Basic geometry, trigonometric functions

1 Circle

All points are the same distance from the center. The **Radius** is the distance from the center outwards. The **Diameter** goes straight across the circle, through the center. The **Circumference** is the distance once around the circle.

\[
\frac{\text{Circumference}}{\text{Diameter}} = \pi = 3.14159...
\]

When we divide the circumference by the diameter we get 3.141592654... which is the number \( \pi \).

The area of a circle is \( \pi \) times the radius squared, which is written: \( A = \pi r^2 \). The circumference is \( 2 \pi r \).

**Special lines and parts of a circle**

**Lines:**

A line that "just touches" the circle as it passes by is called a **Tangent**. It always forms a right angle with the circle's radius.

A line that cuts the circle at two points is called a **Secant**.

A line segment that goes from one point to another on the circle's circumference is called a **Chord**.

If it passes through the center it is called a **Diameter**.

And a part of the circumference is called an **Arc**.

**Slices:**

The "pizza" slice is called a **Sector**.

And the slice made by a chord is called a **Segment**.

**Common Sectors:**

The Quadrant and Semicircle are two special types of Sector:

- Quarter of a circle is called a Quadrant.
- Half a circle is called a Semicircle.

1.1 **Arc length**

If we know the central angle corresponding to the arc, we can use its proportion to 360° and the circumference of the circle. Let the central angle of the arc be \( \Theta \) degrees and the radius of the circle be \( r \):

\[
L = \frac{\Theta}{360^\circ} \cdot 2 \cdot r \cdot \pi = \frac{\Theta}{180^\circ} \cdot r \cdot \pi
\]
If we have $\Theta$ in degrees (which we do), then the $\Theta/180^\circ$ part of the equation means that we are converting our angle from degrees to radians. If we already have our angle in radians, we can simply write:

$$L = \Theta [\text{rad}] \cdot r$$

Surveyors must know several relationships between an angular value and its corresponding subtended distance. Linear error caused by a small angular error at a specific distance can be approximated by the arc length of the angle. The precision of instruments for angular measurement are given in seconds of arc.

Let’s calculate the linear errors caused by small angular errors given in arc seconds! For this we should change the angle given in seconds to radians first, then multiply it by the distance. First let’s calculate how many seconds corresponds to one radian?

$$\pi \text{rad} = 180^\circ \quad \text{and} \quad 1^\circ = 3600^\prime$$

$$1 \text{rad} = \left(\frac{180^\circ}{\pi}\right)^\circ = \left(\frac{180 \cdot 3600}{\pi}\right)^\prime = 206264,806^\prime$$

$$\rho^\prime (\text{rho seconds}) = 206264,806^\prime \approx 200000^\prime$$

When changing angles from radians to arc seconds we should multiply the value given in radians with this $\rho^\prime$, and when changing from arc seconds to radian we should divide the given value with $\rho^\prime$. When we are working with small angles and not very long distances this $\rho^\prime$ can be rounded to $200000^\prime = 2 \cdot 10^5^\prime$.

**Example 2.1:** How big is the linear error in millimeters caused by $10''$ angular error in 100 m?

$$L = \Theta [\text{rad}] \cdot r = \left(\frac{10''}{\rho^\prime}\right) \cdot 100 \text{ m} \approx \left(\frac{10''}{200000^\prime}\right) \cdot 100 \text{ m} = 0.005 \text{ m} = 5 \text{ mm}$$

<table>
<thead>
<tr>
<th>Standard error of angular measurement</th>
<th>Linear error in 1000 Units. (Meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>05’</td>
<td>1.454</td>
</tr>
<tr>
<td>01’</td>
<td>0.291</td>
</tr>
<tr>
<td>30’</td>
<td>0.145</td>
</tr>
<tr>
<td>20’</td>
<td>0.097</td>
</tr>
<tr>
<td>10’</td>
<td>0.048</td>
</tr>
<tr>
<td>05’</td>
<td>0.024</td>
</tr>
<tr>
<td>01’</td>
<td>0.005</td>
</tr>
</tbody>
</table>

Comparison of Angular and Linear Errors in 1000 m

### 1.2 Circle Theorems

**Inscribed Angle**

Inscribed Angle: an angle made from points sitting on the circle's circumference. A and C are "end points" B is the "apex point"

Equivalently, an inscribed angle is defined by two chords of the circle sharing an endpoint.

The inscribed angle theorem relates the measure of an inscribed angle to that of the central angle subtending the same arc.

**Inscribed Angle Theorems**

The inscribed angle theorem (or Angle at the Center Theorem) states that an angle $\theta$ inscribed in a circle is half of the central angle $2\theta$ that subtends the same arc on the circle. Therefore, the angle does not change as its vertex is moved to different positions on the circle.
Angle at the Center Theorem: An inscribed angle $a^\circ$ is half of the central angle $2a^\circ$ and (keeping the endpoints fixed) the angle $a^\circ$ is always the same, no matter where it is on the circumference:

**Thales' Theorem (or Angle in a Semicircle)**

The diameter of a circle always subtends a right angle to any point on the circle. (An angle inscribed in a semicircle is always a right angle.)

