PLASTIC LIMIT ANALYSIS

THIS LECTURE

Repetition: Plastic limit theorems

Heyman's limit state theory for masonry

→ assumptions about the material

→ The Static Theorem („Safe Theorem")

→ The Kinematic Theorem („Unsafe Theorem")

Examples when the Safe Theorem fails

Why does the Safe Theorem fail?

Example when the Unsafe Theorem fails

Practical engineering codes

→ Archie-M

→ LimitState:Ring

Questions

Repetition: Plastic limit theorems

The structure and the loads:

perfectly rigid – perfectly plastic material;

single-parameter load

statically admissible force system: (with λ_S)

→ satisfies the equilibrium conditions, and

→ does not violate the constitutive constraints

[i.e. the stresses do not exceed the plastic limit]

The static / „lower bound" / „safe" / theorem:

If a statically admissible force system can be found,
then the structure with the given geometry is safe under the given loads.

Remarks:

→ If such a force system is found, this is not necessarily „the" force system that acts

→ the collapse load multiplier is larger or equal than the λ_S load multiplier that was found to be statically admissible

Repetition: Plastic limit theorems

The structure and the loads:

perfectly rigid – perfectly plastic material;

single-parameter load

kinematically admissible virtual displacement system: SMALL

$\rightarrow$ displs and deformations are compatible, and

$\rightarrow$ do not violate the constraints given by the supports

The kinematic / „upper bound” / „unsafe” / theorem:

> If a kinematically admissible virtual displacement system can be found for which the external forces (with λ_K) make larger or equal work than done by the internal forces, then the structure with the given geometry will collapse under the considered load.

Remarks:

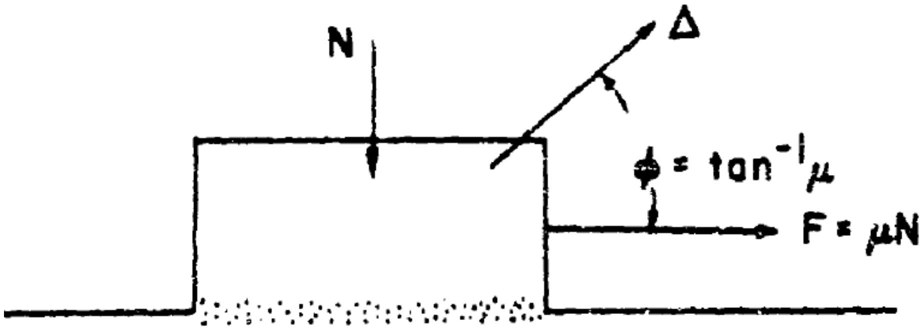
$\rightarrow$ if such a displ system is found, this is not necessarily „the” collapse mode

$\rightarrow$ the collapse load multiplier is smaller or equal than the λ_K load multiplier that was considered above

Repetition: Plastic limit theorems

<u>Drucker (1954):</u>

„associated flow rule":

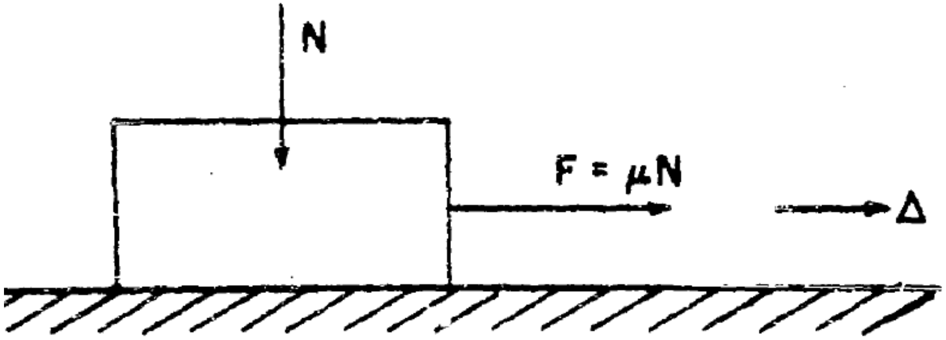


max of λ_S = min of λ_K = collapse load

„non-associated flow rule":

e.g. Coulomb-friction

with no dilation:



duality gap occurs ; history dependence

<u>stone block surfaces:</u> friction angle ≈ 35 - 50° ; dilation angle ≈ 0 - 10°

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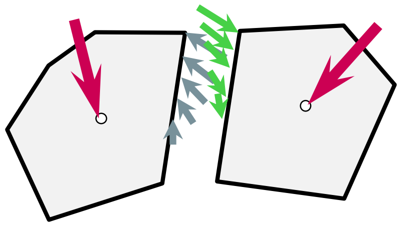
Questions

Heyman’s limit state theory for masonry

The question to answer:

→ the structure is a system of ***rigid blocks***
and ***frictional contacts***;



Task Type 1: → given geometry, given loads (e.g. selfweight);
→ Can the structure equilibrate the given loads
with the given geometry?

Task Type 2: → given geometry;
→ load magnitude to cause collapse?

Heyman's limit state theory for masonry

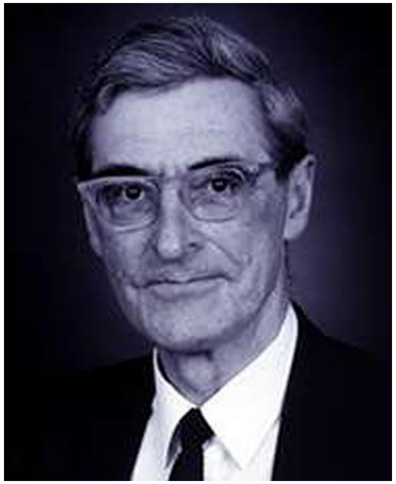
Inspiration: Kooharian (1952):

→ the idea to apply limit state analysis for masonry

→ analysis of a circular arch

Heyman (1966 and later on, Cambridge University):

"The first thing we were realizing about engineering is that it is impossible to obtain an exact solution to any problem in engineering." (Heyman, 2011)



https://www.youtube.com/watch?v=DI-leSI68dM

assumptions about the material:

- → stone blocks have infinite strength
- → contacts have zero resistance to tension
- → contacts do not slide: they have infinite resistance to friction
- → [implicitly also assumed: stone blocks have infinite stiffness]

see these assumptions in detail:

Heyman’s limit state theory for masonry

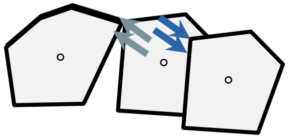
Assumptions about the material:

→ „stone blocks have infinite strength”

justification: max height of tower to carry its own weight:

granite: ≈ above 6-8 km; weak stones: ≈ hundreds of m; km

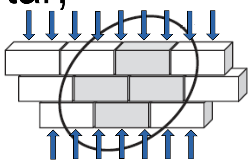
<u>but:</u> infinite strength is not possible

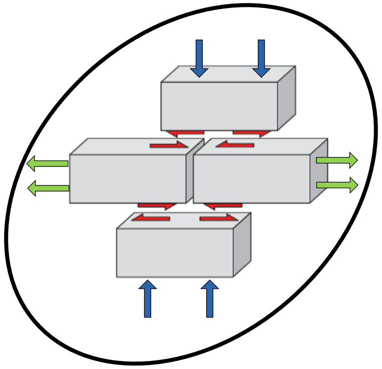


→ „contacts have zero resistance to tension”

justification: dry contacts or old weak mortar;

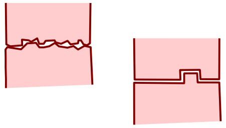
<u>but:</u> crosswise tensile resistance due to friction





→ „contacts do not slide: they have infinite resistance to friction”

justification: friction angle often high (35-50°); blocks may be interlocked



<u>but:</u> sliding AND tangential rel. trans. BOTH should be excluded

→ [implicitly also assumed: stone blocks have infinite stiffness]

Heyman's limit state theory for masonry

<u>Inspiration: Kooharian (1952):</u>

$\rightarrow$ the idea to apply limit state analysis for masonry
$\rightarrow$ analysis of a circular arch

<u>Heyman (1966):</u>

assumptions about the material:
$\rightarrow$ stone blocks have infinite strength
$\rightarrow$ contacts have zero resistance to tension
$\rightarrow$ contacts do not slide: they have infinite resistance to friction
$\rightarrow$ [implicitly also assumed: stone blocks have infinite stiffness]

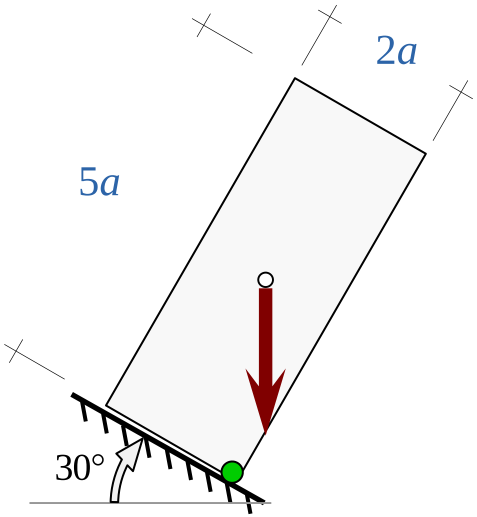
1. The static theorem [formulated without proof]:

If a force system can be found for the given set of external loads which satisfies the material criteria and equilibrates the given external loads, then the structure with the given geometry is safe under these loads.

Heyman's limit state theory for masonry

<u>Example:</u> Is it safe?

Try to find equilibrating reactions!



[realize after a few unsuccessful tries:]

$\rightarrow$ moment about lower right corner cannot be balanced

$\Downarrow$

COLLAPSES!

