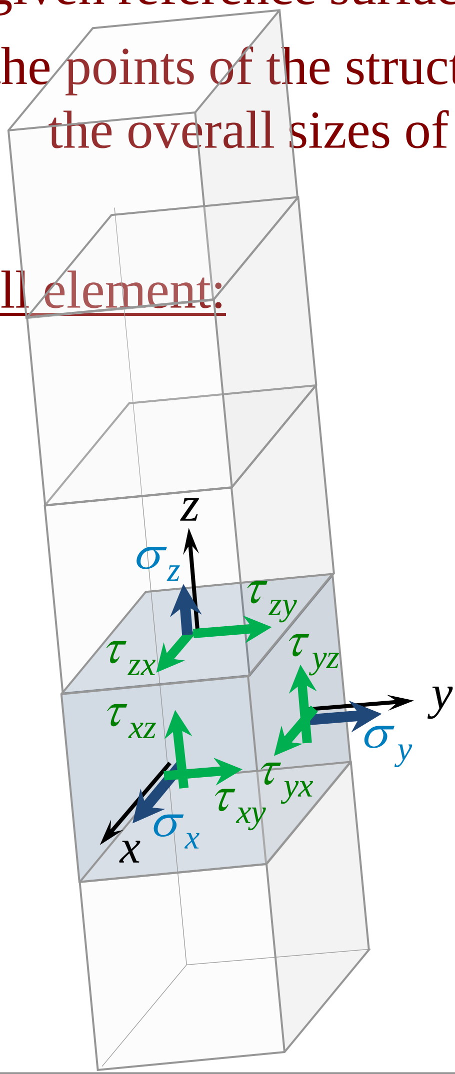
Questions

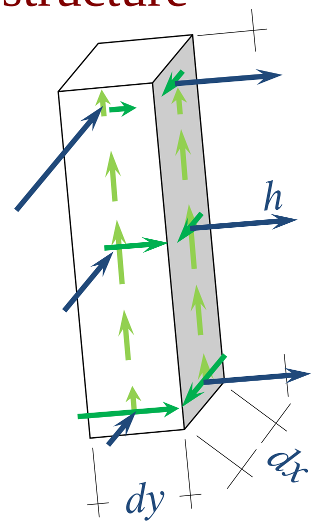
Definition of shells

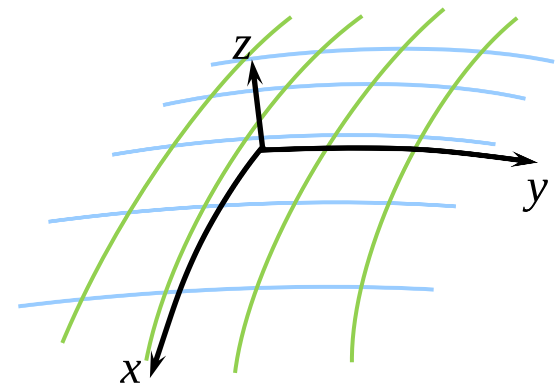
→ given reference surface;

→ the points of the structure are within a small distance from it, compared to the overall sizes of the structure

Shell element:







Note: top and bottom elementary volume:

$$\tau_{zx} = 0 \; ; \; \tau_{zy} = 0 \quad \text{(free surface)}$$

⇒ at the intrados and extrados:

$$\tau_{xz} = 0 \; ; \; \tau_{yz} = 0$$

Definition of shells

→ given reference surface;

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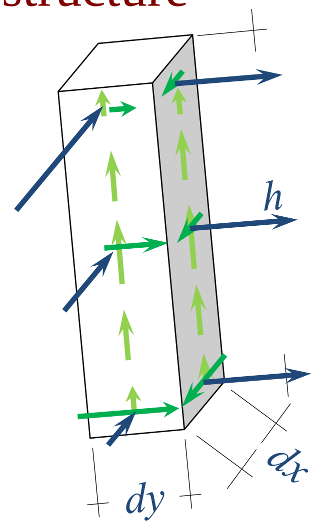

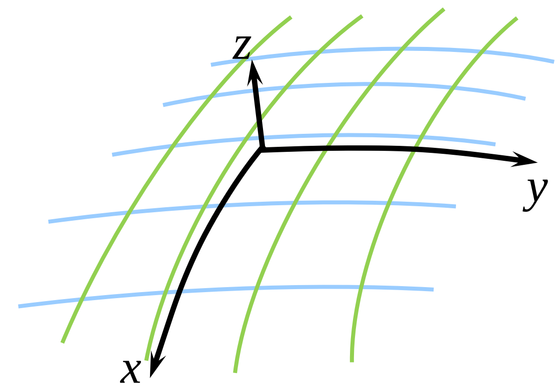

Shell element:

The most common models depending on the thickness:

→ membrane

→ Kirchhoff – Love shell („classical ” shell theory)

→ Mindlin – Reissner shell („linear shear deformation theory”)

→ third-order shear deformation model

→ thick shells: 3D continuum or DEM

increasing h

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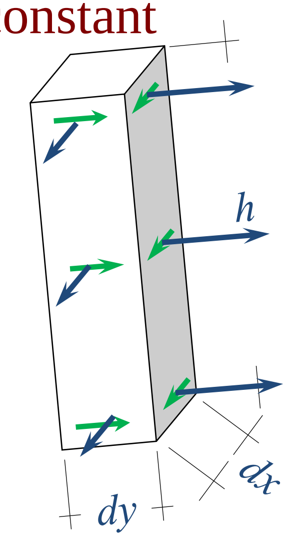
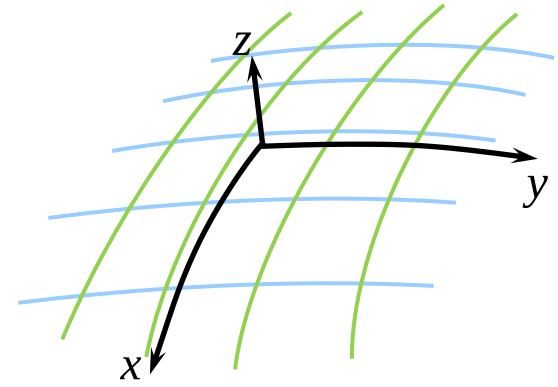
Membranes:

→ stresses:

along h: σ_x, σ_y and τ_{xy} are constant

σ_z, τ_{xz} and τ_{yz} are zero

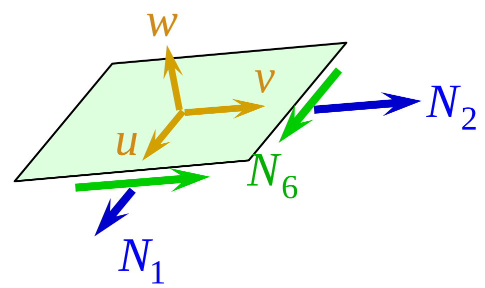




→ displacements:

translations of the points of the reference surface:

$$u(x,y),\ v(x,y),\ w(x,y)$$

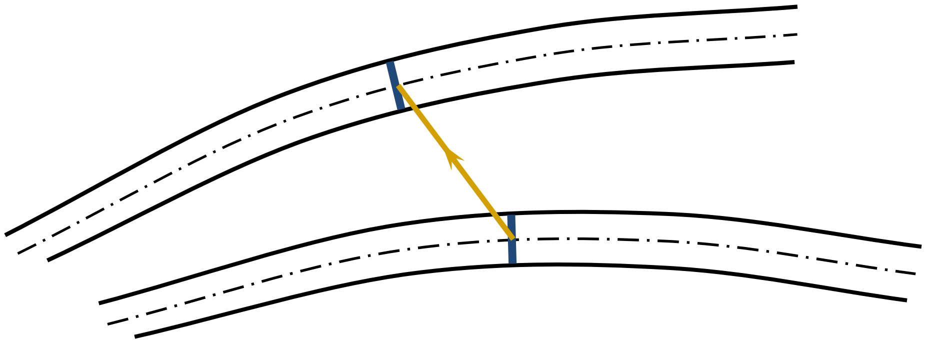


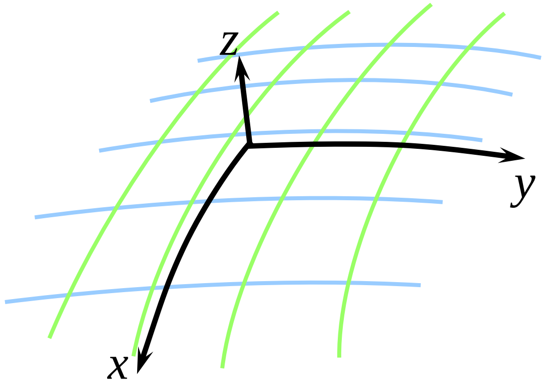
translations of the other points:

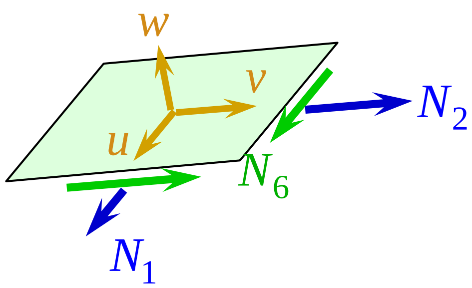
$$u_1(x,y,z)=u(x,y),\ u_2(x,y,z)=v(x,y),\ u_3(x,y,z)=w(x,y)$$

The most important shell theories

Membranes:







→ displacements :

translations of the points of the reference surface :

$u(x,y)$, $v(x,y)$, $w(x,y)$

translations of the other points:

$u_1(x,y,z)=u(x,y)$, $u_2(x,y,z)=v(x,y)$, $u_3(x,y,z)=w(x,y)$

The most important shell theories

Membranes:

Special case: „SOAP FILMS”

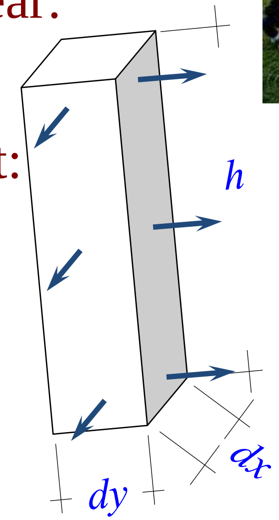
if the material cannot carry shear:

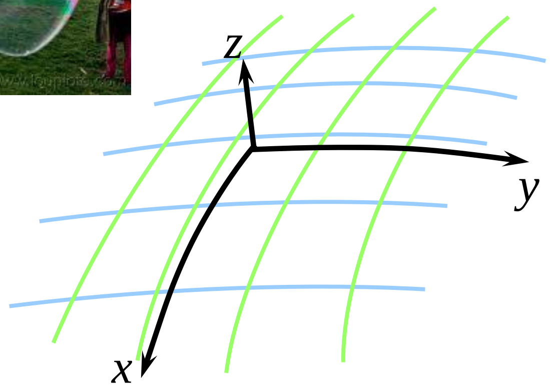
→ stresses:

along h: σ_x, σ_y, τ_{xy}: constant:

σ_x and σ_y are equal, $\tau_{xy} = 0$

σ_z, τ_{xz}, τ_{yz} zero





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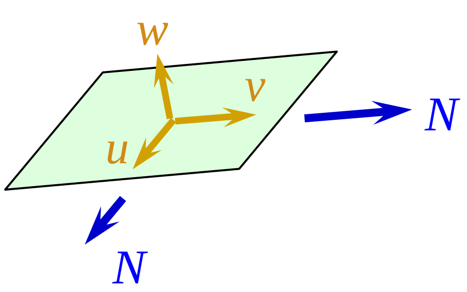

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Membrane solution: Examples

- Repetition from Maths: Principal curvatures at a surface point
- Membrane solution for spherical domes
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Questions

The most important shell theories

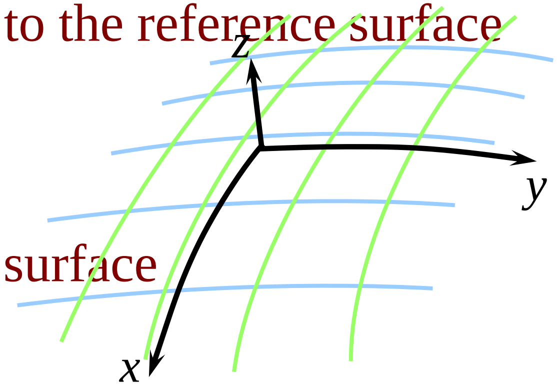
Kirchhoff – Love – shells: „classical shells", „shell bending"

Kirchhoff-hypothesis:

the planar cross sections being perpendicular to the reference surface will, after deformation, remain

→ planar, and

→ perpendicular to the (deformed) reference surface



Basic assumptions of Love:

(1) the shell is thin

(2) translations and rotations are small

(3) straight material lines being orthogonal to the reference surface will remain orthogonal also after deformation

(4) crosswise shear stresses are negligible

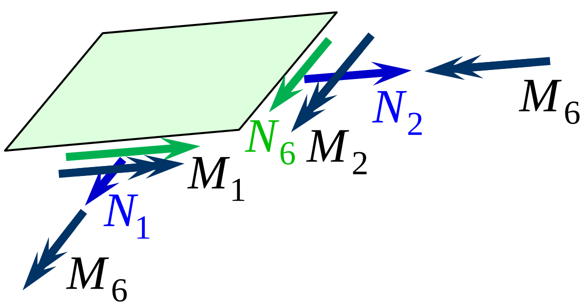
The most important shell theories

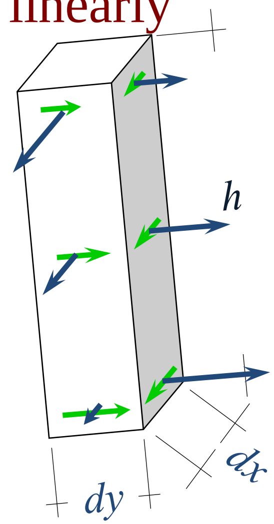
Kirchhoff – Love – shells: „classical shells”, „shell bending”

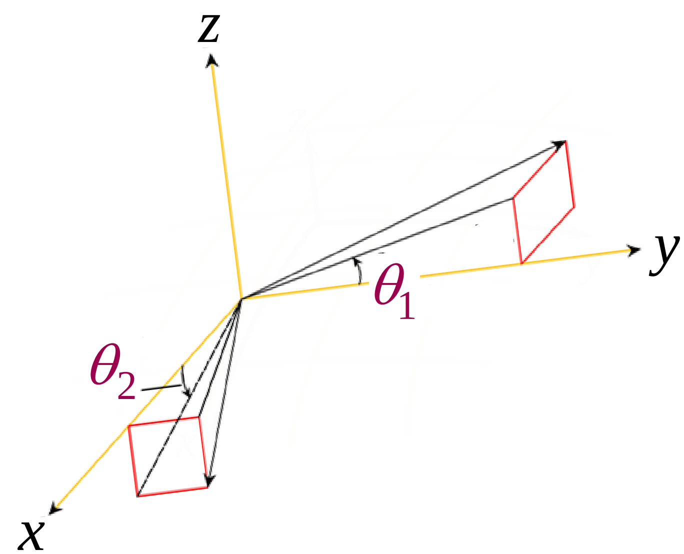
→ stresses:

along h : σ_x, σ_y and τ_{xy} vary linearly

σ_z, τ_{xz} are τ_{yz} ***zero***



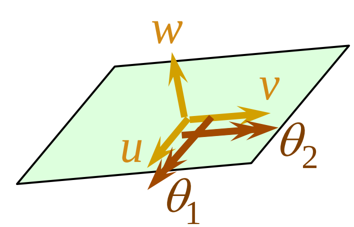




→ displacements:

points of the reference surface:

$$u(x,y),\ v(x,y),\ w(x,y),\quad \Rightarrow \theta_1(x,y),\ \theta_2(x,y)$$



translations of other points:

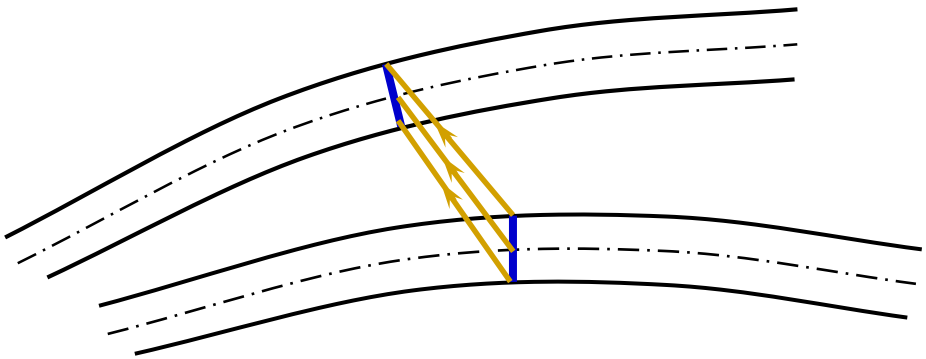
$$u_1(x,y,z) = u(x,y) + z\cdot\theta_2$$

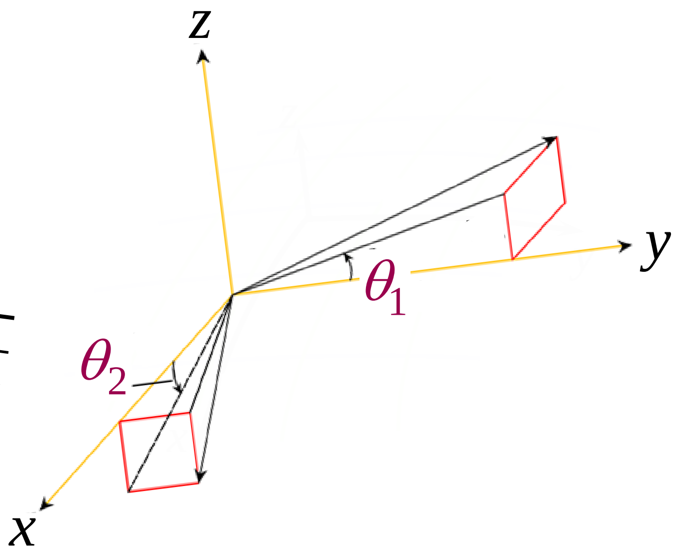
$$u_2(x,y,z) = v(x,y) - z\cdot\theta_1$$

$$u_3(x,y,z) = w(x,y)$$

The most important shell theories

Kirchhoff – Love – shells: „classical shells”, „shell bending”





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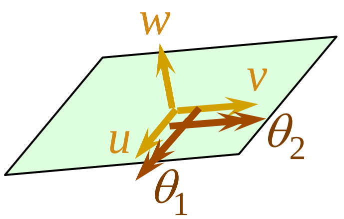

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The most important shell theories