Put another way: If a triangle has, as one side, the diameter of a circle, and the third vertex of the triangle is any point on the circumference of the circle, then the triangle will always be a right triangle.
2 Angles

2.1 Types of angles

Acute Angle is less than 90°
Right Angle is 90° exactly
Obtuse Angle is greater than 90° but less than 180°
Straight Angle is 180° exactly
Reflex Angle is greater than 180°
Full Rotation is 360° exactly

Parts of angles:
The corner point of an angle is called the vertex
And the two straight sides are called arms
The angle is the amount of turn between each arm.

2.2 Pairs of angles

Classification of angles using the figure above
- Opposite angles: a-d, b-c, e-h, g-f (equal)
- Corresponding angles: a-e, b-f, c-g, h-d (equal)
- Alternate interior angles: c-f, d-e (equal)
- Alternate exterior angles: a-h, b-g (equal)
- Consecutive interior angles: c-e, d-f (c + e = 180°)
- Two angles are Supplementary when they add up to 180 degrees: a-b, b-d, d-c, c-a, e-f, f-h, h-g, g-e (Notice that together they make a straight angle. These two (60° and 120°) are supplementary because 60° + 120° = 180°)
- Two angles are Complementary when they add up to 90 degrees (Notice that together they make a right angle. These two (27° and 63°) are complementary because 27° + 63° = 90°)
- +1: angles with perpendicular arms (equal)

3 Triangles

3.1 Types of triangles

Triangles Contain 180°. In a triangle, the three interior angles always add to 180°: A + B + C = 180°

Classification by angles
- Obtuse: has one angle which is > 90°
- Acute: all the angles are < 90°
- Right: one angle is $90°$
- Classification by angles and sides
  - Equilateral: equal sides, equal angles (left figure)
  - Isosceles: two equal sides, two equal angles (center figure)
  - Scalene: no equal sides, no equal angles (right figure)

### 3.2 Similar and congruent triangles

Two triangles are similar if they have the same angles, or the length of the sides are in proportion.

When two triangles are congruent they will have exactly the same three sides and exactly the same three angles.

If $\triangle ADE$ is any triangle and $BC$ is drawn parallel to $DE$, then

$$\frac{AB}{BD} = \frac{AC}{CE}$$

**Example 7.1:** Calculate the height of Pepito’s house considering the given distances and Pepito’s height in the picture:

\[ \frac{54}{132} = \frac{124}{d} \implies d = \frac{124 \cdot 34}{54} = 458.6 \]

The height of the house is 458.6 m.
Example 7.2: Calculate the height of the tallest tree:
Solutions: 45.8 m

4 The right-angled triangle (right triangle)

In the right triangle $a, b$ are called legs (or arms) of the triangle, $c$ is the hypotenuse.

In a right angled triangle, the two non-right angles are complementary, because in a triangle the three angles add to $180^\circ$, and $90^\circ$ has already been taken by the right angle. The most important theorem concerning the right triangle is Pythagoras’s theorem.

4.1 Pythagoras’ theorem

Inside a right triangle like the one in the figure below, one of the most fundamental theorems is the Pythagorean Theorem that tells us that:

$$c^2 = a^2 + b^2$$

or namely, that the square of the hypotenuse is equal to the sum of the squares of the two legs of the triangle

Pythagorean triples: 3, 4, 5. They solve the Pythagorean equation. How can we tell if a triangle is a right triangle? Example 1: $a = 28.2$ m, $b = 34.4$ m, $c = 44.5$ m.

Example 8.1: Tell the correct length of $c$! (check the correct precision!)

a) $a = 16.12$ m  $b = 12.34$ m
b) $a = 16.120$ m  $b = 12.340$ m

In mathematics: 20.3 m or 20.3009852 m is also a good answer for the length of $c$, but in geodesy, the correct answer for question a) is 20.30 m, and for question b) is 20.301 m.
4.2 Right triangle trigonometry

Let the acute angle at \( A \) be \( \theta \). Using the right triangle formed by this angle, we can form exactly six ratios of the different sides:

\[
\begin{array}{cccc}
& b & a & b & a & c & c \\
\overrightarrow{c} & \overrightarrow{c} & \overrightarrow{a} & \overrightarrow{b} & \overrightarrow{b} & \overrightarrow{a}
\end{array}
\]

These ratios do not depend on the size of the triangle, only on the angle \( \theta \), as the figure below shows.