Heyman’s limit state theory for masonry

Inspiration: Kooharian (1952):

→ the idea to apply limit state analysis for masonry
→ analysis of a circular arch

Heyman (1966):

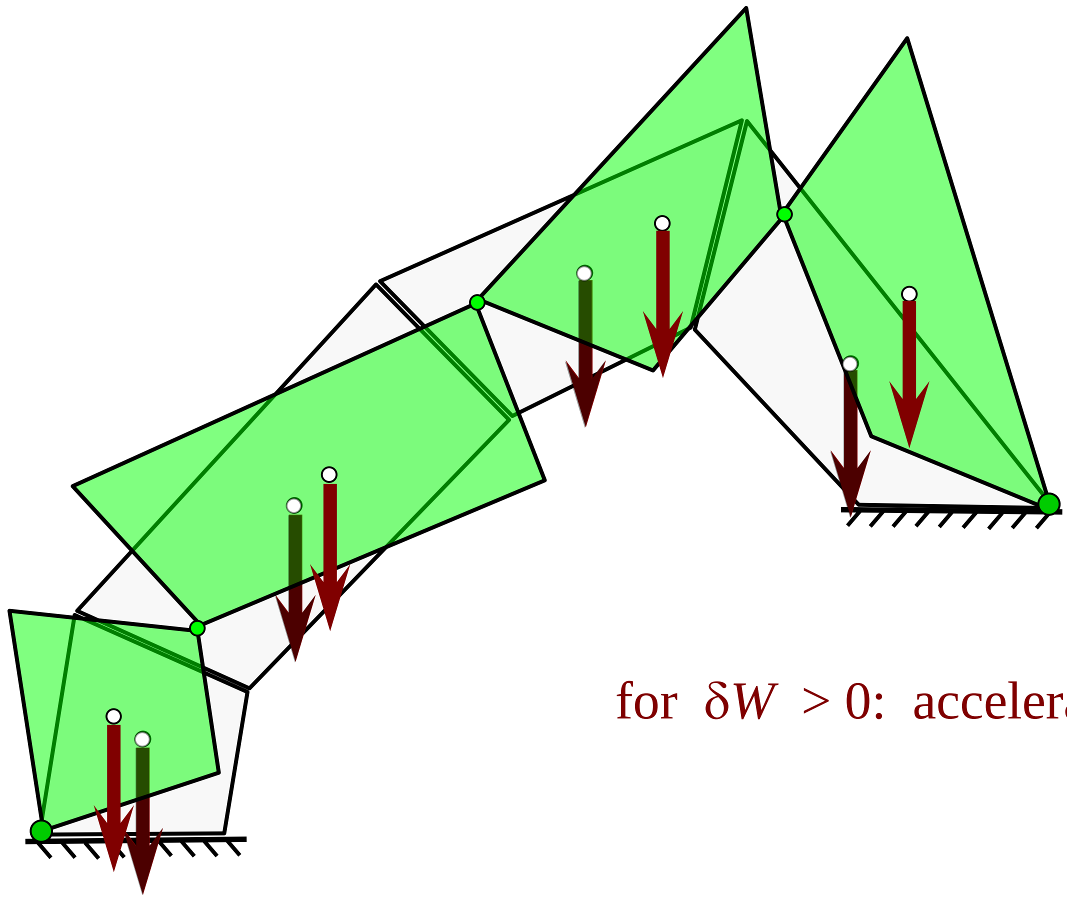
assumptions about the material:
→ stone blocks have infinite strength
→ contacts have zero resistance to tension
→ contacts do not slide: they have infinite resistance to friction
→ [implicitly also assumed: stone blocks have infinite stiffness]

2. The kinematic theorem [formulated without proof]:   **! internal work is zero !**

If a mechanism (a virtual displacement system) can be found for the given set of external loads which satisfies the material criteria and produces non-negative work with the given external loads, then the structure will collapse under these loads.

Heyman's limit state theory for masonry

Meaning of the kinematic theorem:

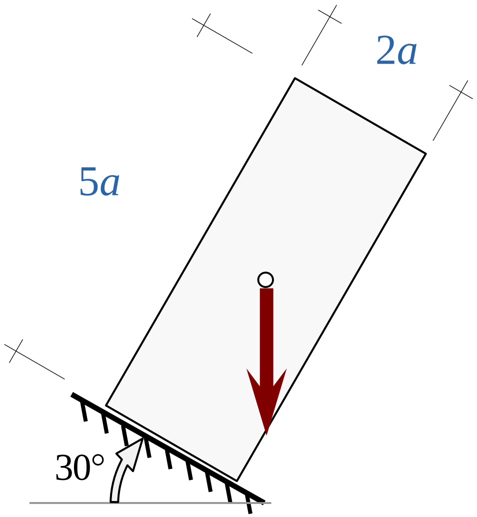


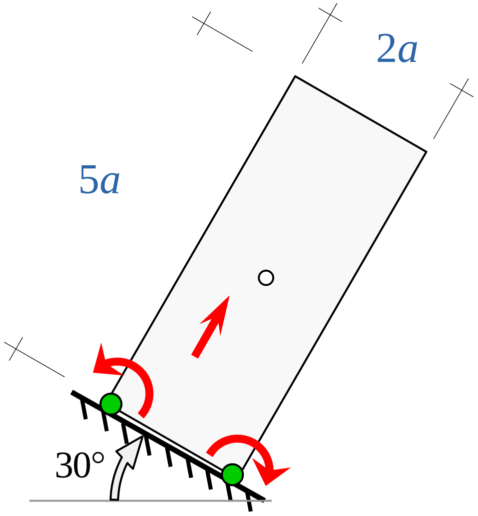
for $\delta W > 0$: accelerating collapse

Heyman's limit state theory for masonry

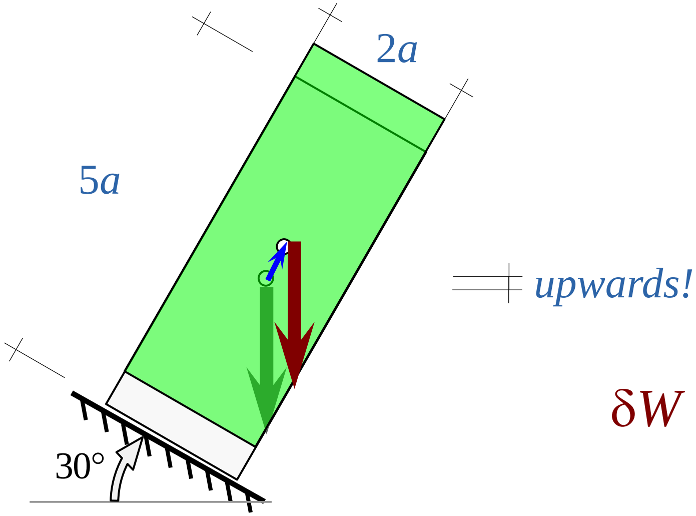
<u>Example:</u> Is it safe?

Try possible displs!





← degrees of freedom:

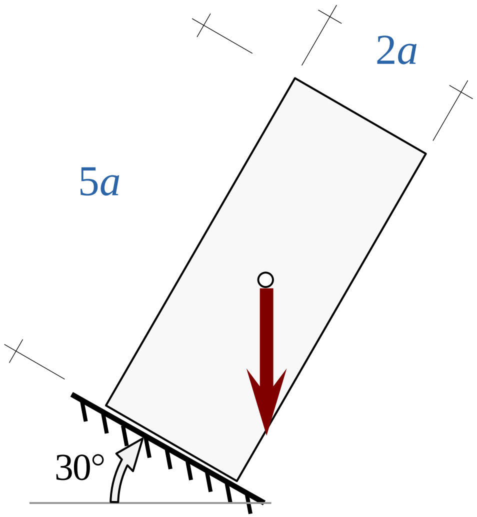


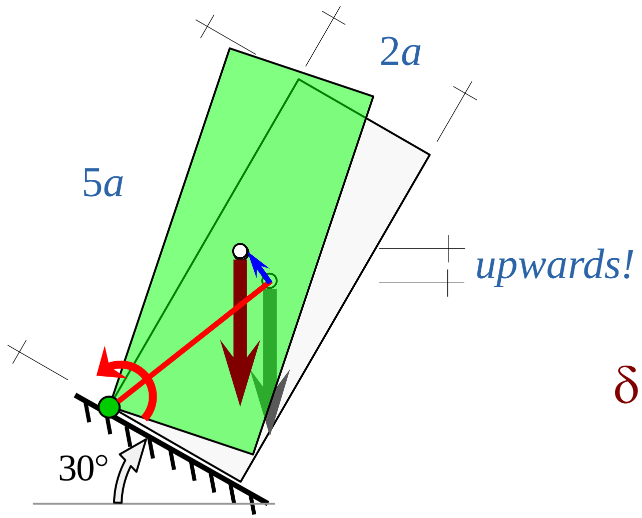
$\delta W < 0$: not decided yet

Heyman's limit state theory for masonry

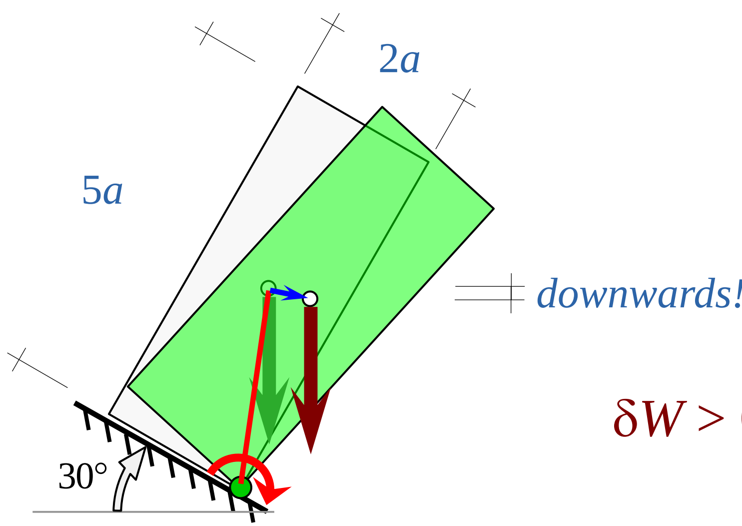
Example: Is it safe?

Try possible displs!





$\delta W < 0$: still not decided



$\delta W > 0$: surely collapses

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Examples when the Safe Theorem fails

Why does the Safe Theorem fail?

Example when the Unsafe Theorem fails

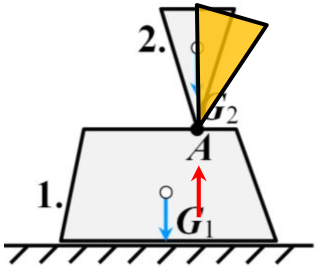
Practical engineering codes

→ Archie-M

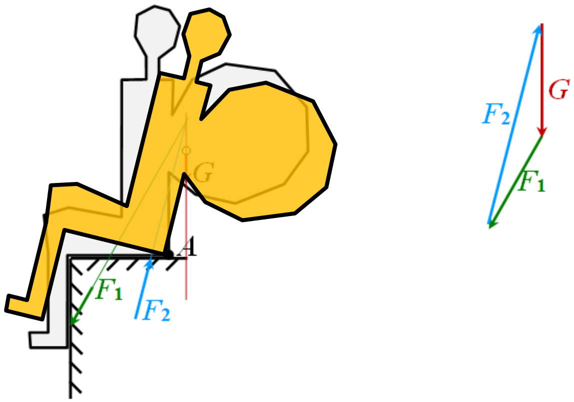
→ LimitState:Ring

Questions

Examples when the Safe Theorem Fails



Pyramid Upwards Down



Rimon's Backpack

„*If there exists any system of forces satisfying the material conditions and being in equilibrium with the loads, then the structure is safe.*"

???

The coming slides:

→ theoretical analysis: why does the Safe Theorem fail?