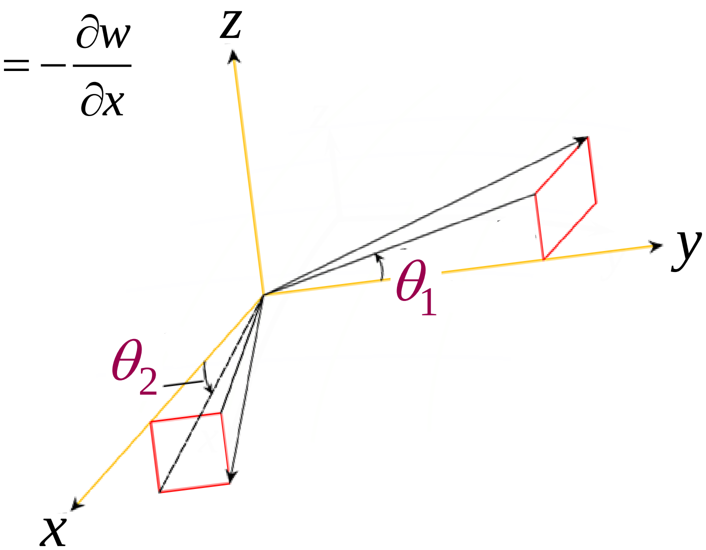
<u>Kirchhoff – Love – shells:</u> „classical shells”, „shell bending”

Remark:

for planar reference surface: $\theta_1 = \frac{\partial w}{\partial y}$; $\theta_2 = -\frac{\partial w}{\partial x}$

for curved reference surface:

θ_1 and θ_2 can uniquely be determined from the curvatures and from $\frac{\partial w}{\partial y}$; $\frac{\partial w}{\partial x}$



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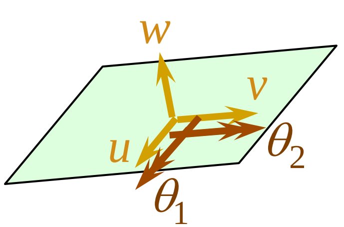

translations of other points: $u_1(x,y,z) = u(x,y) + z \cdot \theta_2$

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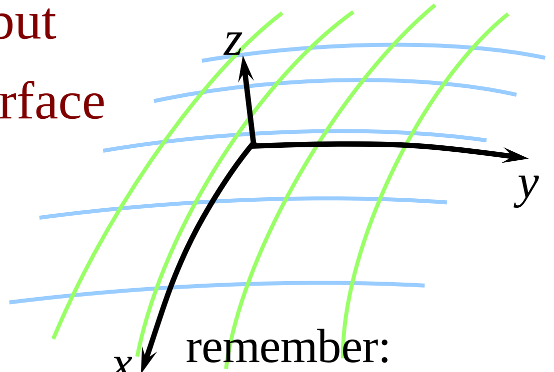
<u>Reissner – Mindlin – shells:</u> „linear shear deformation theory"

planar sections perpendicular to the reference surface will, after deformation,

→ remain planar (linear wharping function), but

→ not perpendicular to the (deformed) ref. surface



crosswise shear effects ***are*** taken into account!

remember:

$$\theta_1 = \frac{\partial w}{\partial y} + \ldots ; \theta_2 = -\frac{\partial w}{\partial x} + \ldots$$

→ displacements:

points of the reference surface:

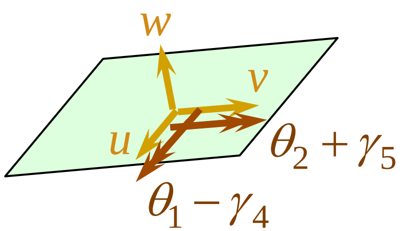
$u(x,y)$, $v(x,y)$, $w(x,y)$, $\Rightarrow \theta_1(x,y)$, $\theta_2(x,y)$; $\gamma_4(x,y)$, $\gamma_5(x,y)$

translations of other points: $g(z) := z$

$$u_1(x,y,z) = u(x,y) + z \cdot \theta_2 + g(z) \cdot \gamma_5$$

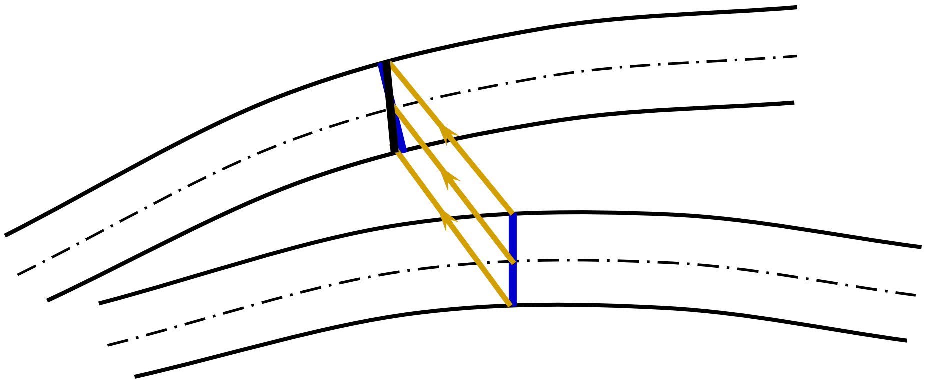
$$u_2(x,y,z) = v(x,y) - z \cdot \theta_1 + g(z) \cdot \gamma_4$$

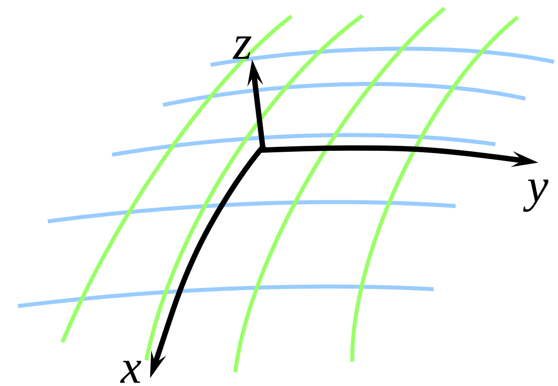
$$u_3(x,y,z) = w(x,y)$$



The most important shell theories

<u>Reissner – Mindlin – shells:</u> „linear shear deformation theory”





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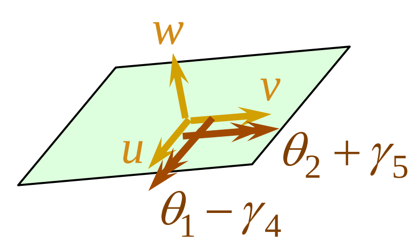

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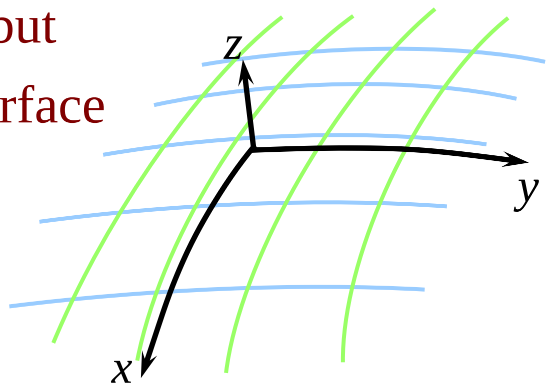
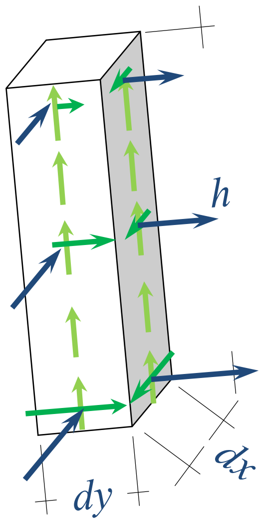


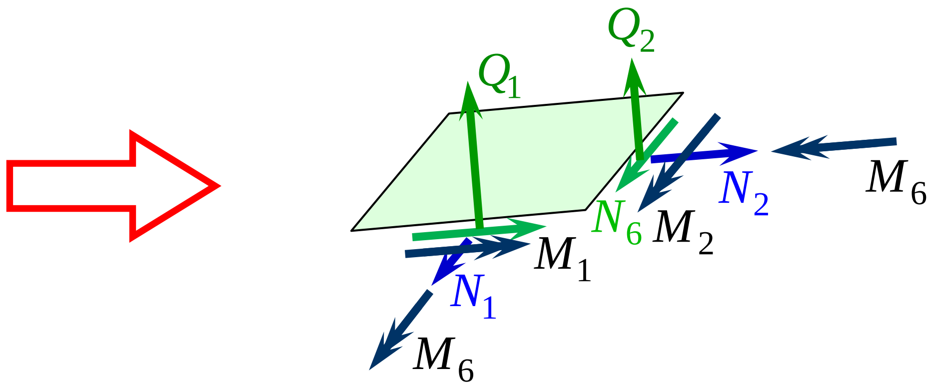
crosswise shear effects taken into account!

