

PLASTIC LIMIT ANALYSIS





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THIS LECTURE

Repetition: Plastic limit theorems

Heyman's limit state theory for masonry

 \rightarrow assumptions about the material

→ The Static Theorem ("Safe Theorem")

→ The Kinematic Theorem (,,Unsafe Theorem'')

Examples when the Safe Theorem fails

Why does the Safe Theorem fail?

Example when the Unsafe Theorem fails

Practical engineering codes

- \rightarrow Archie-M
- \rightarrow LimitState:Ring

Questions

Repetition: Plastic limit theorems

The structure and the loads:

perfectly rigid – perfectly plastic material;

single-parameter load

<u>statically admissible</u> force system: (with λ_{S})

- $\rightarrow\,$ satisfies the equilibrium conditions, and
- \rightarrow does not violate the constitutive constraints

[i.e. the stresses do not exceed the plastic limit]

The static / "lower bound" / "safe" / theorem:

If a statically admissible force system can be found, then the structure with the given geometry is safe under the given loads.

Remarks:

- \rightarrow If such a force system is found, this is not necessarily "the" force system that acts
- → the collapse load multiplier is larger or equal than the λ_S load multiplier that was found to be statically admissible

Repetition: Plastic limit theorems

The structure and the loads:

perfectly rigid – perfectly plastic material; single-parameter load

kinematically admissible virtual displacement system: SMALL

- \rightarrow displs and deformations are compatible, and
- \rightarrow do not violate the constraints given by the supports

The kinematic / "upper bound" / "unsafe" / theorem:

If a kinematically admissible virtual displacement system can be found for which the external forces (with λ_{κ}) make larger or equal work than done by the internal forces, then the structure with the given geometry will collapse under the considered load.

Remarks:

- \rightarrow if such a displ system is found, this is not necessarily "the" collapse mode
- → the collapse load multiplier is smaller or equal than the λ_{κ} load multiplier that was considered above



<u>stone block surfaces</u>: friction angle $\approx 35 - 55^{\circ}$; dilation angle $\approx 0 - 10^{\circ}$ _{5/52}

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The question to answer:

 \rightarrow the structure is a system of *rigid blocks* and *frictional contacts*;



Task Type 1: → given geometry, given loads (e.g. selfweight);
→ Can the structure equilibrate the given loads with the given geometry?

Task Type 2: \rightarrow given geometry; \rightarrow load magnitude to cause collapse?

Inspiration: Kooharian (1952):

 \rightarrow the idea to apply limit state analysis for masonry

 \rightarrow analysis of a circular arch

Heyman (1966 and later on, Cambridge University):

"The first thing we were realizing about engineering is that it is impossible to obtain an exact solution to any problem in engineering." (Heyman, 2011)

assumptions about the material:

- \rightarrow stone blocks have infinite strength
- \rightarrow contacts have zero resistance to tension
- \rightarrow contacts do not slide: they have infinite resistance to friction

 \rightarrow [implicitly also assumed: stone blocks have infinite stiffness] see these assumptions in detail:



https://www.youtube.com/ watch?v=DI-leSI68dM

Assumptions about the material:



 \rightarrow ,,contacts do not slide: they have infinite resistance to friction"

justification: friction angle often high ($\approx 35-55^{\circ}$); blocks may be interlocked <u>but:</u> sliding AND tangential rel. trans. BOTH should be excluded

→ [implicitly also assumed: stone blocks have infinite stiffness]

Inspiration: Kooharian (1952):

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 \rightarrow analysis of a circular arch

Heyman (1966):

assumptions about the material:

- \rightarrow stone blocks have infinite strength
- \rightarrow contacts have zero resistance to tension
- \rightarrow contacts do not slide: they have infinite resistance to friction
- \rightarrow [implicitly also assumed: stone blocks have infinite stiffness]
- 1. The static theorem [formulated without proof]:
- If a force system can be found for the given set of external loads which satisfies the material criteria and equilibrates the given external loads, then the structure with the given geometry is safe under these loads.

Example: Is it safe?