→ improved formulation of the Safe Theorem ⇒ conclusion: restricted validity!

→ simulated experiments to illustrate the danger

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Examples when the Safe Theorem fails

Why does the Safe Theorem fail?

← skipping the lengthy derivations: slide 21-22; slide 30; …

Example when the Unsafe Theorem fails

Practical engineering codes

- → Archie-M
- → LimitState:Ring

Questions

Why does the Safe Theorem fail?

<u>The geometry and the material</u>

→ blocks: *b*

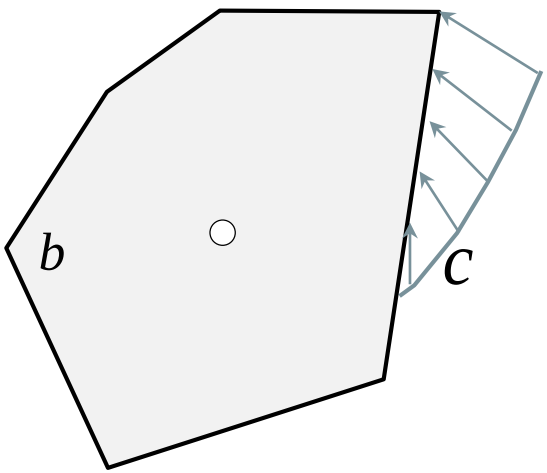
any polyhedral shapes

rigid with infinite strength

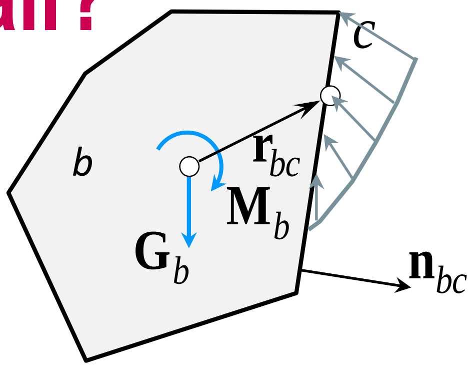
→ contacts: *c*

planar (may be multiple)

resist compression & Coulomb-friction



Why does the Safe Theorem fail?



Force systems

→ External forces and moments [given loads]:

$(\mathbf{G}_b, \mathbf{M}_b)$ for every block b

→ Contact forces:

$(\mathbf{Q}_{bc}, \mathbf{T}_{bc})$ for every contact c of block b

distributed normal forces along the contact:

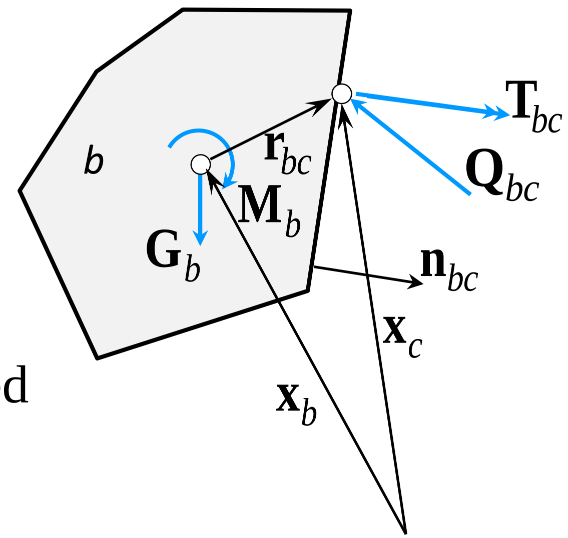
⇒ resultant: „contact point"

⇒ compressional force, $\mathbf{Q}_{bcN} \cdot \mathbf{n}_{bc} \leq 0$

distributed tangential forces along the contact:

⇒ anywhere: friction limit cannot be exceeded

⇒ frictional force + torsional moment



Important:

location of the contact point:

characteristic of the actual contact force system !

Why does the Safe Theorem fail?

Displacement sytems

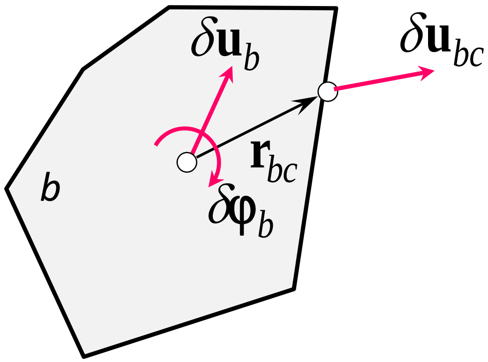
(1) Virtual displacement system:

virtual translation and rotation of block *b*:

$$\delta\mathbf{u}_b \ ; \quad \delta\boldsymbol{\varphi}_b$$

translation of the contact point *c* on block *b*:

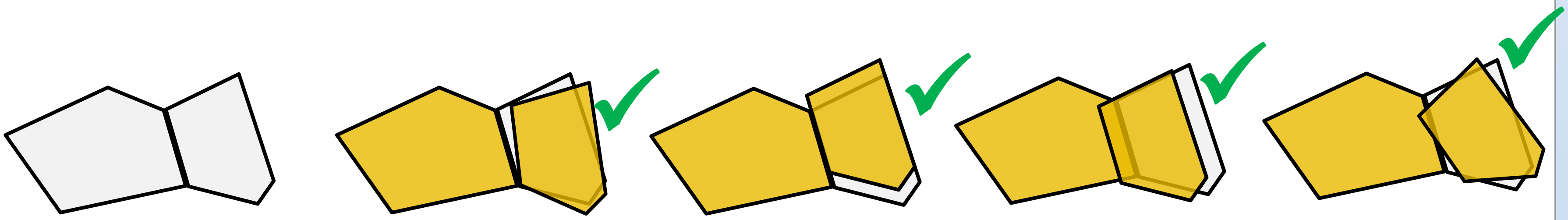
$$\delta\mathbf{u}_{bc} = \delta\mathbf{u}_b + \delta\boldsymbol{\varphi}_b \times \mathbf{r}_{bc}$$



contact deformation: relative translation and rotation at contact c:

$$\delta\mathbf{d}_c = \delta\mathbf{u}_{b_1c} - \delta\mathbf{u}_{b_2c} = \delta\mathbf{u}_{b_1} + \delta\boldsymbol{\varphi}_{b_1} \times \mathbf{r}_{b_1c} - \delta\mathbf{u}_{b_2} + \delta\boldsymbol{\varphi}_{b_2} \times \mathbf{r}_{b_2c}$$

$$\delta\boldsymbol{\theta}_c = \delta\boldsymbol{\varphi}_{b_1} - \delta\boldsymbol{\varphi}_{b_2}$$



Why does the Safe Theorem fail?

Displacement sytems

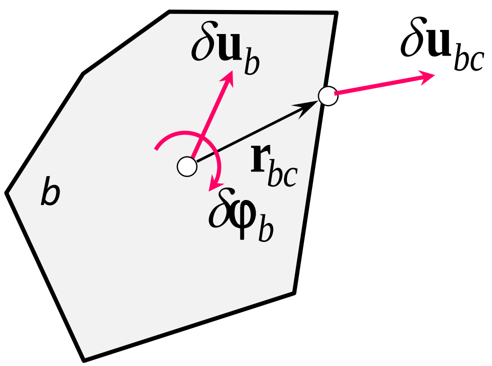
(1) Virtual displacement system:

virtual translation and rotation of block *b*:

$$\boldsymbol{\delta}\mathbf{u}_b \;; \quad \boldsymbol{\delta\varphi}_b$$

translation of the contact point *c* on block *b*:

$$\boldsymbol{\delta}\mathbf{u}_{bc} = \boldsymbol{\delta}\mathbf{u}_b + \boldsymbol{\delta\varphi}_b \times \mathbf{r}_{bc}$$



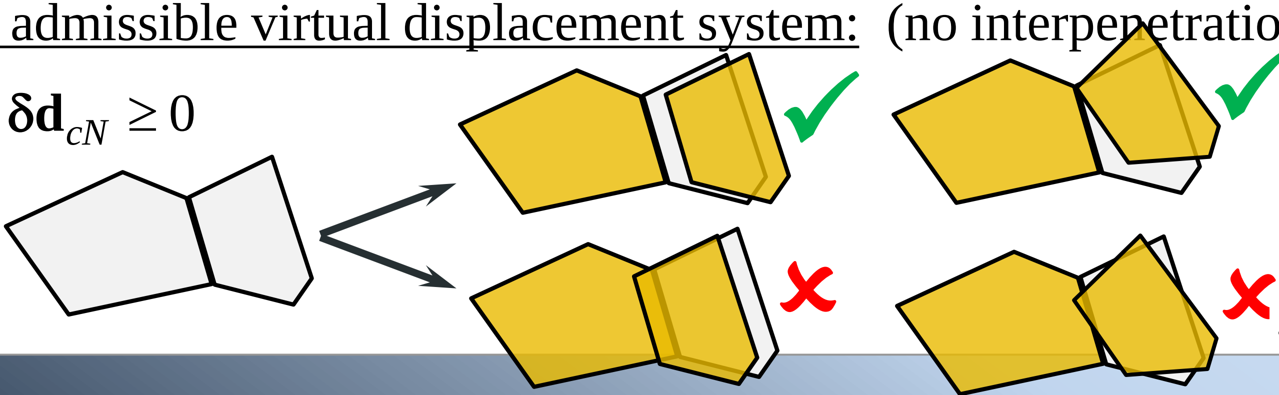
contact deformation: relative translation and rotation at contact c:

$$\boldsymbol{\delta}\mathbf{d}_c = \boldsymbol{\delta}\mathbf{u}_{b_1c} - \boldsymbol{\delta}\mathbf{u}_{b_2c} = \boldsymbol{\delta}\mathbf{u}_{b_1} + \boldsymbol{\delta\varphi}_{b_1} \times \mathbf{r}_{b_1c} - \boldsymbol{\delta}\mathbf{u}_{b_2} + \boldsymbol{\delta\varphi}_{b_2} \times \mathbf{r}_{b_2c}$$

$$\boldsymbol{\delta\theta}_c = \boldsymbol{\delta\varphi}_{b_1} - \boldsymbol{\delta\varphi}_{b_2}$$

(2) Mechanically admissible virtual displacement system: (no interpepetration)

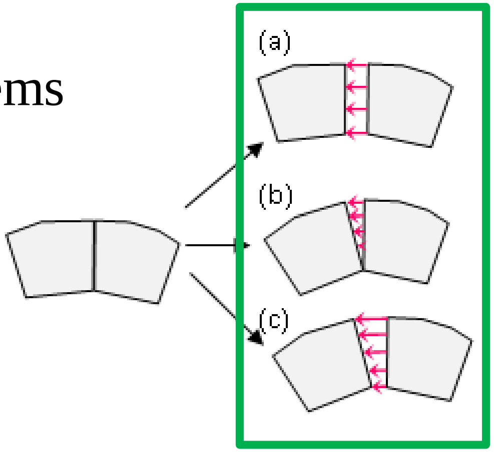
$$\mathbf{Q}_{cN} \cdot \boldsymbol{\delta}\mathbf{d}_{cN} \geq 0$$



Why does the Safe Theorem fail?