→ stresses: along h: σ_x, σ_y and τ_{xy} vary linearly

σ_z is zero

τ_{xz} and τ_{yz} : linear for curved surface (constant for planar shell)





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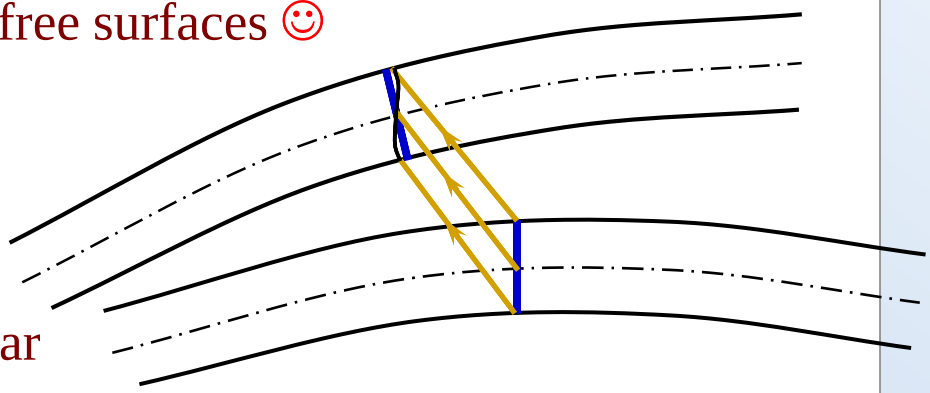
Third-order shear deformation theory:

$$g(z) := z - \frac{4z^3}{3h^2}$$

→ stresses:

τ_{xz} and τ_{yz} : able to be zero on the free surfaces ☺



→ displacements:

like before, only $g(z)$ is modified

⇒ planar sections do not remain planar

$$u_1(x,y,z) = u(x,y) + z \cdot \theta_2 + g(z) \cdot \gamma_5$$

$$u_2(x,y,z) = v(x,y) - z \cdot \theta_1 + g(z) \cdot \gamma_4$$

$$u_3(x,y,z) = w(x,y)$$

Remark: Thick shells:

most often: continuum models,

e.g. FEM: 3D elements (one layer or multi-layered)

masonry vaults: Discrete Element Method

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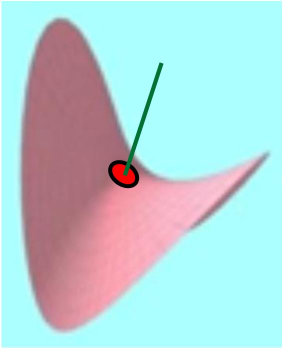
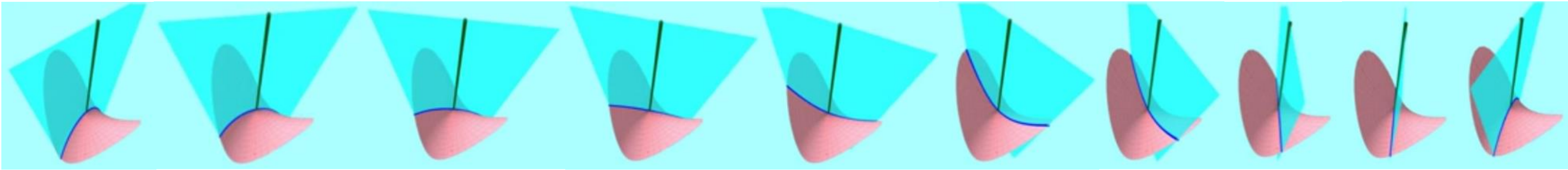
Repetition from Maths: Principal curvatures



Principal curvatures at a point of a surface:

- → draw a straight axis;
- → lay a plane along this axis ⇒ intersection along a curve;
- → rotate the plane and produce these curves:

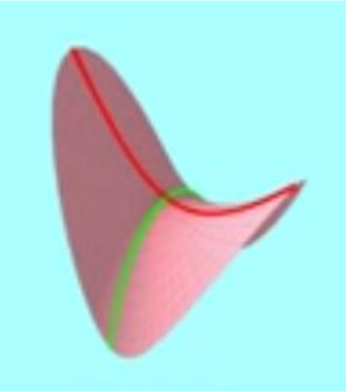


https://www.youtube.com/watch?v=HUzOPbZk8Pg

- → take that two curves having largest / smallest curvature:

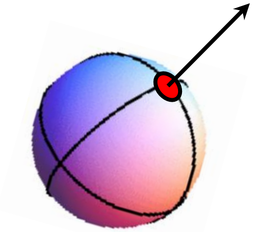
radii of curvature at the point:

R_{min} and R_{max} are received



for any point on a sphere:

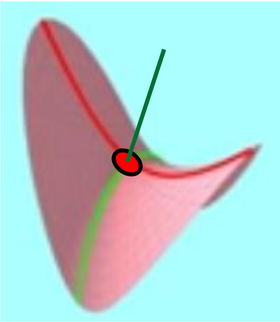
$$R_{min} = R_{max} = R$$



https://slideplayer.com/slide/8958879/

Repetition from Maths: Principal curvatures

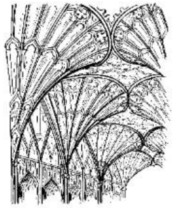
Product of the two principal curvatures: „Gaussian curvature”



→ if it is (+) : „elliptical point”

→ if it is (–) : „hyperbolic point”

→ if it is 0 : „parabolic point”



Its importance in the mechanical behaviour:

> „… *shells of nonpositive Gaussian curvature are generally speaking weak structures, sensitive to disturbances at the boundary, which tend to penetrate deep into the structure. Also, they have no possibility of supporting concentrated forces in the membrane state, in contrast with shells of positive Gaussian curvature, where the membrane state will provide an approximately correct result at sufficient distance from the point where a concentrated force is applied.*” (Niordson, 1985)

„a local problem propagates to global collapse”

https://www.youtube.com/watch?v=vAavRx7uoeA Ph. Block, from 13:21

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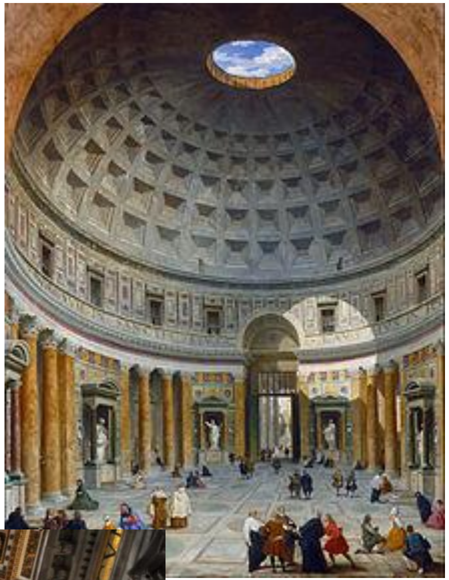
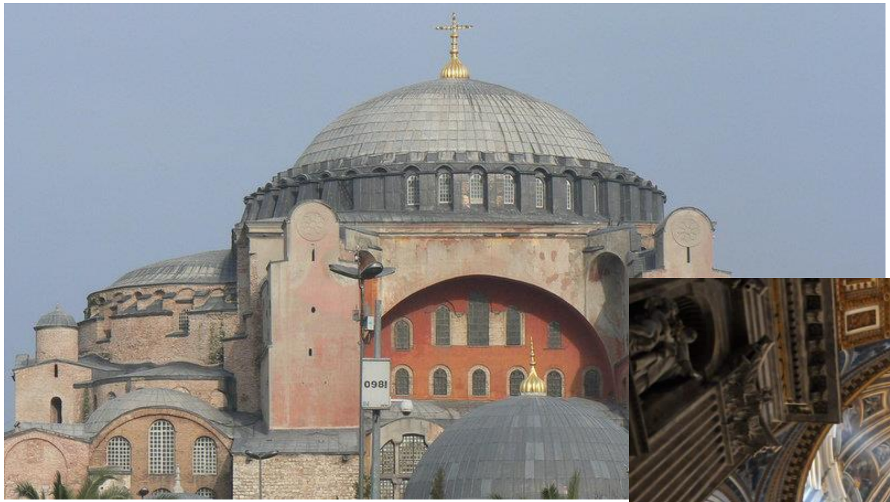


Pantheon



Hagia Sophia

St Peter’s Basilica

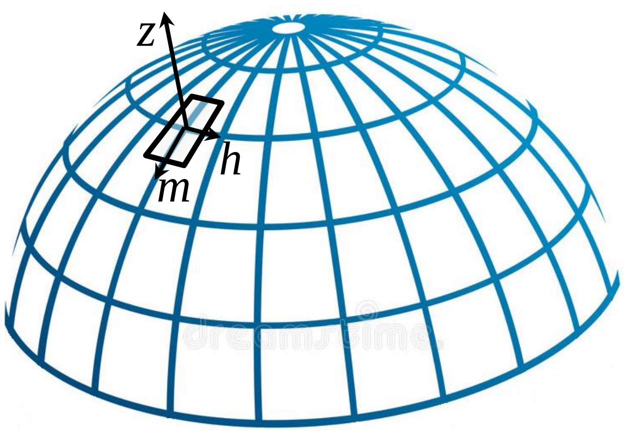
Membrane solution for spherical domes

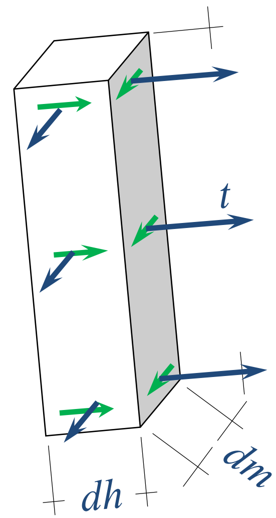
Notations:

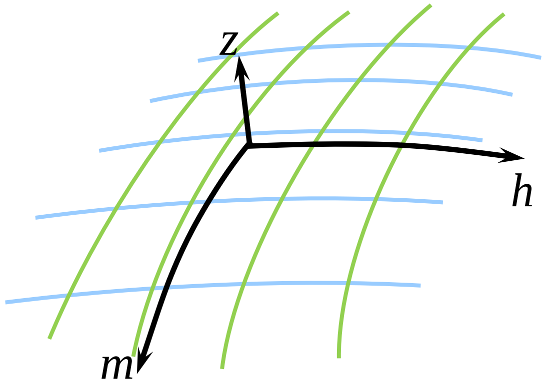
m: meridional direction

h: hoop direction

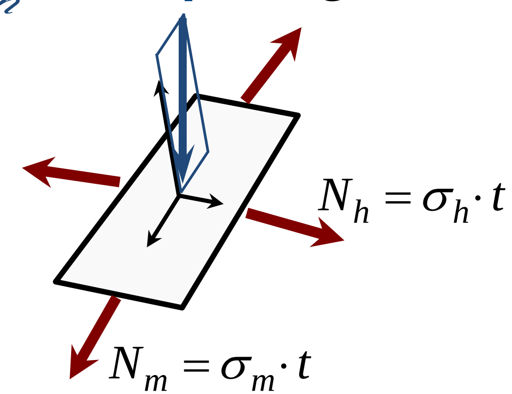
z: normal direction







q : weight for unit area



→ load: selfweight, only in vertical direction

→ stresses:

along *t*: σ_m, σ_h are constant

σ_z , τ_{mh} , τ_{mz} and τ_{hz} are zero

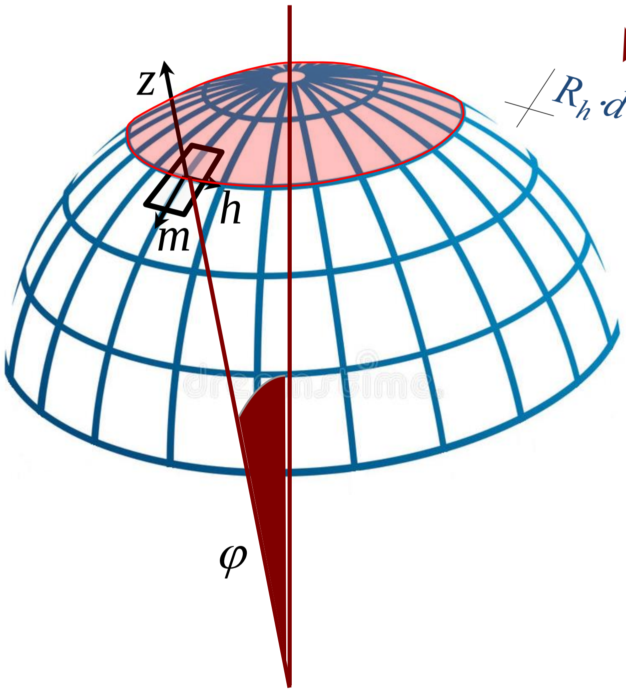
Membrane solution for spherical domes

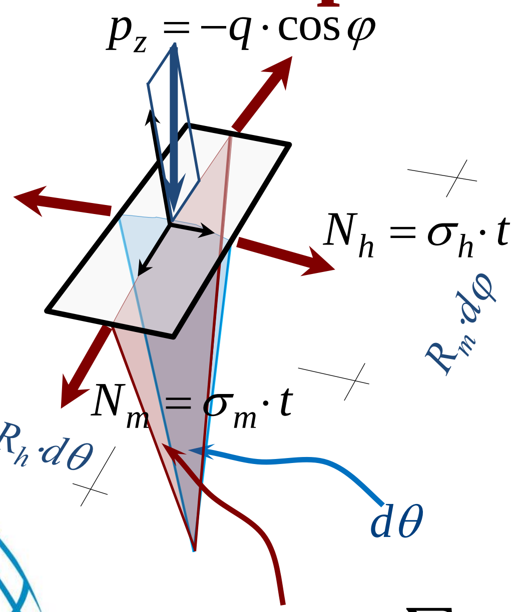
Notations:

m: meridional direction

h: hoop direction

z: normal direction





$\sum F_{iz}$: [in normal dir.]

$$2\left(N_h \sin\frac{d\theta}{2}\right)\cdot R_m d\varphi + 2\left(N_m \sin\frac{d\varphi}{2}\right)\cdot R_h d\theta + p_z \cdot R_m d\varphi \cdot R_h d\theta = 0$$

$$\frac{N_h}{R_h} + \frac{N_m}{R_m} = -q\cdot\cos\varphi \qquad N_m = ?$$

$\sum F_{i,vertical}$ for the cap:

$$\left(N_m \sin\varphi\right)\cdot 2\pi\left(R_m \cdot \sin\varphi\right) + q\cdot\left(A_{cap}\right) = 0$$

$\leftarrow$ for sphere: $A_{cap} = 2\pi R^2 (1-\cos\varphi)$

$$N_m = -\frac{q\cdot R_m(1-\cos\varphi)}{\sin^2\varphi} = -\frac{q\cdot R_m(1-\cos\varphi)}{1-\cos^2\varphi}$$

$$N_h = -R_h \cdot q\left(\cos\varphi - \frac{1}{(1+\cos\varphi)}\right)$$

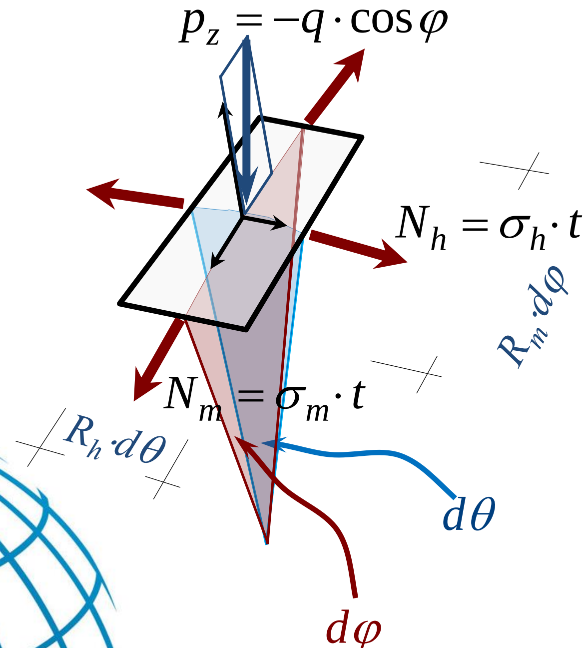
Membrane solution for spherical domes

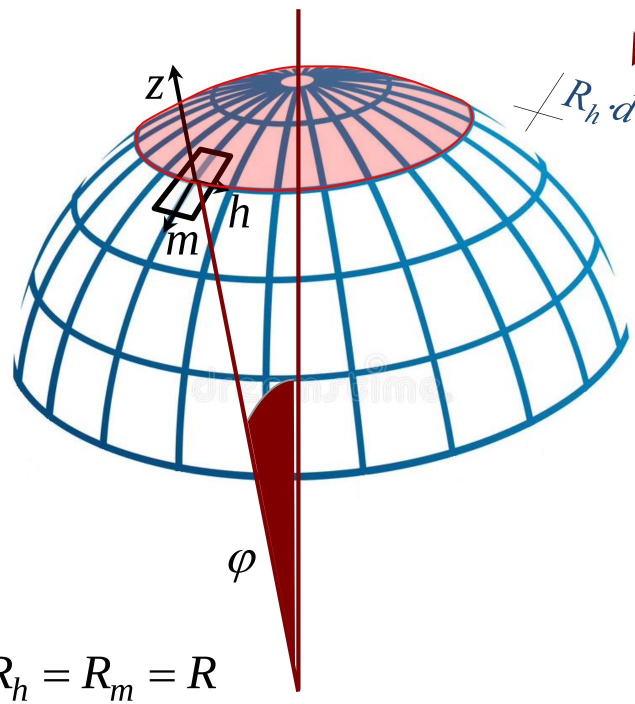
Notations:

m: meridional direction

h: hoop direction

z: normal direction





$R_h = R_m = R$

$$\sigma_m = \frac{N_m}{t} = -\frac{q}{t} R \frac{1}{1+\cos\varphi}$$

$$\sigma_h = \frac{N_h}{t} = $$
$$= -\frac{q}{t} R \left(\cos\varphi - \frac{1}{(1+\cos\varphi)} \right)$$

$$N_m = -\frac{q \cdot R_m (1-\cos\varphi)}{\sin^2\varphi} = -\frac{q \cdot R_m (1-\cos\varphi)}{1-\cos^2\varphi}$$