Try to find equilibrating reactions!



[realize after a few unsuccessful tries:]
 → moment about lower right corner cannot be balanced

↓ COLLAPSES!

Inspiration: Kooharian (1952):

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Heyman (1966):

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- \rightarrow contacts have zero resistance to tension
- \rightarrow contacts do not slide: they have infinite resistance to friction
- \rightarrow [implicitly also assumed: stone blocks have infinite stiffness]

2. The kinematic theorem [formulated without proof]: ! internal work is zero !

If a mechanism (a virtual displacement system) can be found for the given set of external loads which satisfies the material criteria and produces non-negative work with the given external loads, then the structure will collapse under these loads.

Meaning of the kinematic theorem:







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Questions

Examples when the Safe Theorem Fails



Pyramid Upwards Down



Rimon's Backpack

, If there exists any system of forces satisfying the material conditions and being in equilibrium with the loads, then the structure is safe. "???

The coming slides:

- \rightarrow theoretical analysis: why does the Safe Theorem fail?
- \rightarrow improved formulation of the Safe Theorem \Rightarrow conclusion: restricted validity!

 \rightarrow simulated experiments to illustrate the danger

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← skipping the lengthy derivations: slide 21-22; slide 30; ...

The geometry and the material

 \rightarrow blocks: b

any polyhedral shapes rigid with infinite strength

 \rightarrow contacts: c

planar (may be multiple)
resist compression & Coulomb-friction



Force systems

→ External forces and moments [given loads]: $(\mathbf{G}_b, \mathbf{M}_b)$ for every block *b*

\rightarrow Contact forces:

 $(\mathbf{Q}_{bc}, \mathbf{T}_{bc})$ for every contact *c* of block *b* distributed normal forces along the contact:

 \Rightarrow resultant: "contact point"

 \Rightarrow compressional force, $\mathbf{Q}_{bcN} \cdot \mathbf{n}_{bc} \leq 0$

distributed tangential forces along the contact:

- \Rightarrow anywhere: friction limit cannot be exceeded
- \Rightarrow frictional force + torsional moment

Important:

location of the contact point:

characteristic of the actual contact force system !





Displacement systems: Definitions

(1) Virtual displacement system:

virtual translation and rotation of block *b*:

δu_b;**δφ**_btranslation of the contact point *c* on block *b*:

$$\delta \mathbf{u}_{bc} = \delta \mathbf{u}_b + \delta \boldsymbol{\varphi}_b \times \mathbf{r}_{bc}$$



contact deformation: relative translation and rotation at contact c:

$$\delta \mathbf{d}_{c} = \delta \mathbf{u}_{b_{1}c} - \delta \mathbf{u}_{b_{2}c} = \delta \mathbf{u}_{b_{1}} + \delta \varphi_{b_{1}} \times \mathbf{r}_{b_{1}c} - \delta \mathbf{u}_{b_{2}} + \delta \varphi_{b_{2}} \times \mathbf{r}_{b_{2}c}$$
$$\delta \Theta_{c} = \delta \varphi_{b_{1}} - \delta \varphi_{b_{2}}$$

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$$\delta \boldsymbol{\theta}_{c} = \delta \boldsymbol{\varphi}_{b_{1}} - \delta \boldsymbol{\varphi}_{b_{2}}$$

(2) <u>Mechanically admissible virtual displacement system</u>: (no interperetration)



Displacement systems: Definitions

(2) Mechanically admissible virtual displacement systems

\rightarrow Heymanian:

no tangential relative translation occurs at any point of any contact!