Displacement sytems

(2) Mechanically admissible virtual displacement systems

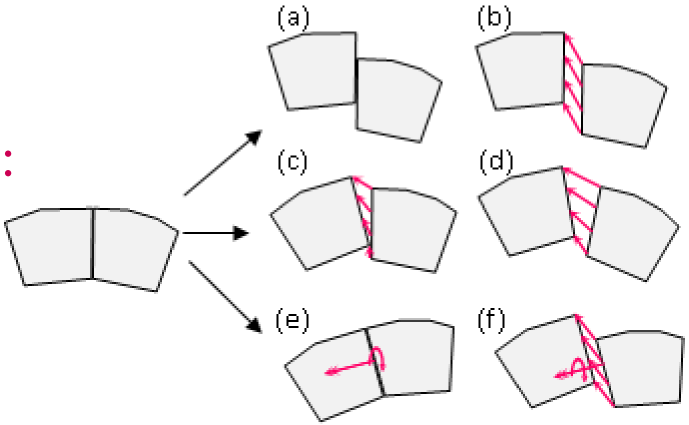


→ Heymanian:

no tangential relative translation occurs

at any point of any contact!

→ Non-Heymanian:

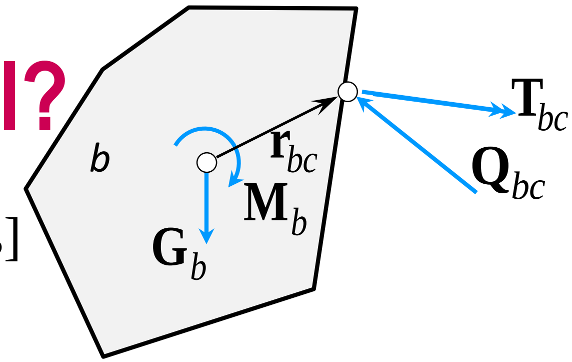


(3) Mechanically admissible small but finite displacement systems

Heymanian / non-Heymanian:

according to their first-order approximation

Why does the Safe Theorem fail?



Equilibrium of a force system: [loads & contact forces]

for every block b:

$$\begin{cases} \mathbf{G}_b + \sum_{(bc)} \mathbf{Q}_{bc} = 0 \\ \mathbf{M}_b + \sum_{(bc)} \left(\mathbf{r}_{bc} \times \mathbf{Q}_{bc} + \mathbf{T}_{bc} \right) = 0 \end{cases}$$

for any arbitrary system of virtual displacements the following holds:

$$\boxed{\sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b + \mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) + \sum_{(c)} \left(\mathbf{Q}_c \cdot \delta\mathbf{d}_c + \mathbf{T}_c \cdot \delta\boldsymbol{\theta}_c \right) = 0}$$

Note: for all mechanically admissible Heymanian virtual displacements:

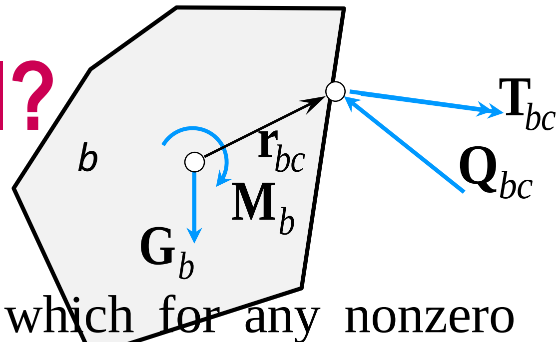
$$\sum_{(c)} \left(\mathbf{Q}_c \cdot \delta\mathbf{d}_c \right) \geq 0 \; ; \quad \sum_{(c)} \left(\mathbf{T}_c \cdot \delta\boldsymbol{\theta}_c \right) = 0$$

$$\Rightarrow \boxed{\sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b + \mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) \leq 0}$$

$W_{ext} = 0$ if
all contacts remain closed

$W_{ext} < 0$ if
any contact opens up

Why does the Safe Theorem fail?



Conclusions:

(i) If an equilibrium force system can be found for which for any nonzero Heymanian virtual displacement system at least one contact point opens up, then the existence of the equilibrated force system ensures that the **external work is negative** on any arbitrarily chosen mechanically admissible Heymanian system.

(ii) If there exists any mechanically admissible Heymanian virtual displacement system for which the structure moves with none of the contact points opening up, then the **external work** along this displacement system **is zero**.

Note: for all mechanically admissible Heymanian virtual displacements:

$$\sum_{(c)}\left(\mathbf{Q}_c \cdot \delta\mathbf{d}_c\right) \geq 0 \;\; ; \quad \sum_{(c)}\left(\mathbf{T}_c \cdot \delta\boldsymbol{\theta}_c\right) = 0$$

$$\Rightarrow \boxed{\sum_{(b)}\left(\mathbf{G}_b \cdot \delta\mathbf{u}_b + \mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b\right) \leq 0}$$

$W_{ext} = 0$ if all contacts remain closed

$W_{ext} < 0$ if any contact opens up

Why does the Safe Theorem fail?

Displacement systems

(3) Mechanically admissible small but finite displacement systems

Heymanian / non-Heymanian:

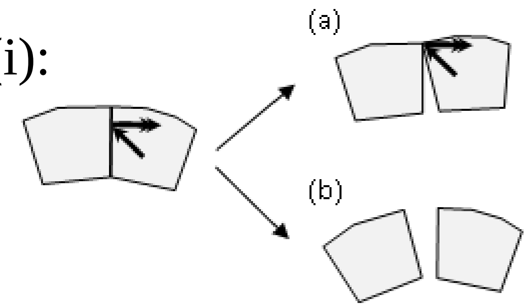
according to their first-order approximation

small: $\rightarrow$ no new contacts;

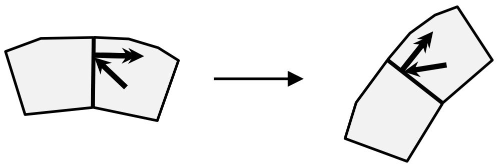
$\rightarrow$ first-order approximation gives the same sign for work

if Heymanian:

Case (i):



Case (ii):



(i) $$\sum_{(b)} \left(\mathbf{G}_b \cdot \Delta\mathbf{u}_b + \mathbf{M}_b \cdot \Delta\boldsymbol{\varphi}_b \right) < 0$$

(ii) $$\sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b + \mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) = 0$$

$$\sum_{(b)} \left(\mathbf{G}_b \cdot \Delta\mathbf{u}_b + \mathbf{M}_b \cdot \Delta\boldsymbol{\varphi}_b \right) \; > ??? < \; 0$$

Why does the Safe Theorem fail?

Stability of the equilibrium:

Definition: The actual state of a masonry system is **stable** if there exists a continuous, finite-sized domain of mechanically admissible finite displacement systems (Δu, $\Delta \varphi$) containing ($\Delta u = 0$, $\Delta \varphi = 0$) as an interior point, for which the **total work** done by the actual external and internal forces along **any** (Δu, $\Delta \varphi$) of the set is **negative**.

Definition: The actual state of a masonry system is **critical** if there exists any mechanically admissible virtual displacement systems (δu, $\delta \varphi$) for which the **total virtual work** done by the actual external and internal forces along (δu, $\delta \varphi$) of the set is **zero**.

[similarly to unstable / neutral equilibrium]

Stability analysis:

Assume that an equilibrated contact force system was found to the given loads!

$\Rightarrow$ conclusions???

Why does the Safe Theorem fail?

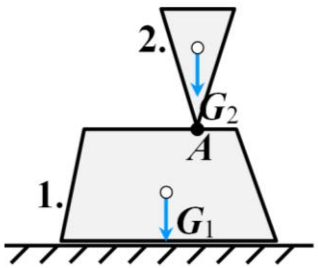
Stability analysis:

Assume that an equilibrated contact force system was found to the given loads!

$\Rightarrow$ conclusions???

Can we find a mechanically admissible Heymanian virtual displacement system in such a way that none of the contact points opens up?

If we can:



Case (ii) found for at least one infinitesimally small perturbation:

$$\text{(ii)} \quad \sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b + \mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) = 0$$

$\Rightarrow$ **CRITICAL STATE!**

If we cannot:

Case (i) found for every perturbation of the position in a small finite neighborhood of the analysed position:

$$\text{(i)} \quad \sum_{(b)} \left(\mathbf{G}_b \cdot \Delta\mathbf{u}_b + \mathbf{M}_b \cdot \Delta\boldsymbol{\varphi}_b \right) < 0$$

$\Rightarrow$ **STABLE STATE against Heymanian displacements!**

Corrected Formulation of the Safe Theorem

<u>Assumptions:</u>

(a) the masonry blocks are polyhedral

(b) the masonry blocks are infinitely rigid and infinitely strong

(c) the contacts transmit no tension

<u>The Safe Theorem:</u>

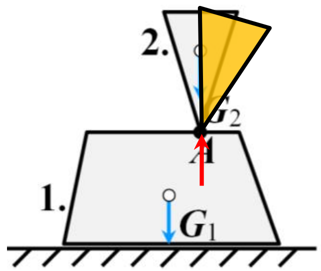
If there exists any system of forces satisfying (a-c) being in equilibrium with the loads, and

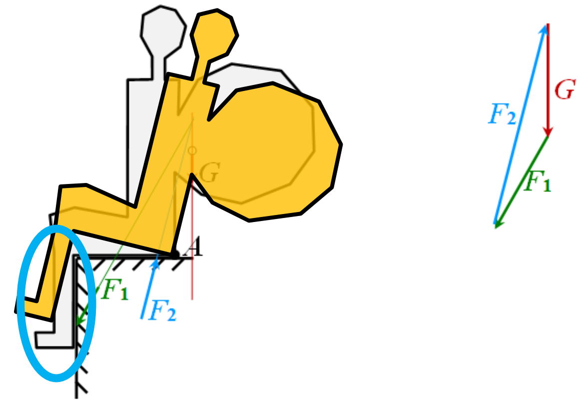
if there does not exist any mechanically admissible Heymanian virtual displacement system for which all contact points of this force system remain closed,

then the structure is safe against collapse along any Heymanian displacements.