$$N_h = -R_h \cdot q \left(\cos\varphi - \frac{1}{(1+\cos\varphi)} \right)$$

Membrane solution for spherical domes

<u>Notations:</u>

m: meridional direction

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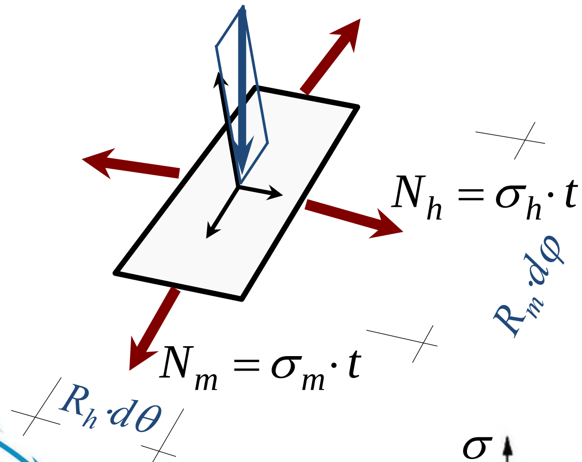
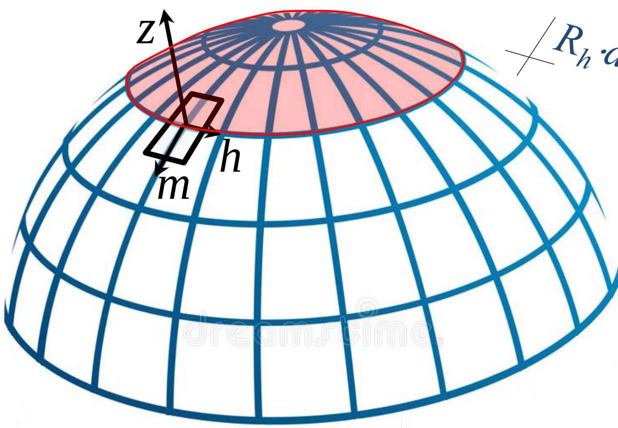
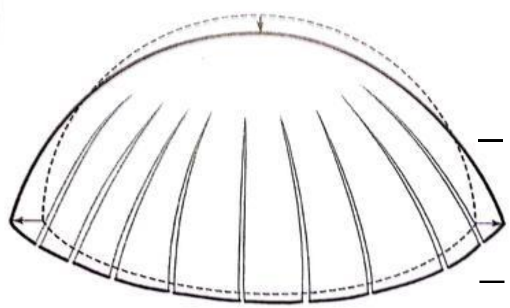
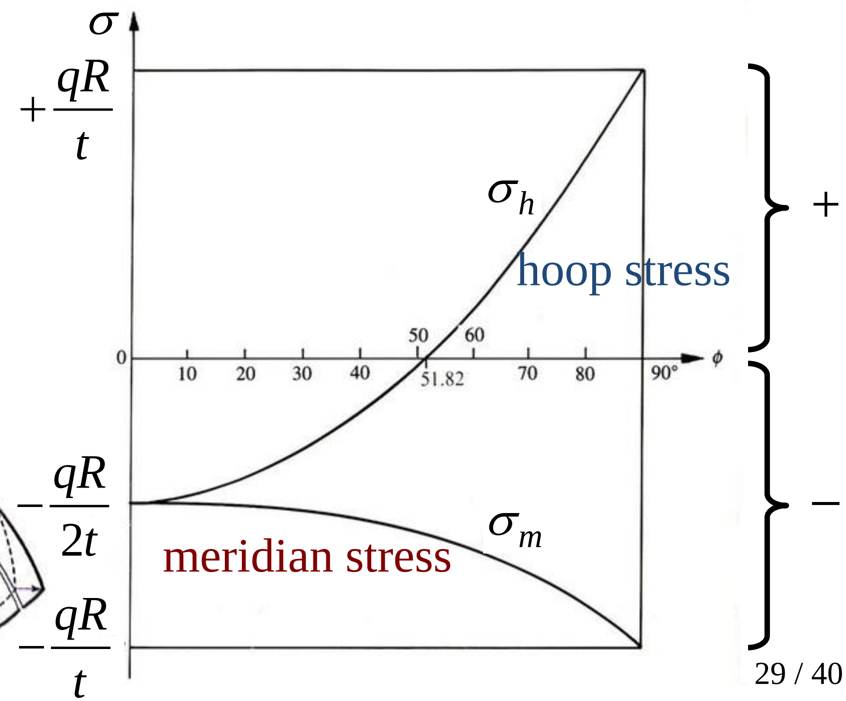




$$\sigma_m = \frac{N_m}{t} = -\frac{q}{t} R \frac{1}{1+\cos\varphi}$$

$$\sigma_h = \frac{N_h}{t} = -\frac{q}{t} R \left(\cos\varphi - \frac{1}{(1+\cos\varphi)} \right)$$





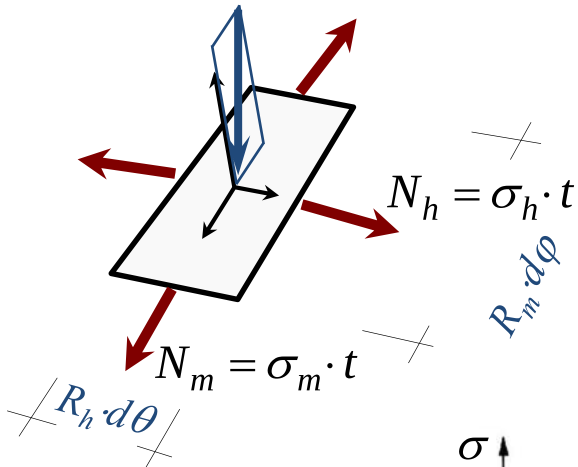
Membrane solution for spherical domes

<u>Notations:</u>

m: meridional direction

h: hoop direction

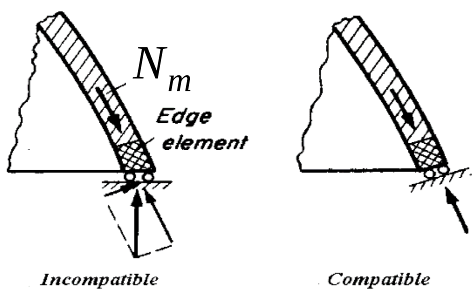
z: normal direction

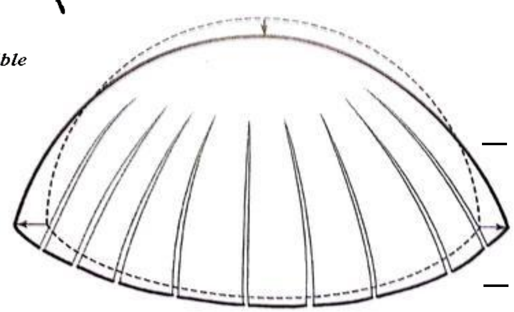


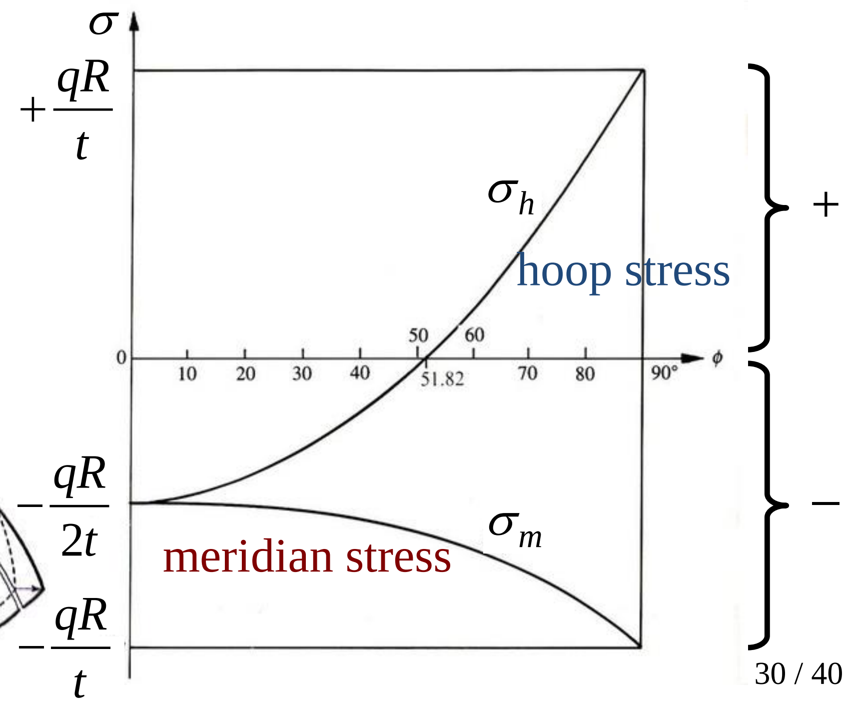
$$\sigma_m = \frac{N_m}{t} = -\frac{q}{t} R \frac{1}{1+\cos\varphi}$$

$$\sigma_h = \frac{N_h}{t} = -\frac{q}{t} R \left(\cos\varphi - \frac{1}{(1+\cos\varphi)} \right)$$

Importance of boundaries:







This Lecture: Shell Theories

Definition of shells and the shell element

The most important shell theories

- Membranes; soap films
- Kirchhoff-Love shells („classical shells”, „shell bending”)
- Reissner-Mindlin shells („linear shear deformation theory”)
- Third-order shear deformation theory

Membrane solution: Examples

- Repetition from Maths: Principal curvatures at a surface point
- Membrane solution for spherical domes
- Membrane solution for fan vaults

Questions

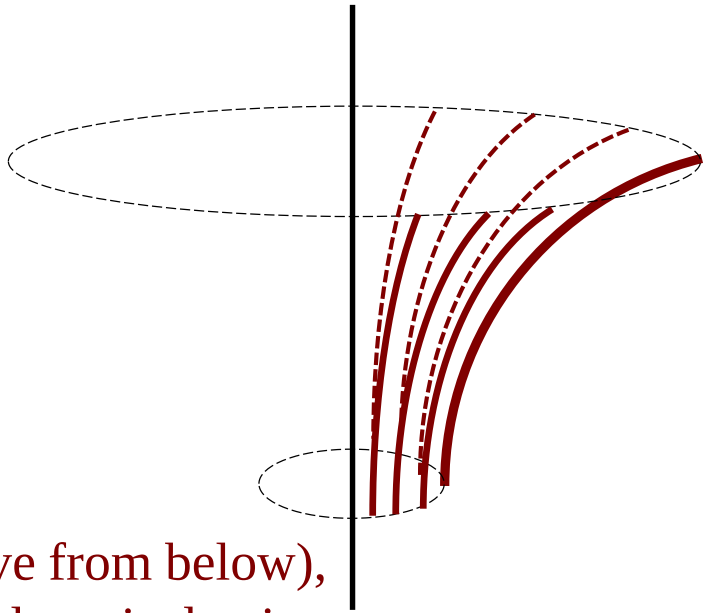
Membrane solution for fan vaults

Gloucester Cathedral cloister walk, slideshare.net/michaelasanda/gloucester-cathedral2

Bath Abbey, quillcards.com/blog/bath-abbey/

King's College Chapel, quora.com/Are-there-any-buildings-with-fanned-vaulting-outside-of-the-US-UK-and-Canada

Membrane solution for fan vaults



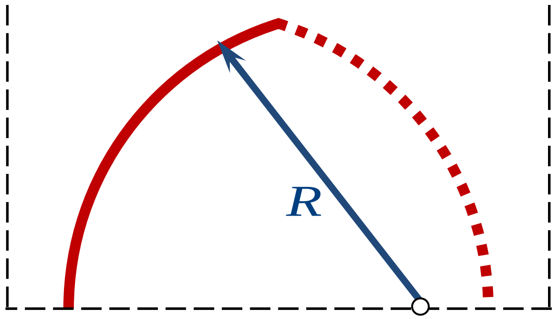
What is a fan vault?