 (3) Mechanically admissible small but finite displacement systems Heymanian / non-Heymanian: according to their first-order approximation

Equilibrium of a force system: [loads & contact forces]

for every block *b*:
$$\begin{bmatrix} \mathbf{G}_b + \sum_{(bc)} \mathbf{Q}_{bc} = 0 \\ \mathbf{M}_b + \sum_{(bc)} \left(\mathbf{r}_{bc} \times \mathbf{Q}_{bc} + \mathbf{T}_{bc} \right) = 0 \end{bmatrix}$$

for any arbitrary system of virtual displacements the following holds:

$$\sum_{(b)} \left(\mathbf{G}_b \cdot \delta \mathbf{u}_b + \mathbf{M}_b \cdot \delta \varphi_b \right) + \sum_{(c)} \left(\mathbf{Q}_c \cdot \delta \mathbf{d}_c + \mathbf{T}_c \cdot \delta \Theta_c \right) = 0$$

Note: for all mechanically admissible Heymanian virtual displacements:

Conclusions:

- (i) If an equilibrium force system can be found for which for any nonzero Heymanian virtual displacement system at least one contact point opens up, then the existence of the equilibrated force system ensures that the **external work is negative** on any arbitrarily chosen mechanically admissible Heymanian system.
- (ii) If there exists any mechanically admissible Heymanian virtual displacement system for which the structure moves with none of the contact points opening up, then the **external work** along this displacement system **is zero**.

Note: for all mechanically admissible Heymanian virtual displacements:

$$\sum_{(c)} (\mathbf{Q}_c \cdot \delta \mathbf{d}_c) \ge 0 \; ; \quad \sum_{(c)} (\mathbf{T}_c \cdot \delta \mathbf{\theta}_c) = 0$$

$$\Rightarrow \sum_{(b)} (\mathbf{G}_b \cdot \delta \mathbf{u}_b + \mathbf{M}_b \cdot \delta \mathbf{\varphi}_b) \le 0$$

$$W_{ext} = 0 \quad \text{if}$$
all contacts remain closed
$$W_{ext} < 0 \quad \text{if}$$
any contact opens up
$$25/5$$

Displacement systems: Definitions

(3) Mechanically admissible small but finite displacement systems

Heymanian / non-Heymanian:

according to their first-order approximation

small: \rightarrow no new contacts;

 \rightarrow first-order approximation gives the same sign for work

if Heymanian:



Stability of the equilibrium:

- <u>Definition</u>: The actual state of a masonry system is **stable** if there exists a continuous, finite-sized domain of mechanically admissible finite displacement systems (Δu , $\Delta \phi$) containing ($\Delta u = 0$, $\Delta \phi = 0$) as an interior point, for which the **total work** done by the actual external and internal forces along **any** (Δu , $\Delta \phi$) of the set is **negative**.
- <u>Definition</u>: The actual state of a masonry system is **critical** if there exists any mechanically admissible virtual displacement systems (δu , $\delta \phi$) for which the **total virtual work** done by the actual external and internal forces along (δu , $\delta \phi$) of the set is **zero**.

[similarly to unstable / neutral equilibrium]

Stability analysis:

Assume that an equilibrated contact force system was found to the given loads! \Rightarrow conclusions???

Stability analysis:

Assume that an equilibrated contact force system was found to the given loads! \Rightarrow conclusions???

Can we find a mechanically admissible Heymanian virtual displacement system in such a way that **none of the contact points** opens up?

If we can:



Case (ii) found for at least one infinitesimally small perturbation:

ii)
$$\sum_{(b)} \left(\mathbf{G}_b \cdot \delta \mathbf{u}_b + \mathbf{M}_b \cdot \delta \boldsymbol{\varphi}_b \right) = 0$$
$$\Rightarrow \mathbf{CRITICAL \ STATE}$$

If we cannot:

Case (i) found for every perturbation of the position in a small finite neighborhood of the analysed position:

(i)
$$\sum_{(b)} \left(\mathbf{G}_b \cdot \Delta \mathbf{u}_b + \mathbf{M}_b \cdot \Delta \mathbf{\varphi}_b \right) < 0 \implies \mathbf{STABLE \ STATE \ against} \\ \mathbf{Heymanian \ displacements!}_{28/52}$$

Corrected Formulation of the Safe Theorem

Assumptions:

- (a) the masonry blocks are polyhedral
- (b) the masonry blocks are infinitely rigid and infinitely strong
- (c) the contacts transmit no tension

The Safe Theorem:

- If there exists any system of forces satisfying (a-c) being in equilibrium with the loads, and
- *if there does not exist any mechanically admissible Heymanian virtual displacement system for which all contact points of this force system remain closed,*
- then the structure is safe against collapse along any Heymanian displacements.