Examples when the Safe Theorem Fails

Explanation:





Pyramid Upwards Down

Rimon’s Backpack

Pyramid Upwards Down: *Safe Theorem is not valid for this !*

$\rightarrow$ a Heymanian virtual displacement system exists so that no contact opens up

$\rightarrow$ the equilibrium state is critical; higher-order analysis reveals: unstable

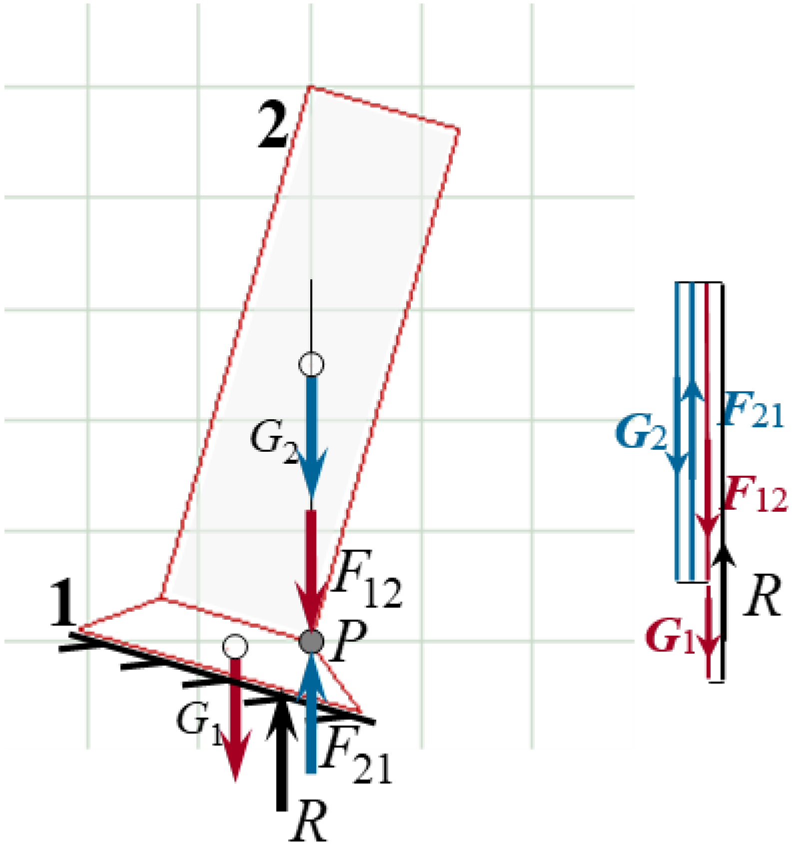
Rimon’s Backpack: *Safe Theorem is only for Heymanian disps !*

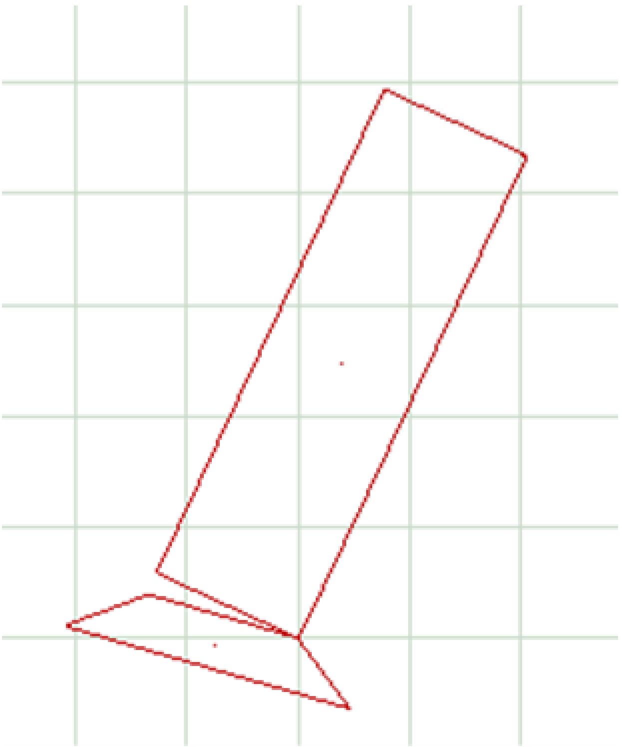
$\rightarrow$ contact always opens up $\Rightarrow \oplus$ external work for any Heymanian systems

$\rightarrow$ Safe Theorem: no protection against collapse along non-Heymanian disps

Examples

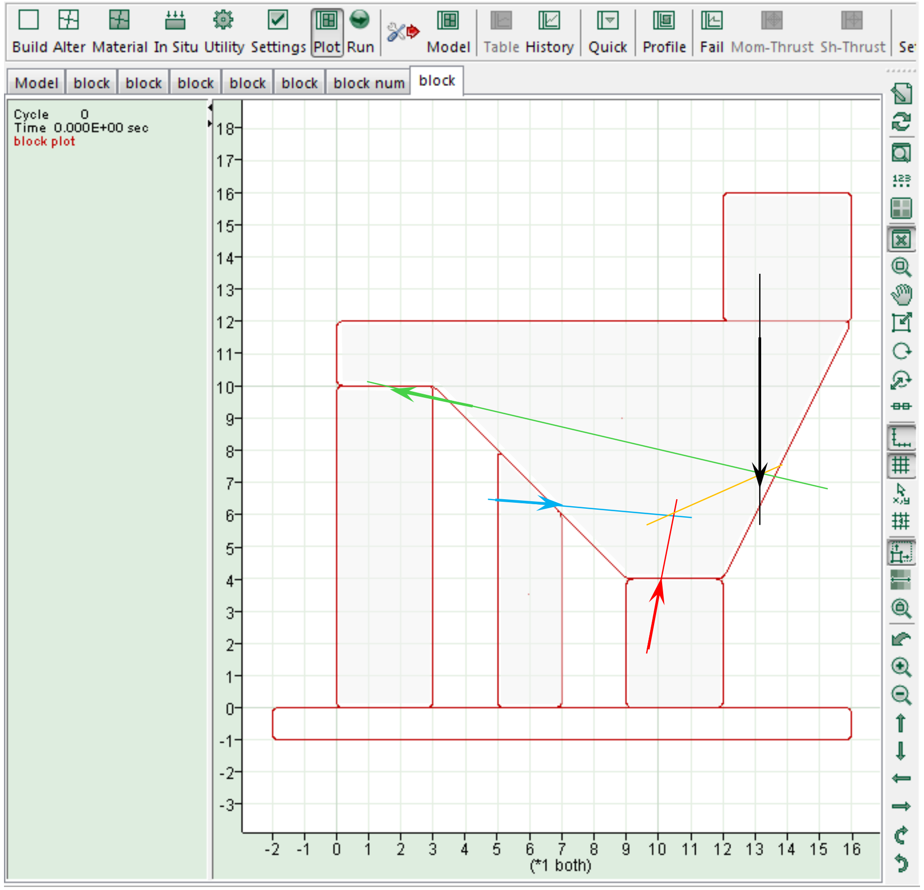
„Pisa Tower":

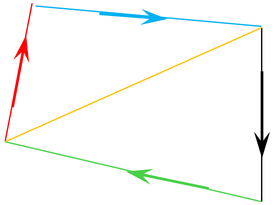




Examples

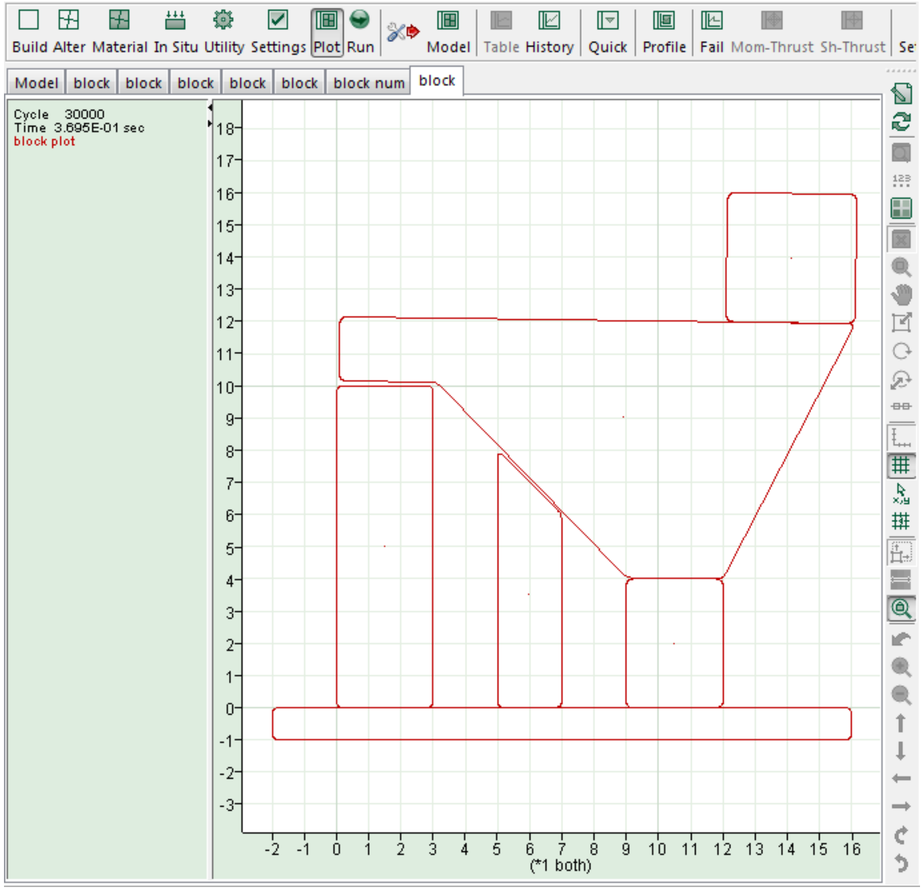
Three-Coloumn Roof:





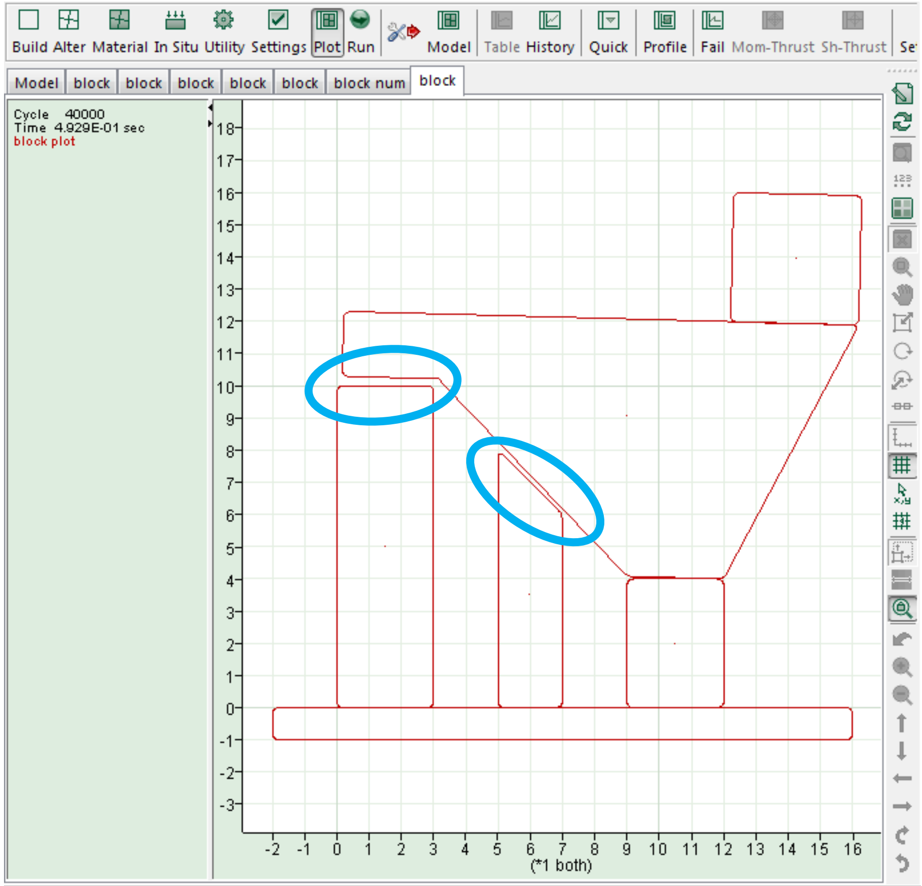
Examples

Three-Coloumn Roof:



Examples

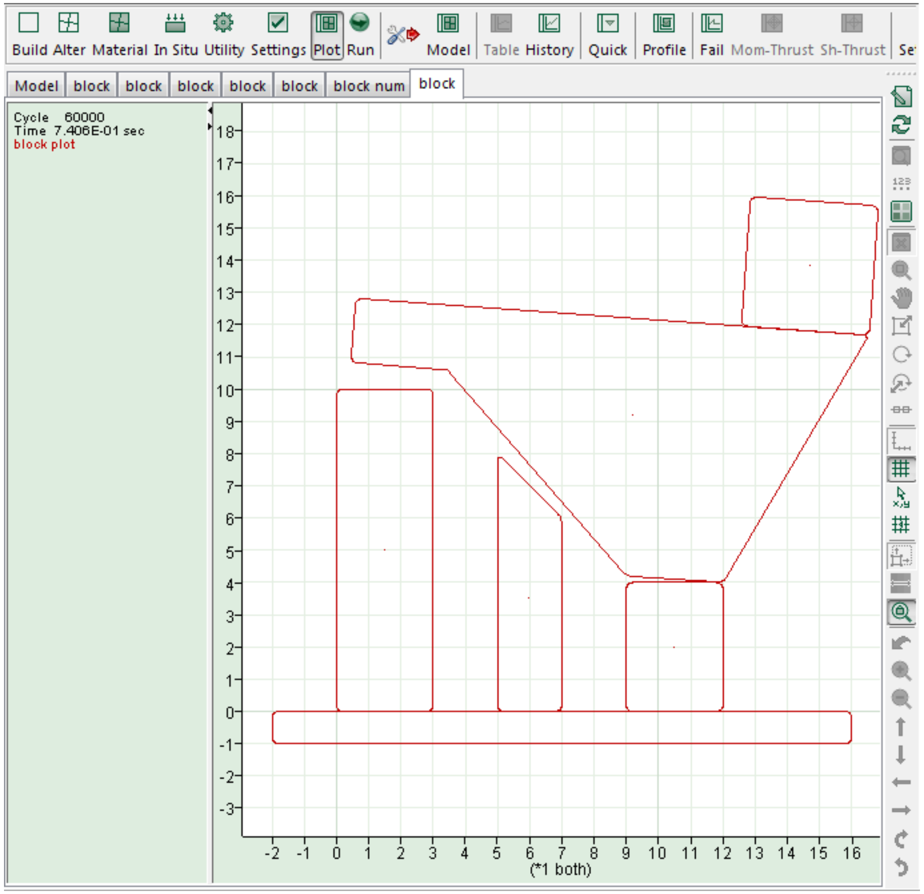
Three-Coloumn Roof:



Non-Heymanian displacement system!

Examples

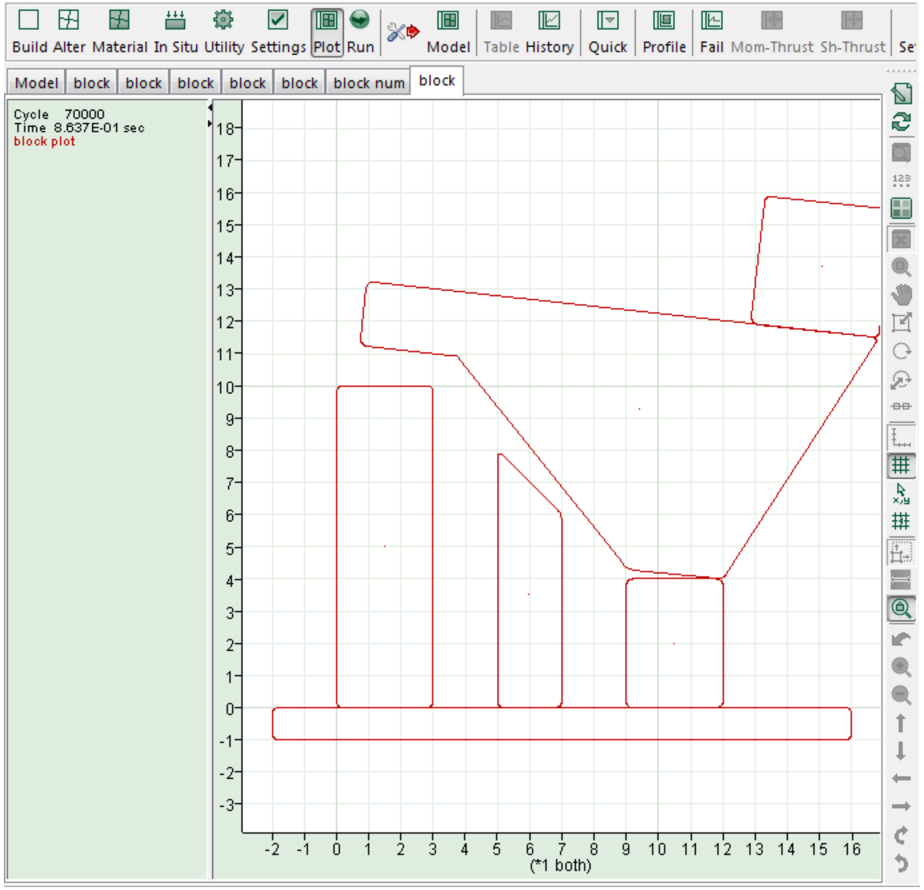
Three-Coloumn Roof:



Non-Heymanian displacement system!

Examples

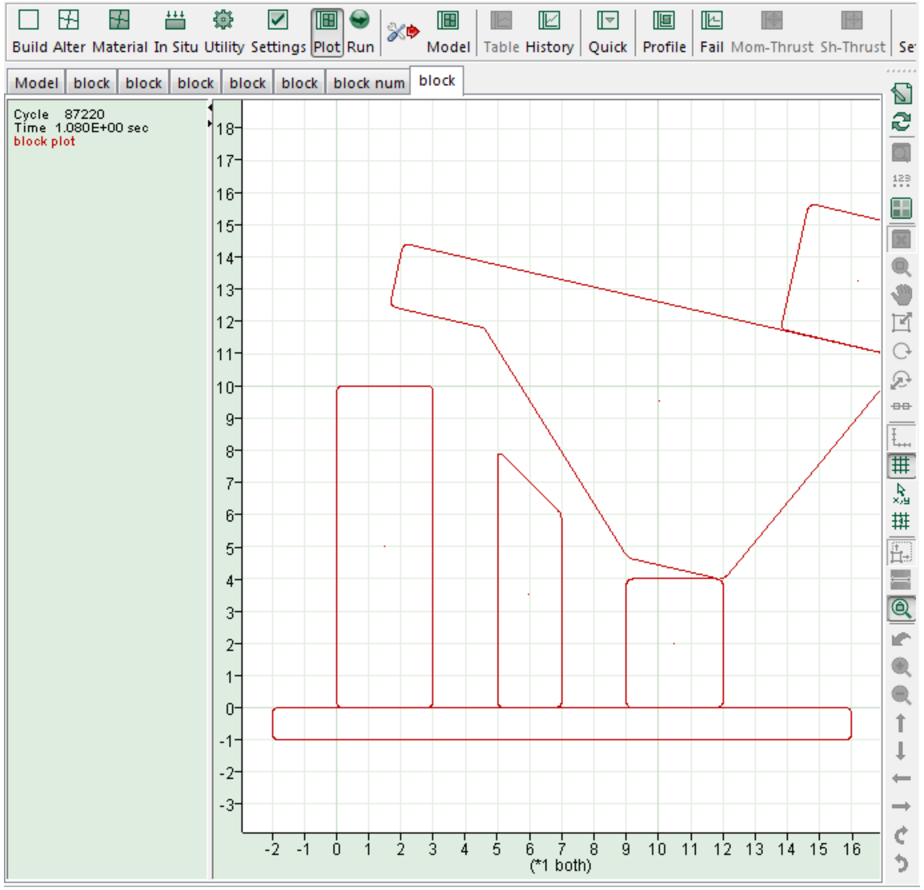
Three-Coloumn Roof:



Non-Heymanian displacement system!