As the ratios only depend on the angle \( \theta \), we can give them unique names that involve \( \theta \), respectively: sine of \( \theta \), cosine of \( \theta \), tangent of \( \theta \), cotangent of \( \theta \), cosecant of \( \theta \) and the secant of \( \theta \). The table below gives an overview of the these six ratios and their names:

<table>
<thead>
<tr>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>opposite ( b )</td>
<td>adjacent ( a )</td>
<td>( \frac{b}{a} )</td>
</tr>
<tr>
<td>hypotenuse ( c )</td>
<td>hypotenuse ( c )</td>
<td>( \frac{a}{c} )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} & \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} & \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a} \\
\csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b} & \sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a} & \cot \theta &= \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}
\end{align*}
\]

4.3 Fundamental identities

Observing the definition of the trigonometric functions, we can find a couple of fundamental identities.

Reciprocal identities:

\[
csc(\theta) = \frac{c}{b} = \frac{1}{\sin(\theta)} \quad \sec(\theta) = \frac{c}{a} = \frac{1}{\cos(\theta)} \quad \cot(\theta) = \frac{a}{b} = \frac{1}{\tan(\theta)}
\]

Quotient identities:

\[
\tan(\theta) = \frac{b}{a} = \frac{\sin(\theta)}{\cos(\theta)} \quad \cot(\theta) = \frac{a}{b} = \frac{\cos(\theta)}{\sin(\theta)}
\]

A few more fundamental identities can be derived from the Pythagorean Theorem. These are called the Pythagorean Identities.

If we take the equation

\[b^2 + a^2 = c^2\]

and divide each side by \( c^2 \), we get

\[
\frac{b^2}{c^2} + \frac{a^2}{c^2} = 1 \Rightarrow \left( \frac{b}{c} \right)^2 + \left( \frac{a}{c} \right)^2 = 1 \Rightarrow (\sin(\theta))^2 + (\cos(\theta))^2 = 1
\]

The terms \((\sin(\theta))^2\) and \((\cos(\theta))^2\) are usually written as \(\sin^2(\theta)\) and \(\cos^2(\theta)\).

\[\sin^2(\theta) + \cos^2(\theta) = 1\]
Another identity can be obtained from the one above, if we divide each side by \( \cos^2(\theta) \):

\[
\frac{\sin^2(\theta)}{\cos^2(\theta)} + 1 = \frac{1}{\cos^2(\theta)}
\]

Replacing the identities in the equation, we can arrive at:

\[
\tan^2(\theta) + 1 = \sec^2(\theta)
\]

Similarly, if we divide by \( \sin^2(\theta) \) instead of \( \cos^2(\theta) \), we get:

\[
\cot^2(\theta) + 1 = \csc^2(\theta)
\]

The figure below shows all the identities mentioned above:

![Identity Diagram]

**Example 8.2:** Find the exact values of the expressions:

(a) \( \tan(20^\circ) - \frac{\sin(20^\circ)}{\cos(20^\circ)} \)  
(b) \( \sin^2\left(\frac{\pi}{12}\right) + \frac{1}{\sec^2\left(\frac{\pi}{12}\right)} \)

Solution:

a) 
\[
\tan(20^\circ) - \frac{\sin(20^\circ)}{\cos(20^\circ)} = \tan(20^\circ) - \tan(20^\circ) = 0
\]

b) 
\[
\sin^2\left(\frac{\pi}{12}\right) + \frac{1}{\sec^2\left(\frac{\pi}{12}\right)} = \sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right) = 1
\]

**Example 8.3:** Find the exact values of the remaining trigonometric functions given that \( \sin(\theta) = \frac{1}{3} \).

Solution: If \( \sin(\theta) \) is \( \frac{1}{3} \), the opposite side is 1 unit long, while the hypotenuse is 3 units long. Using the Pythagorean Theorem, we can find the length of the adjacent side:

\[
a^2 + 1^2 = 3^2 \Rightarrow a = \sqrt{8}
\]

\[
a = 2\sqrt{2}
\]

From here, the values of the other five trigonometric functions:

\[
\cos(\theta) = \frac{a}{c} = \frac{2\sqrt{2}}{3} \quad \quad \tan(\theta) = \frac{b}{a} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}
\]
\[
\cot(\theta) = \frac{a}{b} = 2\sqrt{2} \quad \csc(\theta) = \frac{c}{b} = 3
\]
\[
\sec(\theta) = \frac{c}{a} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}
\]

Another solution could be to compute the value of \(\cos(\theta)\) using
\[
\sin^2(\theta) + \cos^2(\theta) = 1
\]
and then use the identities to find the exact values of the functions.

4.4 Complementary angle theorem

Two angles are called complementary if their sum is 90°. As the sum of the angles in any triangle is 180°, it follows that in a right triangle the sum of the acute angles is always 90°, that is, they are complementary.