Middle surface:

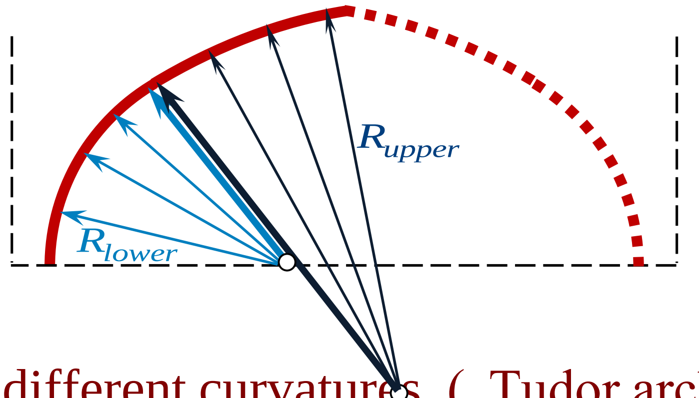
→ generator curve (concave from below),

→ rotated about an external vertical axis

The generator curve:

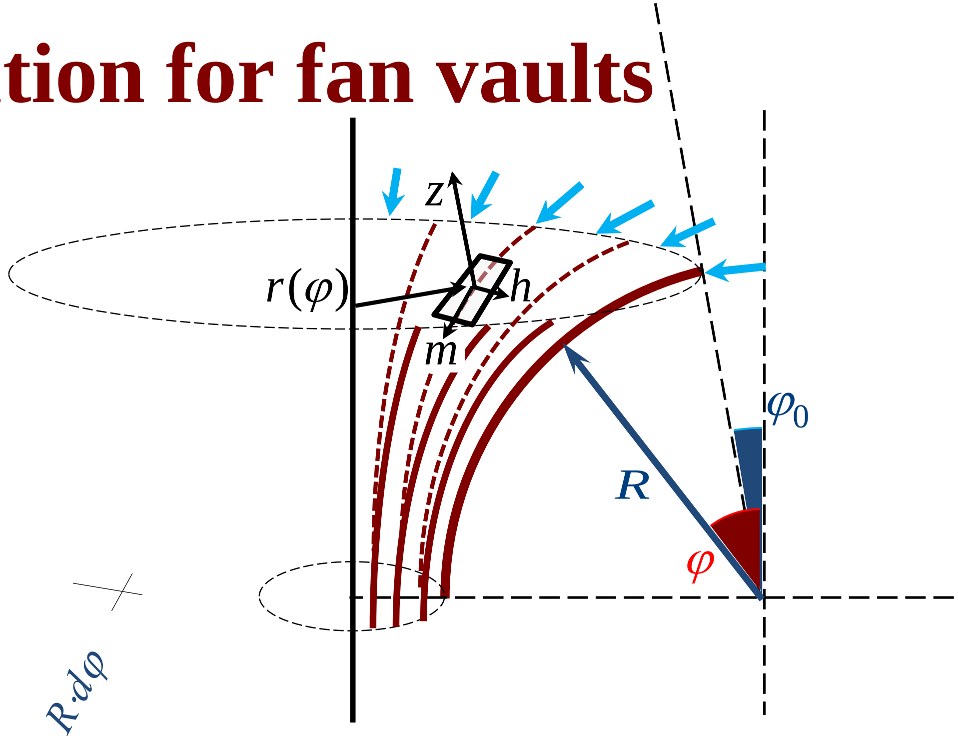


uniform curvature



two different curvatures („Tudor arch”)

Membrane solution for fan vaults

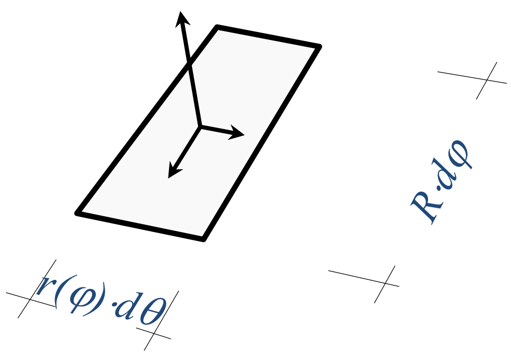


Notations:

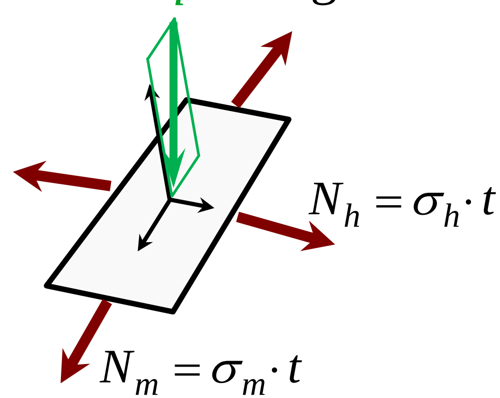
m: meridional direction

h: hoop direction

z: normal direction



q : weight for unit area



→ load: selfweight, vertical direction

spandrel load, meridional direction

→ stresses:

along *t*: σ_m, σ_h are constant

σ_z , τ_{mh} , τ_{mz} and τ_{hz} are zero

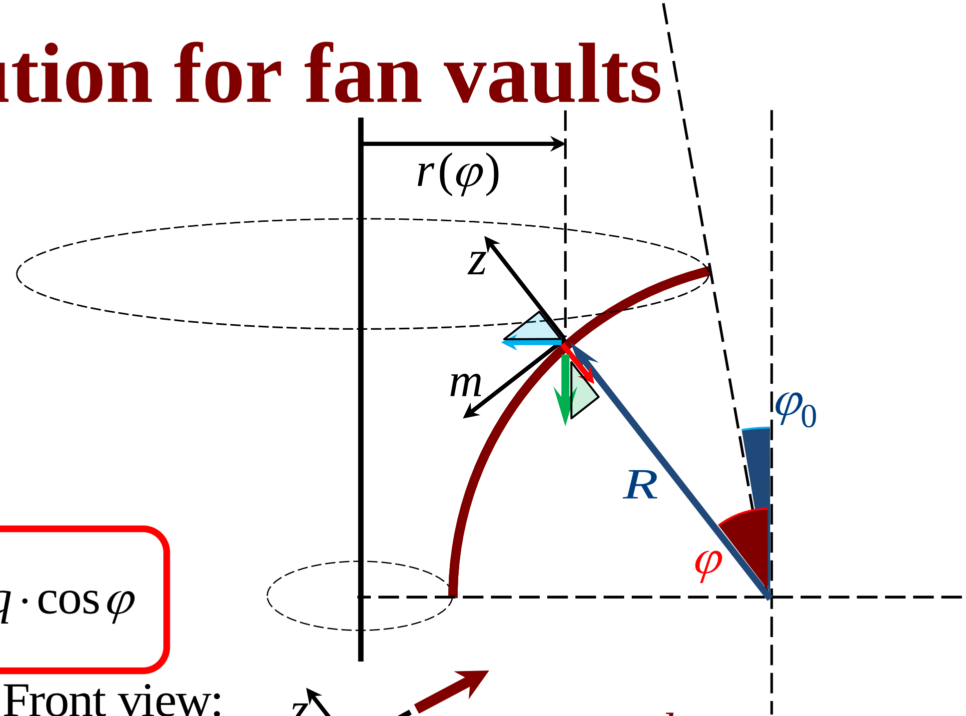
Membrane solution for fan vaults

<u>Notations:</u>

m: meridional direction

h: hoop direction

z: normal direction

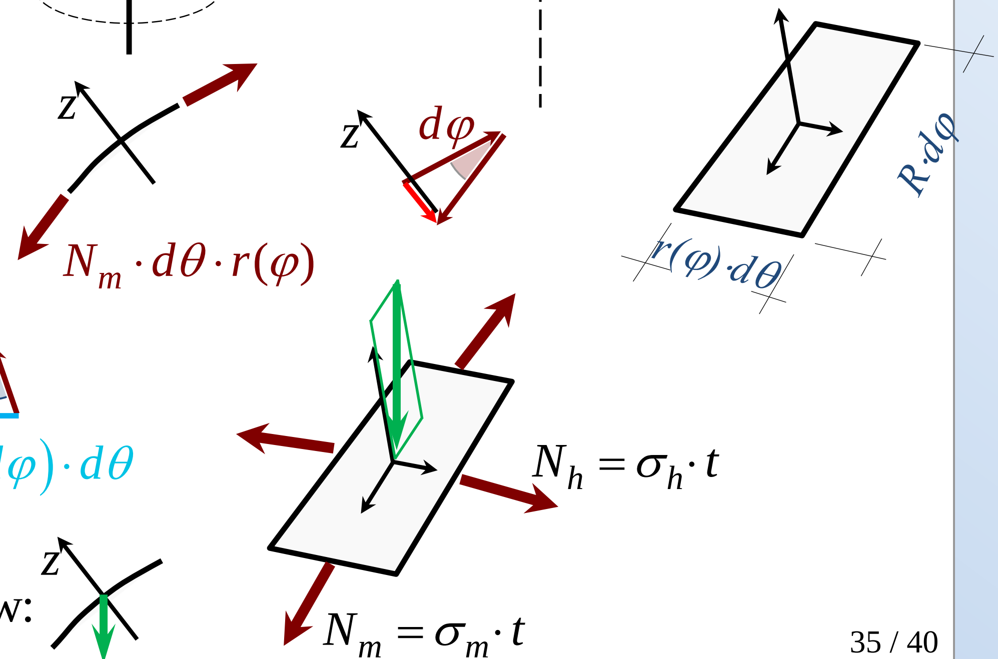


$$\sum F_{iz}: \quad \boxed{\frac{N_h}{r(\varphi)}\sin\varphi = \frac{N_m}{R} + q\cdot\cos\varphi}$$