Examples

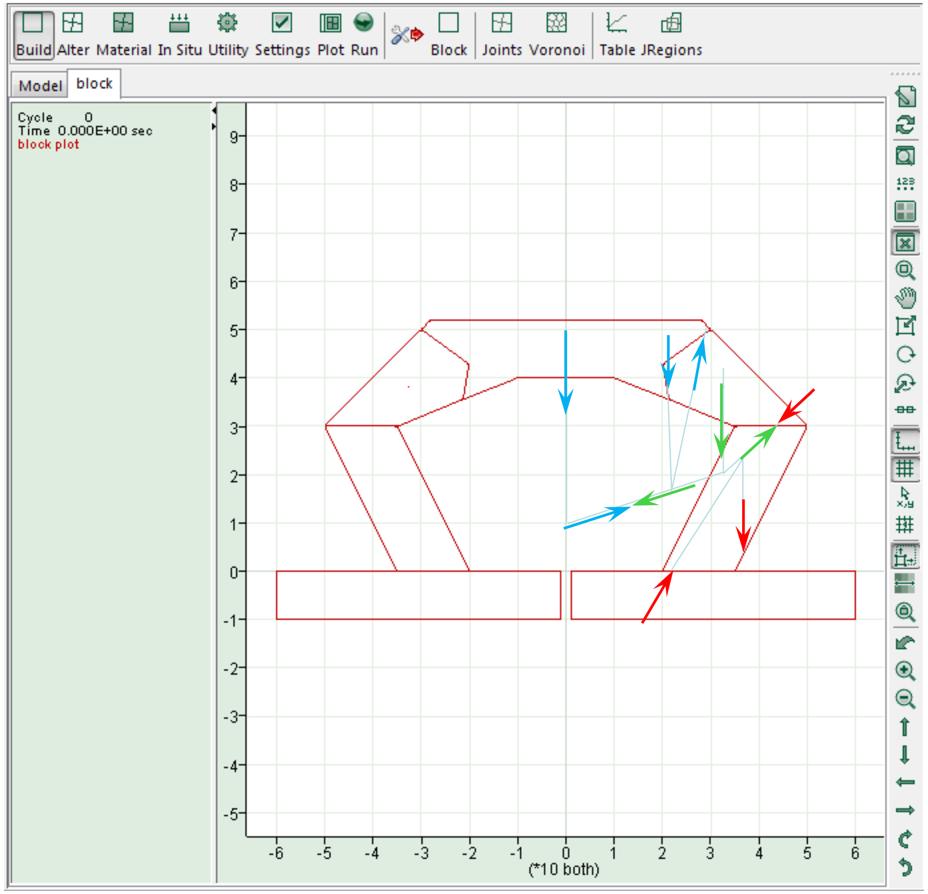
Three-Coloumn Roof:

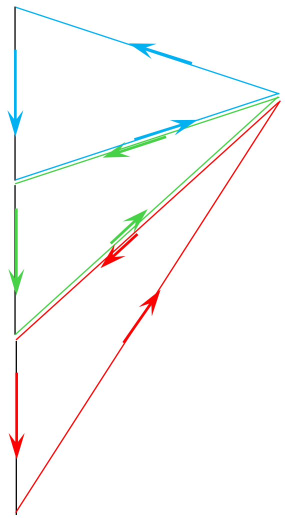


Non-Heymanian displacement system!

Examples

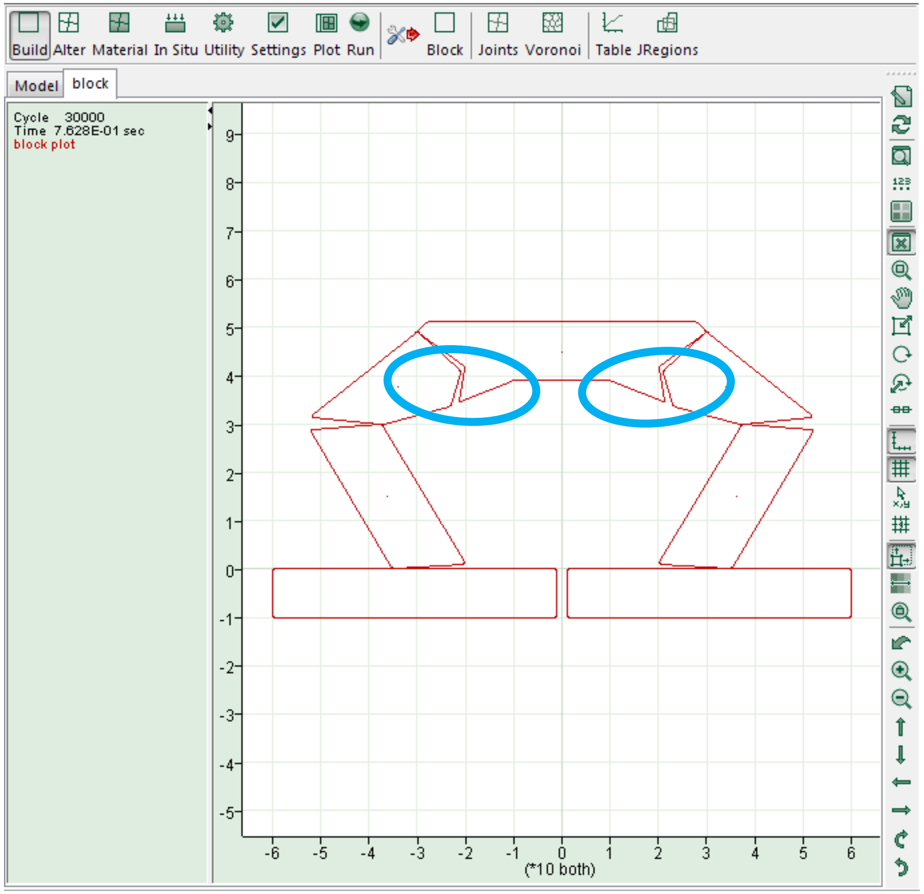
Buckling Arch:





Examples

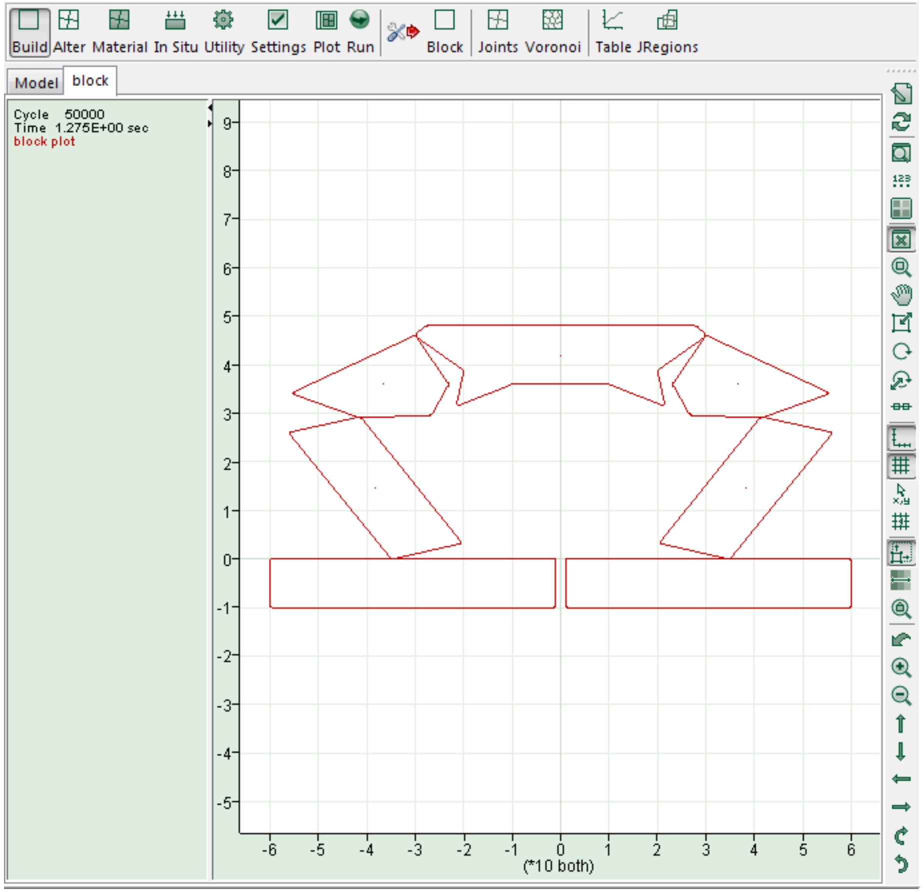
Buckling Arch:



Non-Heymanian displacement system!

Examples

Buckling Arch:



Non-Heymanian displacement system!

Conclusion:

the Safe Theorem does not give protection against collapses along non-Heymanian displacements!

THIS LECTURE

Repetition: Plastic limit theorems

Heyman's limit state theory for masonry

→ assumptions about the material

→ The Static Theorem („Safe Theorem")

→ The Kinematic Theorem („Unsafe Theorem")

Examples when the Safe Theorem fails

Why does the Safe Theorem fail?

Example when the Unsafe Theorem fails

Practical engineering codes

→ Archie-M

→ LimitState:Ring

Questions

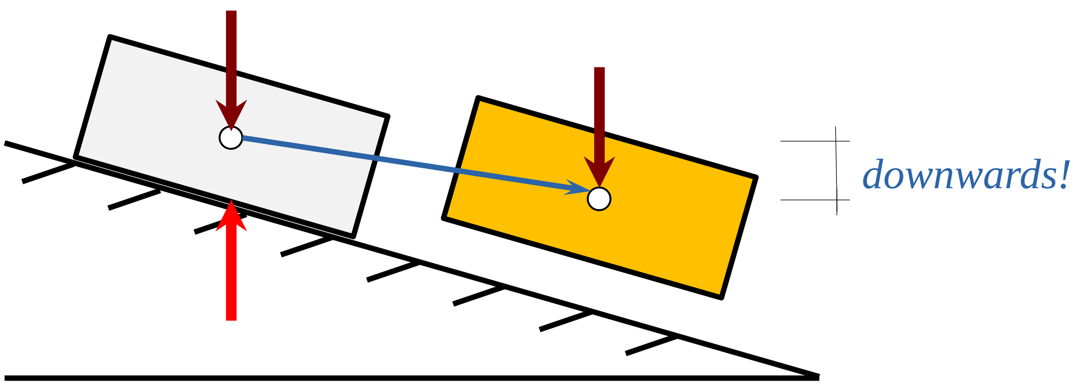
Example when the Unsafe Theorem Fails

Brick on an Inclined Plane:

$\delta W > 0$: collapses according to the Unsafe Theorem

BUT: for [friction angle] > [angle of the slope]:

in fact THE STRUCTURE IS SAFE!



notice again: presence of tangential relative translation

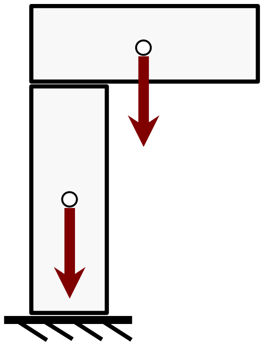
$\Rightarrow$ the Unsafe Theorem fails

Conclusion:

the Unsafe Theorem is
unnecessarily too conservative if
non-Heymanian displacements are present!

