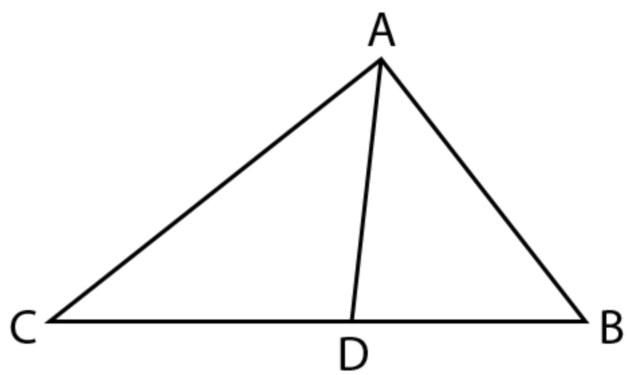
Basic geometry and trigonometry II.

1. Fundamental theorems of the triangle

The angle bisector theorem

If we have an angle bisector in the triangle ΔABC below, that is, $\angle CAD = \angle DAB$, then the side opposite to the angle is divided proportionally to the other two sides. In other words:

$$\frac{\overline{CD}}{\overline{DB}} = \frac{\overline{AC}}{\overline{AB}}$$

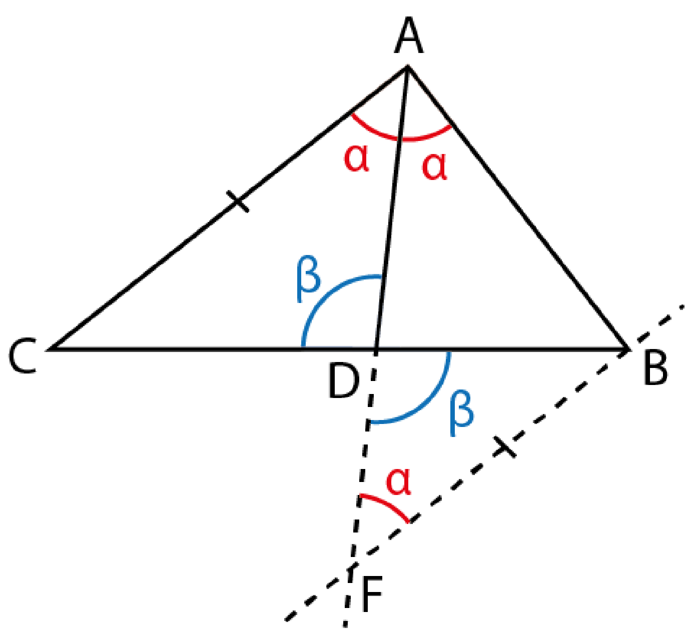


Proof:

Extend the line $\overline{AD}$ and create a parallel line with $\overline{AC}$ from point B. This line with intersect with the extended line in point F. The angles $\angle CAD$ and $\angle DFB$ are alternate interior angles which makes them equal. This means that the triangle ΔAFB is an isosceles triangle and that:

$$\overline{AB} = \overline{BF}$$

The two β angles at point D are opposing angles. The two triangles ΔADC and ΔDBF are similar triangles as all their angles are the same (they both have the same α and β angles, which means that their third angles are the same as well).



As they are similar, we can write:

$$\frac{\overline{CD}}{\overline{AC}} = \frac{\overline{DB}}{\overline{FB}}$$

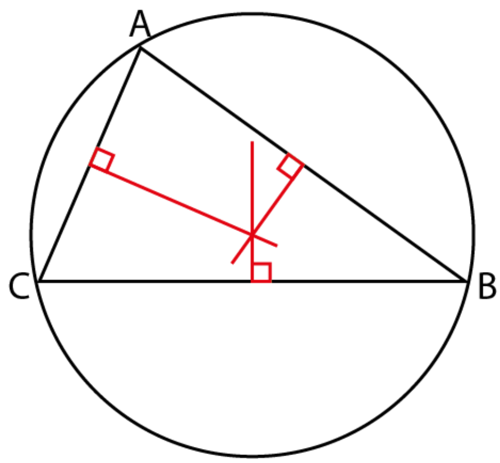
If we now substitute $\overline{FB} = \overline{AB}$, and rearrange the equation, we get:

$$\frac{\overline{CD}}{\overline{DB}} = \frac{\overline{AC}}{\overline{AB}}$$

which is the angle bisector theorem.