As shown in the figure above, if we are talking about the sine of angle \(B\), we mean the ratio given by the opposite side divided by the hypotenuse. The same ratio is given by taking the adjacent side to angle \(A\) and dividing it by the hypotenuse. It follows from this that the sine of \(B\) is equal to the cosine of \(A\). This is true for all the trigonometric functions:

\[
sin(B) = \frac{b}{c} = \cos(A) \quad \cos(B) = \frac{a}{c} = \sin(A)
\]
\[
tan(B) = \frac{b}{a} = \cot(A) \quad cot(B) = \frac{a}{b} = \tan(A)
\]
\[
csc(B) = \frac{c}{b} = \sec(A) \quad sec(B) = \frac{c}{a} = csc(A)
\]

Because of this property, these trigonometric functions are called the cofunctions of each other.

The theorem in words: **Cofunctions of complementary angles are equal.**

**Example 8.4:** Find the exact values of the following expressions.

a) \(\cos(35^\circ)\sin(55^\circ) + \cos(55^\circ)\sin(35^\circ)\)  

b) \(\tan(35^\circ) \cdot \sec(55^\circ) \cdot \cos(35^\circ)\)  

c) \(\cot(40^\circ) - \frac{\sin(50^\circ)}{\sin(40^\circ)}\)  

d) \(1 - \cos^2(20^\circ) - \cos^2(70^\circ)\)

**Solution:**

a) \(\cos(35^\circ)\sin(55^\circ) + \cos(55^\circ)\sin(35^\circ) = \sin(55^\circ)\sin(55^\circ) + \cos(55^\circ)\cos(55^\circ) = \sin^2(55^\circ) + \cos^2(55^\circ) = 1\)

b) \(\tan(35^\circ) \cdot \sec(55^\circ) \cdot \cos(35^\circ) = \frac{\sin(35^\circ)}{\cos(35^\circ)} \cdot \cos(35^\circ) \cdot \csc(35^\circ) = \sin(35^\circ) \cdot \frac{1}{\sin(35^\circ)} = 1\)
c) \[ \cot(40^\circ) - \frac{\sin(50^\circ)}{\sin(40^\circ)} = \cot(40^\circ) - \frac{\cos(40^\circ)}{\sin(40^\circ)} = \cot(40^\circ) - \cot(40^\circ) = 0 \]

d) \[ 1 - \cos^2(20^\circ) - \cos^2(70^\circ) = 1 - (\cos^2(20^\circ) + \sin^2(20^\circ)) = 1 - 1 = 0 \]

Example 8.5: From a parking lot we want to walk to a house on the ocean. The house is located 500 meters down a paved path that parallels the beach, which is 150 meter wide. Along the path you can walk 100 meters per minute, but in the sand on the beach you can only walk 30 meters per minute. See the illustration.

a) Calculate the time \( T \) if we walk 500 meters along the paved path then walk 150 meters in the sand to the house.

b) Calculate the time \( T \) if we walk in the sand directly towards the ocean for 150 meters and then turn left and walk along the beach for 500 to the house.

c) Express the time \( T \) to get from the parking lot to the beachhouse as a function of the angle \( \theta \) shown in the illustration.

d) Calculate the time \( T \) if we walk directly from the parking lot to the house.

e) Graph \( T = T(\theta) \). For what angle \( \theta \) is \( T \) the least? What is the value of \( x \) for this angle. What is the minimum time?

Solution:

a) \( T = \frac{500}{100} + \frac{150}{30} = 5 + 5 = 10 \) min

b) \( T = \frac{150}{30} + \frac{50}{3} = 5 + \frac{50}{3} = \frac{65}{3} \approx 21.67 \) min

c) The value of \( x \) as a function of \( \theta \):

\[ x = 150 \cdot \cot(\theta) \]

The length of the diagonal path \( P \) in the sand as a function of \( \theta \):

\[ \sin(\theta) = \frac{150}{P} \Rightarrow P = \frac{150}{\sin(\theta)} \]

The total time \( T \):

\[ T = \frac{500 - x}{100} + \frac{P}{30} = \frac{500 - 150 \cdot \cot(\theta)}{100} + \frac{150}{30} \]

\[ = 5 - 1.5 \cdot \cot(\theta) + \frac{5}{\sin(\theta)} = 5 \left( 1 - \frac{0.3}{\tan(\theta)} + \frac{1}{\sin(\theta)} \right) \]

d) Using the Pythagorean Theorem:

\[ P^2 = 150^2 + 500^2 \Rightarrow P \approx 522 \text{ m} \]
\[ T = \frac{522}{30} = 17.4 \text{ min} \]

e) The maximum value of \( \theta \) is 90°, that is the case we computed in a). If consider the case in d), where the value of \( \theta \) is minimal, its tangent is

\[ \tan(\theta) = \frac{150}{500} = \frac{3}{10} \]

from here \( \theta = \arctan\left(\frac{3}{10}\right) \approx 16.7° \).