Heyman's limit state theory for masonry

For home:

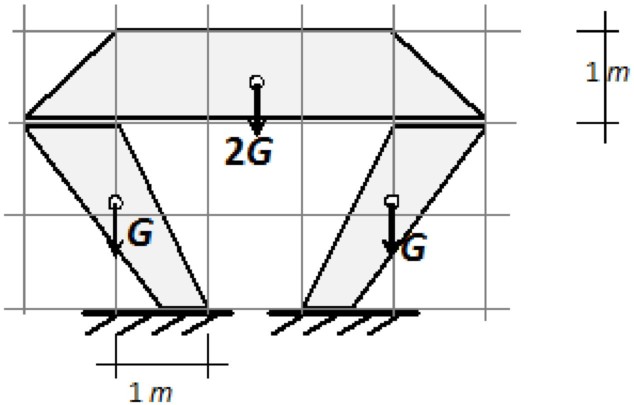


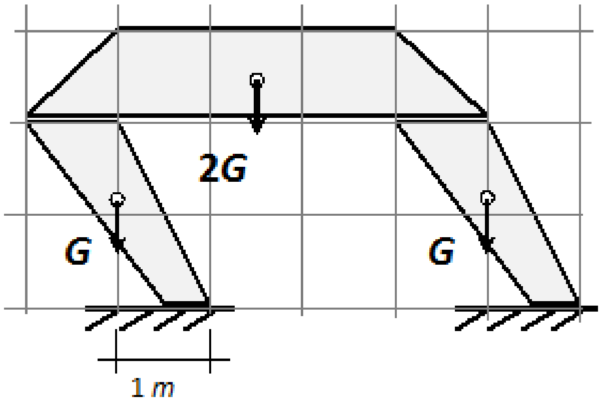
Try to apply both theorems:

Is the structure safe?

Under what geometrical conditions can the structure be safe?

For home:





Try to apply both theorems: are these structures safe?

Answer:

↑ safe ↑ collapses

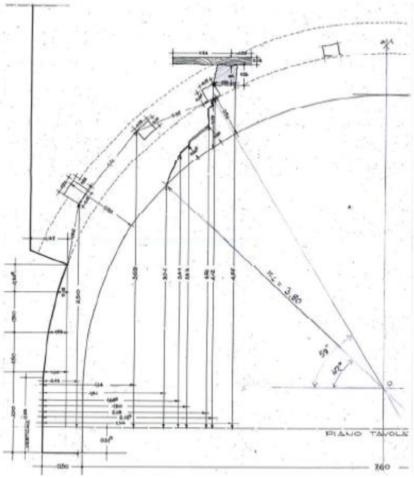
Shear failure of historic structures

http://viafrancigenatoscana.it

Beatini et al (under review):

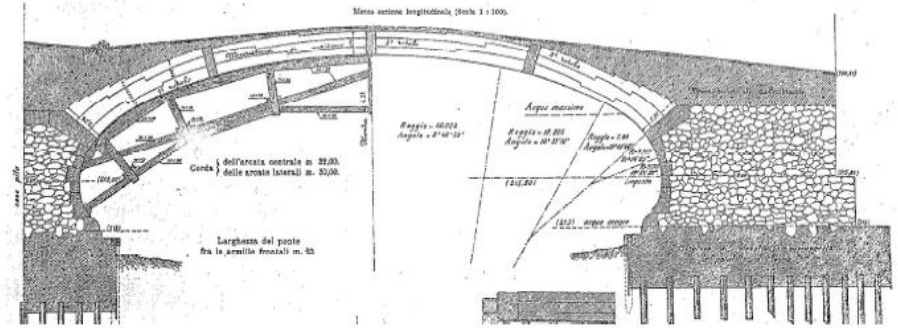
http://wikimapia.org/13139959/it/Duomo-di-San-Martino

Cathedral of Pietrasanta, Italy:



Umberto Bridge, Turin, Italy:

„multiring arch”



rings slide on each other;
protection against
shear failure

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Practical engineering codes

$\rightarrow$ Archie-M

$\rightarrow$ LimitState:Ring

Questions

Practical engineering codes

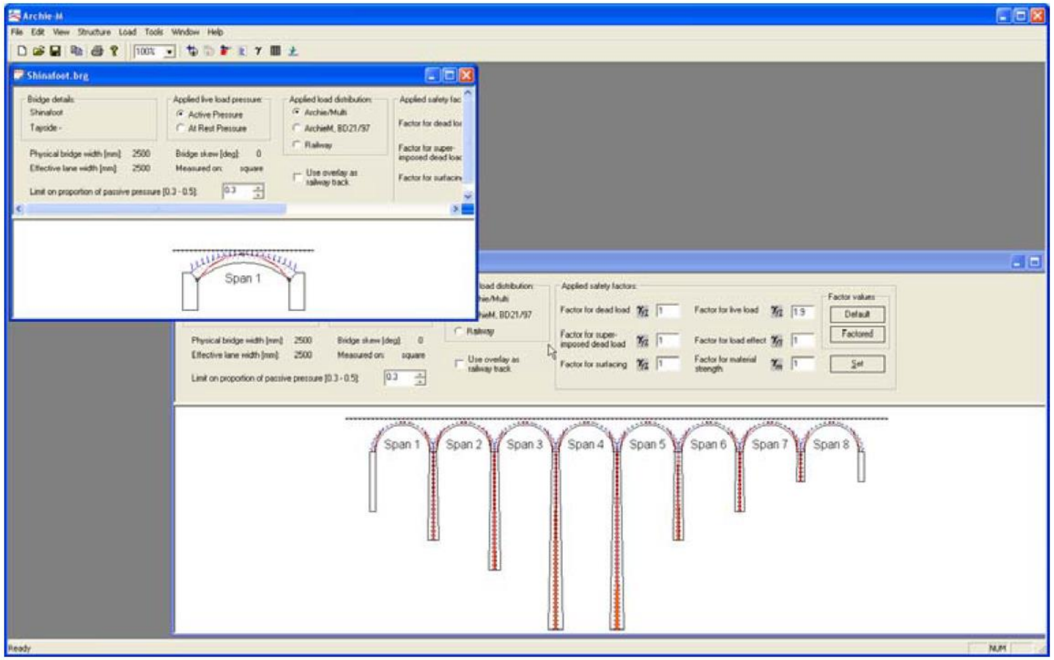
e.g. Archie-M: http://www.obvis.com

Bill Harvey, developed since 1981

based on the Static Theorem: frictional sliding excluded;

an equilibrum force system is searched for

~~thrust line~~ $\rightarrow$ thrust zone, according to the finite strength



single-span or multispan

Practical engineering codes

e.g. Archie-M: http://www.obvis.com

Bill Harvey, developed since 1981

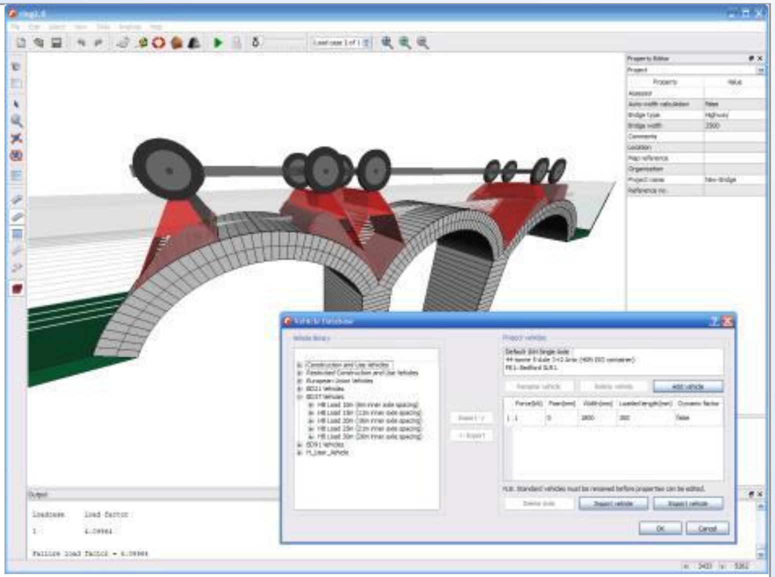
based on the Static Theorem: frictional sliding excluded;

an equilibrum force system is searched for

~~thrust line~~ → thrust zone, according to the finite strength

skew bridges: formally, as if they were straight

Practical engineering codes



e.g. LimitState:Ring:

Gilbert & Melbourne (1994); Gilbert (2007)

based on the parallel use of the

kinematic AND the static theorem

The basic line of thought for live load in addition to selfweight:

→ the analysis would be this, in case of manual calculations:

(i) choose a likely mechanism of collapse;

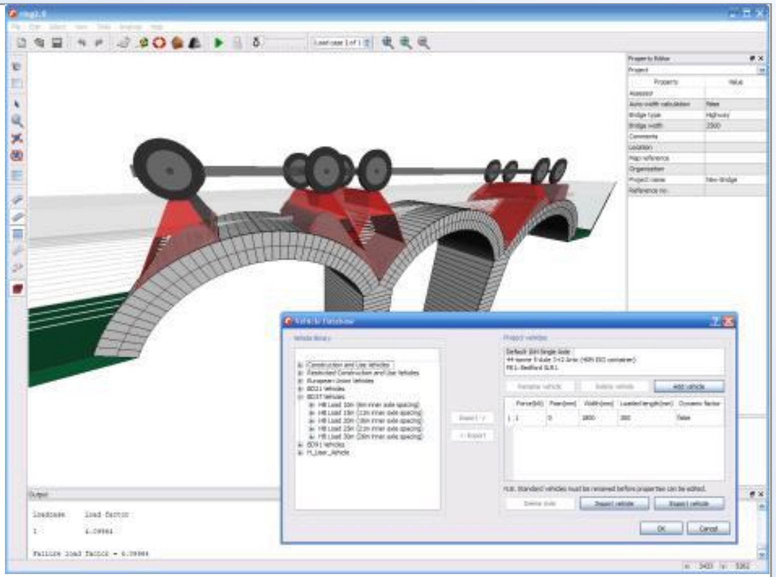
(ii) compile the work, to calculate the collapse load;

(iii) try other likely collapse mechanisms until the critical one is found.

→ instead:

a computerized solution technique to systematically find the collapse mode with smallest load, for which the equilibrium eqs are also satisfied

Practical engineering codes



e.g. LimitState:Ring:

Gilbert & Melbourne (1994); Gilbert (2007)

→ associated contact friction model:

dilation equals to friction angle

experiences for multiring arches:

„RING is usually on the safe side"

→ block crushing taken into account: „thrust zone" instead of „thrust line"

→ multispan bridges; multiring arches with frictional contacts between layers

→ distributing effect of backfill taken into account

→ experimental and industrial validations:

„RING is usually on the safe side"

http://www.limitstate.com/ring/experimental-validation

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Questions

QUESTIONS

1. In case of general plasticity, what does the static theorem state, and what does the kinematic theorem state? Under what condition will the largest λ_S surely coincide with the smallest λ_K ?

2. What are the basic assumptions in Heyman's theory for the material behaviour? Justify / criticize them.

3. What does Heyman's static theorem state? What does Heyman's kinematic theorem state? Illustrate their use on an example. Show examples when these theorems fail.

4. Introduce the fundaments of the LimitState:Ring and Archie-M codes.