From meridional forces:

$-\left(N_m \cdot r(\varphi)d\theta\right)\cdot d\varphi$

Front view:



From hoop forces:

$+\left(N_h \cdot Rd\varphi\right)\cdot d\theta \cdot \sin\varphi$

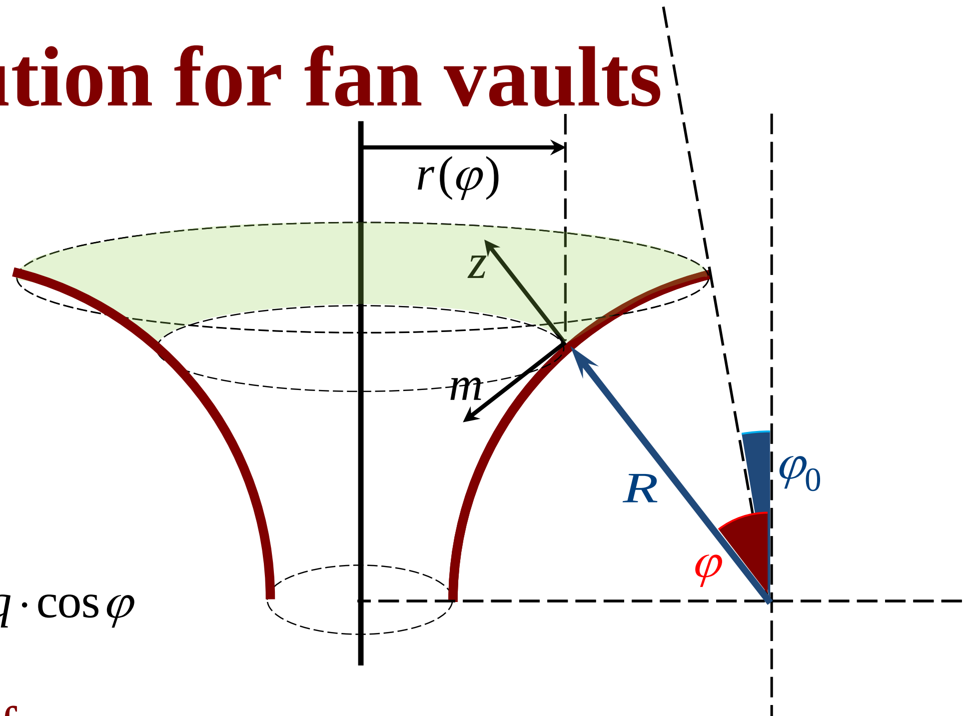
Top view:

From selfweight:

$-\left(q\cdot Rd\varphi \cdot r(\varphi)d\theta\right)\cdot\cos\varphi$

Front view:

Membrane solution for fan vaults



Notations:

m: meridional direction

h: hoop direction

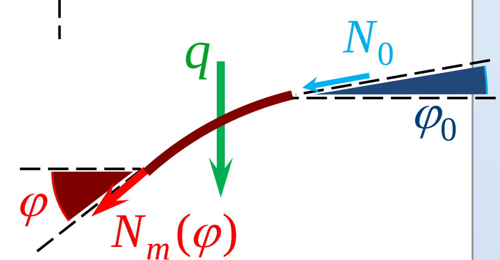
z: normal direction

$$\sum F_{iz}: \quad \frac{N_h}{r(\varphi)} \sin\varphi = \frac{N_m}{R} + q \cdot \cos\varphi$$

Determine the meridional force:

$\sum F_{i,vertical}$:

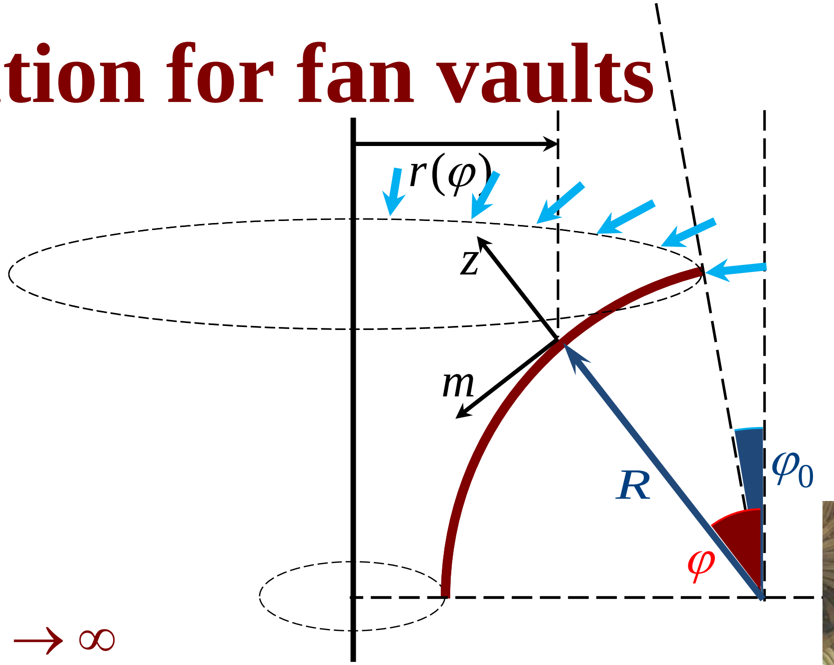
- Weight of the cap above φ: $A_{cap}(\varphi) \cdot q$
- Membrane load at top, at φ_0: $N_0 \cdot \sin\varphi_0 \cdot \pi \cdot r(\varphi_0)$
- Meridional force at φ: $N_m(\varphi) \cdot \sin\varphi \cdot \pi \cdot r(\varphi)$

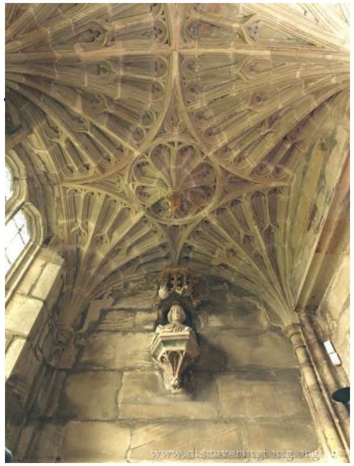


$$N_m(\varphi) \cdot \sin\varphi \cdot \pi \cdot r(\varphi) + N_0 \cdot \sin\varphi_0 \cdot \pi \cdot r(\varphi_0) + A_{cap}(\varphi) \cdot q = 0$$

$$N_m(\varphi) = \frac{-1}{\sin\varphi \cdot r(\varphi)} \left(N_0 \cdot \sin\varphi_0 \cdot r(\varphi_0) + \frac{A_{cap}(\varphi)}{\pi} \cdot q \right) \Rightarrow N_h(\varphi) = \frac{r(\varphi)}{\sin\varphi} \left(\frac{N_m(\varphi)}{R} + q \cdot \cos\varphi \right)$$

Membrane solution for fan vaults





Tong Church Chapel, http://www.discoveringtong.org/tong600/ChapelRoof.htm

Remarks:

1) if $\varphi_0 = 0$:

$\sin \varphi_0 = 0 \Rightarrow N_h(\varphi_0) \to \infty$

$\Rightarrow$ the conoid must be ***truncated*** !

2) if $\varphi_0 \neq 0$:

if $N_0 = 0$: $N_m(\varphi_0) = 0 \Rightarrow N_h > 0$

$\Rightarrow$ ***spandrel load*** is needed to avoid hoop tension !

$$N_m(\varphi) = \frac{-1}{\sin\varphi \cdot r(\varphi)}\left(N_0 \cdot \sin\varphi_0 \cdot r(\varphi_0) + \frac{A_{cap}(\varphi)}{\pi} \cdot q\right) \qquad N_h(\varphi) = \frac{r(\varphi)}{\sin\varphi}\left(\frac{N_m(\varphi)}{R} + q \cdot \cos\varphi\right)$$

SUGGESTED VIDEOS

https://www.youtube.com/watch?v=DI-leSI68dM (Jacques Heyman: The membrane analysis of thin masonry shells, 50:46)

https://www.youtube.com/watch?v=r-tG68WvNDM&t=185s (John Ochsendorf, MIT, „Form and Forces”, 1:17:17)

THIS LECTURE:
SHELL THEORIES

Definition of shells and the shell element

The most important shell theories

- Membranes; soap films
- Kirchhoff-Love shells („classical shells”, „shell bending”)
- Reissner-Mindlin shells („linear shear deformation theory”)
- Third-order shear deformation theory

Membrane solution:

- Repetition from Maths: Principal curvatures at a surface point
- Membrane solution for spherical domes
- Membrane solution for fan vaults

Questions

Questions

1. Introduce the kinematics of
   - → membranes,
   - → Kirchhoff-Love shells,
   - → Mindlin-Reissner-Hencky shells,
   - → third-order shear deformation shells.
2. Show the distribution of the six stress components along the thickness for
   - → membranes,
   - → Kirchhoff-Love shells,
   - → Mindlin-Reissner-Hencky shells,
   - → third-order shear deformation shells.
3. Introduce the membrane solution for spherical domes under selfweight.
4. Introduce the membrane solution for constant-curvature fan vaults under selfweight.