Below is a table of the function values \( T(\theta) \) for every 5° of \( \theta \) between the maximum and minimum value:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( T(\theta) )</th>
<th>( \theta )</th>
<th>( T(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.7</td>
<td>17.40</td>
<td>55</td>
<td>10.05</td>
</tr>
<tr>
<td>20</td>
<td>15.50</td>
<td>60</td>
<td>9.91</td>
</tr>
<tr>
<td>25</td>
<td>13.61</td>
<td>65</td>
<td>9.82</td>
</tr>
<tr>
<td>30</td>
<td>12.40</td>
<td>70</td>
<td>9.77</td>
</tr>
<tr>
<td>35</td>
<td>11.58</td>
<td>75</td>
<td>9.77</td>
</tr>
<tr>
<td>40</td>
<td>10.99</td>
<td>80</td>
<td>9.81</td>
</tr>
<tr>
<td>45</td>
<td>10.57</td>
<td>85</td>
<td>9.89</td>
</tr>
<tr>
<td>50</td>
<td>10.27</td>
<td>90</td>
<td>10.00</td>
</tr>
</tbody>
</table>

The graph of the function:

From the table above, we can see that the minimum value of \( T \) corresponds to a \( \theta \) value between 70° and 75°. We wouldn’t be far off, if we use average of these two values: 72.5°.

A more precise approximation is 72.54° (acquired using numerical derivation), which is pretty close to our initial guess. The value of \( x \) corresponding to this \( T \):

\[ x = 150 \cdot \cot(72.54°) \approx 47.18 \text{ m} \]

The minimum time \( T_{\text{min}} \) can be found by substituting the found value of \( \theta \) into the function:

\[ T_{\text{min}} = 5 \left( 1 - \frac{0.3}{\tan(72.54°)} + \frac{1}{\sin(72.54°)} \right) \approx 9.8 \text{ min} \]
Example 8.6: (David Halliday: Fundamentals of Physics, I/19.)

Suppose that, while lying on a beach near the equator watching the Sun set over a calm ocean, you start a stopwatch just as the top of the Sun disappears. You then stand, elevating your eyes by a height $H = 1.70$ m, and stop the watch when the top of the Sun again disappears. If the elapsed time is $t = 11.1$ s, what is the radius $r$ of Earth?

19. When the Sun first disappears while lying down, your line of sight to the top of the Sun is tangent to the Earth’s surface at point $A$ shown in the figure. As you stand, elevating your eyes by a height $h$, the line of sight to the Sun is tangent to the Earth’s surface at point $B$.

Let $d$ be the distance from point $B$ to your eyes. From the Pythagorean theorem, we have

$$d^2 + r^2 = (r + h)^2 = r^2 + 2rh + h^2$$

or $d^2 = 2rh + h^2$, where $r$ is the radius of the Earth. Since $r \gg h$, the second term can be dropped, leading to $d^2 \approx 2rh$. Now the angle between the two radii to the two tangent points $A$ and $B$ is $\theta$, which is also the angle through which the Sun moves about Earth during the time interval $t = 11.1$ s. The value of $\theta$ can be obtained by using

$$\frac{\theta}{360^\circ} = \frac{t}{24 \text{ h}}.$$

This yields

$$\theta = \frac{(360^\circ)(11.1 \text{ s})}{(24 \text{ h})(60 \text{ min/h})(60 \text{ s/min})} = 0.04625^\circ.$$

Using $d = r \tan \theta$, we have $d^2 = r^2 \tan^2 \theta = 2rh$, or

$$r = \frac{2h}{\tan^2 \theta}$$

Using the above value for $\theta$ and $h = 1.7$ m, we have $r = 5.2 \times 10^6$ m.

4.5 Angle of elevation and angle of depression

The height of a building, a tree, or any similar object is measured as the length of the perpendicular from the top of the object to the ground. The measurement of height often involves right triangles.

An angle of elevation is an angle such that one ray is part of a horizontal line and the other ray represents a line of sight raised upward from the horizontal. To visualize the angle of elevation of a building, think of some point on the same horizontal line as the base of the building. The angle of
elevation is the angle through which our line of sight would rotate from the base of the building to its top. In the diagram, BAC is the angle of elevation.

An angle of depression is an angle such that one ray is part of a horizontal line and the other ray represents a line of sight moved downward from the horizontal. To visualize the angle of depression of a building, think of some point on the same horizontal line as the top of the building. The angle of depression is the angle through which our line of sight would rotate from the top of the building to its base. In the diagram, ABD is the angle of depression.

When solving right triangles, we can use the ratio of sides given above or we can use the Law of Sines or the Law of Cosines.

5 Trigonometric applications

5.1 Law of sines

In any triangle (scalene) the quotient of the sine of an angle and the opposing side is constant.

\[
\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}
\]

Proof:

1. \(h = a \sin(\beta)\) and \(h = b \sin(\alpha)\)
2. \(a \sin(\beta) = b \sin(\alpha)\)
3. \(\sin(\beta) / b = \sin(\gamma) / c\)

The same formula can be given for another height in the triangle. Combining them gives the law of sines.