Examples when the Safe Theorem Fails

Explanation:





Pyramid Upwards Down

Rimon's Backpack

Pyramid Upwards Down: *Safe Theorem is not valid for this !*

 \rightarrow a Heymanian virtual displacement system exists so that no contact opens up

 \rightarrow the equilibrium state is critical; higher-order analysis reveals: unstable

Rimon's Backpack: Safe Theorem is only for Heymanian disps !

- \rightarrow contact always opens up
- \rightarrow Safe Theorem: no protection against collapse along non-Heymanian disps

Examples

"Pisa Tower":































Buckling Arch:







Buckling Arch:





Buckling Arch:



Conclusion:

the existence of an equilibrated force system **does not give protection against** collapses along **non-Heymanian** displacements!

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Questions

Example when the Unsafe Theorem Fails

Brick on an Inclined Plane:

 $\delta W > 0$: collapses according to the Unsafe Theorem

BUT: for [friction angle] > [angle of the slope]: in fact THE STRUCTURE IS SAFE!



<u>notice again:</u> presence of tangential relative translation \Rightarrow the Unsafe Theorem fails

Conclusion:

the Unsafe Theorem is unnecessarily **too conservative** if **non-Heymanian** displacements are present!

For home:



Try to apply both theorems: Is the structure safe? Under what geometrical conditions can the structure be safe?

 \uparrow collapses

45 / 52

For home:



Try to apply both theorems: are these structures safe?

 \uparrow safe

Answer:

Shear failure of historic structures

Beatini et al (2018):

Cathedral of San Martino, in Pietrasanta, Italy:







http://viafrancigenatoscana.it



http://wikimapia.org/13139959/ it/Duomo-di-San-Martino

Umberto Bridge, Turin, Italy:



"multiring arch"

rings slide on each other; protection needed against shear failure

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Questions

e.g. Archie-M:

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single-span or multispan

e.g. Archie-M: http://www.obvis.com

Bill Harvey, developed since 1981 based on the Static Theorem: frictional sliding excluded; an equilibrum force system is searched for thrust ine → thrust zone, according to the finite strength skew bridges: formally, as if they were straight



e.g. LimitState:Ring: Gilbert & Melbourne (1994); Gilbert (2007)

based on the parallel use of the kinematic AND the static theorem



The basic line of thought for live load in addition to selfweight:

 \rightarrow the analysis would be this, in case of manual calculations:

(i) choose a likely mechanism of collapse;

(ii) compile the work, to calculate the collapse load;

(iii) try other likely collapse mechanisms until the critical one is found.

 \rightarrow instead:

a computerized solution technique to systematically find the collapse mode with smallest load, for which the equilibrium eqs are also satisfied

<u>e.g. LimitState:Ring:</u> Matthew Gilbert Gilbert & Melbourne (1994); Gilbert (2007) youtube.com/watch?v=ciWG36N0kLM

 → associated contact friction model: dilation equals to friction angle experiences for multiring arches: ,,RING is usually on the safe side"



- \rightarrow block crushing taken into account: ",thrust zone" instead of ",thrust line"
- \rightarrow multispan bridges; multiring arches with frictional contacts between layers
- \rightarrow distributing effect of backfill taken into account
- → experimental and industrial validations: ,,RING is usually on the safe side" http://www.limitstate.com/ring/experimental-validation

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QUESTIONS

1. In case of general plasticity, what does the static theorem state, and what does the kinematic theorem state? Under what condition will the largest λ_s surely coincide with the smallest λ_k ?

2. What are the basic assumptions in Heyman's theory for the material behaviour? Justify / criticize them.

3. What does Heyman's static theorem state? What does Heyman's kinematic theorem state? Illustrate their use on an example. Show examples when these theorems fail.

4. Introduce the fundaments of the LimitState:Ring and Archie-M codes.