5.2 Law of cosines

The law of cosines states the following (using the notation in the figure above):

\[
a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha)
\]

Proof, using the figure above:

1. \(h = b \sin(\alpha)\)
2. The side adjacent to \(\beta\) is \(c' = c - b \cdot \cos(\alpha)\)
3. Using Pythagoras’s theorem:

\[
a^2 = h^2 + (c')^2 = (b \cdot \sin(\alpha))^2 + (c - b \cdot \cos(\alpha))^2 = \\
= b^2 \sin^2(\alpha) + (c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha) + b^2 \cos^2(\alpha)) = \\
= b^2 \cdot (\sin^2(\alpha) + \cos^2(\alpha)) + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha) = \\
= b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha)
\]
6 Trigonometric exercises

1) A walking trail is laid out in the shape of a triangle. The lengths of the three paths that make up the trail are 2500 meters, 2000 meters and 1800 meters. Determine, to the nearest arc second, the measure of the greatest angle of the trail.

Solution:
The greatest angle of the triangle is opposite the longest side. As we know all the sides, we can use the law of cosines to determine the value of the angle:

\[ 2500^2 = 2000^2 + 1800^2 - 2 \cdot 2000 \cdot 1800 \cdot \cos(\gamma) \\Rightarrow \gamma = \arccos\left(\frac{2500^2 - 2000^2 - 1800^2}{-2 \cdot 2000 \cdot 1800}\right) = 82-05-48.4 \approx 82-05-48 \]

2) A field is bordered by two pairs of parallel roads so that the shape of the field is a parallelogram. The lengths of the two adjacent sides of the field are 2 kilometers and 3 kilometers, and the length of the shorter diagonal of the field is 3 kilometers.

a) Find the exact value of the cosine of the acute angle of the parallelogram.

b) Find the exact value of the sine of the acute angle of the parallelogram.

c) Find the exact value of the area of the field.

d) Find the area of the field to the nearest integer in km².

Solution:
a) We can use the two adjacent sides and the diagonal with the law of cosines to find the cosine of the acute angle:

\[ 3^2 = 3^2 + 2^2 - 2 \cdot 3 \cdot 2 \cdot \cos(\gamma) \Rightarrow \cos(\gamma) = \frac{3^2 - 3^2 - 2^2}{-2 \cdot 3 \cdot 2} = \frac{-4}{-12} = \frac{1}{3} \]

b) From the Pythagorean Identity:

\[ \sin^2(\gamma) + \cos^2(\gamma) = 1 \Rightarrow \sin(\gamma) = \sqrt{1 - \cos^2(\gamma)} = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3} \]

c) The area of the field using the gamma angle and the two adjacent sides:

\[ 3 \cdot 2 \cdot \sin(\gamma) = 3 \cdot 2 \cdot \frac{2\sqrt{2}}{3} = 4\sqrt{2} \text{ km}^2 \]

d) The area of the field to the nearest integer km²:

\[ 4\sqrt{2} = 5.6568 ... \approx 6 \text{ km}^2 \]

2) Two sides of a triangular lot form angles that measure \( \alpha = 29-10-00 \) and \( \beta = 33-59-25 \) with the third side \( c \), which is 487 meters long. To the nearest dollar, how much will it cost to fence the lot if the fencing costs $5.59 per meter?

Solution:
The third angle: \( \gamma = 180^\circ - ((29-10-00) + (33-59-25)) = 116-50-35 \)

The two unknown sides from the law of sines:

\[ a = 487 \cdot \frac{\sin(29-10-00)}{\sin(116-50-35)} = 266.0029 \approx 266 \text{ m} \]

\[ b = 487 \cdot \frac{\sin(33-59-25)}{\sin(116-50-35)} = 305.13 \approx 305 \text{ m} \]

The perimeter of the triangle:
\[ P = a + b + c = 1058 \text{ m} \]

The total fencing cost

\[ TC = P \cdot $5.59 = $5914.22 \]

3) From point \( C \) at the top of a cliff, two points, \( A \) and \( B \) are sited on the level ground. Points \( A \) and \( B \) are on a straight line with point \( D \), a point directly below point \( C \). The angle of depression of the nearer point, \( A \) is \( \alpha = -72°52'42" \) and the angle of depression of the farther point, \( B \) is \( \beta = -48°12'09" \). If the points \( A \) and \( B \) are 20 meters apart, what is the height of the cliff to the nearest meter?

(Angle of depression: an angle measure from a horizontal line. If the angle is negative, the direction it represents is below the horizontal line, if it’s positive, the direction is above the horizontal line.)

Solution:
The angle in \( \Delta ABC \) at point \( C \) is \( \gamma = \alpha - \beta = 24-40-33 \)
The angle in \( \Delta ABC \) at point \( B \) is \( \beta \) due to being alternate angles.

Law of sines in \( \Delta ABC \):

\[ \frac{\overline{CA}}{\sin(\beta)} = \frac{20}{\sin(\gamma)} \Rightarrow \overline{CA} = \frac{\sin(\beta)}{\sin(\gamma)} \cdot 20 = 35.71 \text{ m} \]

The angle in \( \Delta ACD \) at point \( A \) is \( \alpha \) due to being alternate angles.

In \( \Delta ACD \):

\[ \frac{h}{\overline{CA}} = \sin(\alpha) \Rightarrow h = \overline{CA} \cdot \sin(\alpha) = 34.13 \text{ m} \]

4) A vertical pole is braced by two wires that extend from different points on the pole to the same point on the level ground. One wire is fastened to the pole 0.5 meters from the top of the pole and makes an angle of 61 degrees with the ground. The second wire is fastened to the top of the pole and makes an angle of 66 degrees with the ground. Find the height of the pole!

Solution:
The total height of the pole can be written as \( h = 0.5 + y \), where \( y \) is the distance from the bottom of the pole to the lower wire.

Let \( x \) denote the horizontal distance between point on the ground where the wires are fixed and the pole. In the triangle created by the pole, the ground and the lower wire:

\[ \tan(61°) = \frac{y}{x} \Rightarrow x = \frac{y}{\tan(61°)} \]

In the triangle created by the pole, the ground and the upper wire:

\[ \tan(66°) = \frac{0.5 + y}{x} \]

Combining the two equations:

\[ \tan(66°) = \frac{0.5 + y}{y} \cdot \tan(61°) \]
\[
	\tan(66°) \cdot y = 0.5 \cdot \tan(61°) + y \cdot \tan(61°)
\]
\[
	\gamma \cdot (\tan(66°) - \tan(61°)) = 0.5 \cdot \tan(61°)
\]
\[
	\gamma = 0.5 \cdot \frac{\tan(61°)}{\tan(66°) - \tan(61°)} = 2.04 \text{ m}
\]

The height of the pole
\[
	h = 0.5 + y = 2.54 \text{ m}
\]

5) Imagine a 500 m high mountain on the Equator of the Earth. If a person is standing on top of the mountain (point D on the figure), he will see the first rays of the sun a little bit earlier than as if he were standing at the foot of the mountain (point P). How much is this time difference? (The radius of the Earth is 6378 km.)

Solution:
From the magenta triangle in the figure:
\[
\cos(\theta) = \frac{R}{R + 0.5} \Rightarrow \theta = \arccos\left(\frac{R}{R + h}\right) = 0.43-03
\]

The rotational velocity of the earth is:
\[
\omega = \frac{360°}{24h} = \frac{15°}{h} = 0.25° \frac{\text{min}}{}
\]

The time it takes for the Earth to rotate by \(\theta\):
\[
 t = \frac{\theta}{\omega} = \frac{0.43-03}{0.25°} = 2.87 \text{ min} \approx 3 \text{ min}
\]

6) The top of a lighthouse is 110 meters above sea level. What is the maximum distance where a ship on sea level can still see the light emitted by the lighthouse? Refraction of the light can be omitted. (The radius of the Earth is 6378 km.)

Solution:
In the blue triangle in the figure:
\[
\cos(\theta) = \frac{R}{R + h} \Rightarrow \theta = \cos\left(\frac{6378}{6378 + 0.11}\right) = 0.2011
\]

The length of the arc corresponding to the central angle \(\theta\):

\[s = R \cdot \theta \left[\text{deg}\right] \cdot \frac{\pi}{180} = 37.46 \text{ km}\]

7) A surveillance satellite circles the Earth at a height of \(h\) kilometers above the surface. Suppose that \(d\) is the distance, in kilometers, on the surface of the Earth that can be observed from the satellite. (The radius of the Earth is 6378 km.)

a) Find an equation that relates the central angle \(\Theta\) (in radians) to the height \(h\)!

The length of the line connecting the center of the Earth and satellite is \(R + h\).

In one of the triangles that make up the kite in the figure:

\[
\cos\left(\frac{\Theta}{2}\right) = \frac{R}{R + h} \Rightarrow h = \frac{R}{\cos\left(\frac{\Theta}{2}\right)} - R \quad \text{and} \quad \Theta = 2 \cdot \arccos \left(\frac{R}{R + h}\right)
\]

b) Find an equation that relates the observable distance \(d\) and \(\Theta\)!

\[d = R \cdot \theta \left[\text{rad}\right] = R \cdot \frac{\theta^\circ \cdot \pi}{180^\circ} \Rightarrow \theta^\circ = \frac{d \cdot 180^\circ}{R \cdot \pi}
\]

c) Find an equation that relates \(d\) and \(h\)! Use equations in a) and b)

\[h = \frac{R}{\cos\left(\frac{\Theta}{2}\right)} - R = \frac{R}{\cos\left(\frac{d \cdot 180^\circ}{2 \cdot R \cdot \pi}\right)} - R
\]

d) If \(d\) is to be 4500 kilometers, how high is the satellite orbit above the Earth?

From the equation in c):

\[h = \frac{R}{\cos\left(\frac{d \cdot 180^\circ}{2 \cdot R \cdot \pi}\right)} - R = \frac{6378}{\cos\left(\frac{4500 \cdot 180^\circ}{2 \cdot 6378 \cdot \pi}\right)} - 6378 = \frac{6378}{\cos\left(20.2125^\circ\right)} - 6378 = 418.55 \approx 419 \text{ km}
\]

e) If the satellite orbit is at a height of 1500 kilometers, what distance \(d\) on the surface can be observed?

From the equation a) and b)

\[d = R \cdot \frac{\theta^\circ \cdot \pi}{180^\circ} = R \cdot \frac{2 \cdot \arccos \left(\frac{R}{R + h}\right) \cdot \pi}{180^\circ} = \frac{2 \cdot \pi \cdot 6378}{180^\circ} \cdot \arccos \left(\frac{6378}{6378 + 1500}\right) = 8002.25 \text{ km} \approx 8002 \text{ km}
6.1 Practice Exercises

1) Two points A and B are on the shoreline of Lake George. A surveyor is located at a third point C some distance from both points. The distance from A to C is 180.0 meters and the distance from B to C is 120.0 meters. The surveyor determines that the measure of angle ACB is 56.3°. To the nearest tenth of a meter, what is the distance from A to B? (Solution: 151.1 m)

2) A field is in the shape of a parallelogram. The lengths of two adjacent sides are 48 meters and 65 meters. The measure of one angle of the parallelogram is 100°.
   a) Find, to the nearest meter, the length of the longer diagonal. (Sol: 87 m)
   b) Find, to the nearest meter, the length of the shorter diagonal. (Sol: 74 m)

3) Two sailboats leave a dock at the same time sailing on courses that form an angle of 112° with each other. If one boat sails at 18.5 km per hour and the other sails at 22.2 km per hour, how many kilometers apart are the boats after two hours? Round to the nearest tenth. (Solution: 67.6 km)

4) Two lighthouses are 12 kilometers apart along a straight shore. A ship is 15 kilometers from one light-house and 20 kilometers from the other. Find, to the nearest degree, the measure of the angle between the lines of sight from the ship to each lighthouse. (Solution: 37°)

5) A garden is in the shape of an isosceles trapezoid. The lengths of the parallel sides of the garden are 30 meters and 20 meters and the length of each of the other two sides is 10 meters. If a base angle of the trapezoid measures 60°, find the exact area of the garden. (Solution: $125 \cdot \sqrt{3}$ m²)

6) A telephone pole on a hillside makes an angle of 78 degrees with the upward slope. A wire from the top of the pole to a point up the hill is 4 meters long and makes an angle of 15 degrees with the pole.
   a) Find, to the nearest hundredth, the distance from the foot of the pole to the point at which the wire is fastened to the ground. (Sol.: 1.06 m)
   b) Use the answer to part a to find, to the nearest tenth, the height of the pole. (Solution: 4.1 m)

7) Three streets intersect in pairs enclosing a small park. Two of the angles at which the streets intersect measure 85 degrees and 65 degrees. The length of the longest side of the park is 275 meters. Find the lengths of the other two sides of the park to the nearest tenth meter. (Solution: 138.0 m, 250.2 m)

8) A small park is in the shape of an isosceles trapezoid. The length of the longer of the parallel sides is 3.2 kilometers and the length of an adjacent side is 2.4 kilometers. A path from one corner of the park to an opposite corner is 3.6 kilometers long.
   a) Find, to the nearest tenth, the measure of each angle between adjacent sides of the park. (Sol: 78.6°, 101.4°)
   b) Find, to the nearest tenth, the measure of each angle between the path and a side of the park. (Sol: 37.8°, 40.8°, 60.6°)
   c) Find, to the nearest tenth, the length of the shorter of the parallel sides. (Sol.: 2.3 km)

9) From a point 5 meters from the foot of a vertical monument, the measure of the angle of elevation of the top of the monument is 65 degrees. What is the height of the monument to the nearest tenth of a meter? (Solution: 10.7 m)

10) A distress signal from a ship, S, is received by two coast guard stations located 3.8 kilometers apart along a straight coastline. From station A, the signal makes an angle of 48° with the coastline and from station B the signal makes an angle of 67° with the coastline. Find, to the nearest tenth of a kilometer, the distance from the ship to the nearer station. (Solution: 3.1 km)