Concurrency of the perpendicular bisectors

The theorem states that the perpendicular bisectors of a triangle intersect in one point (they are concurrent) and that point (the circumcenter) is the center of the circumcircle of the triangle.

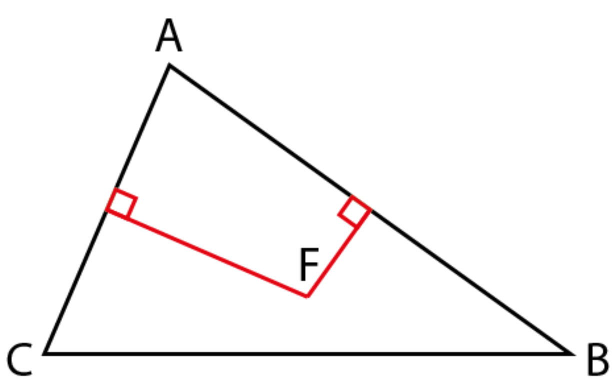


The perpendicular bisector of any side is a line that is perpendicular to that side and divides it into half.

Proof:

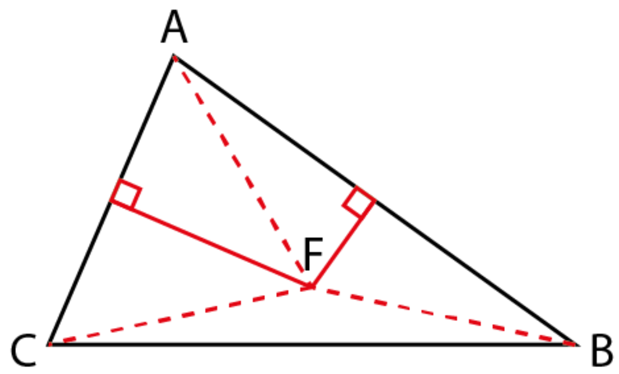
To prove this theorem, we will have to use the perpendicular bisector theorem. This states that if a point lies on the perpendicular bisector of a segment, it is equidistant from the endpoints of the bisected segment.

First, we draw the perpendicular bisectors for two sides. These two lines intersect in point D. If we can show that the third bisector goes through point D, we have proven the theorem.



Using the theorem above, we can state that because point F lies on the perpendicular bisector of side $\overline{AC}$, it is equidistant from both A and C. This means that $\overline{AF}$ is the same length as $\overline{FC}$:

$$\overline{AF} = \overline{FC}$$



As F also lies on the perpendicular bisector of side $\overline{AB}$, by the theorem it is equidistant from A and B as well. This means that:

$$\overline{AF} = \overline{AB}$$

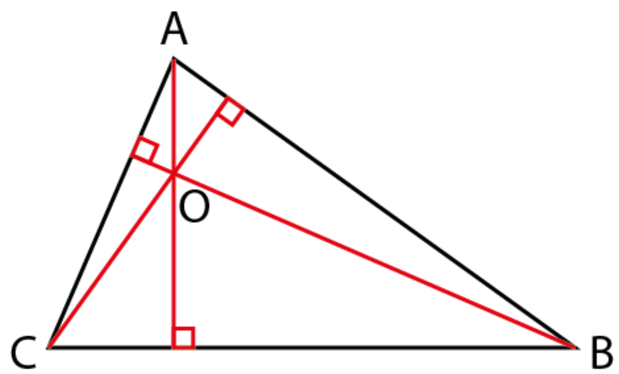
Combining the two, we get that

$$\overline{FC} = \overline{AB}$$

Applying the theorem one last time "in reverse", we can state that F has to line on the perpendicular bisector of side $\overline{CB}$ as it is equidistant from both C and B. This proves the concurrency of the three perpendicular bisectors.

The altitudes of a triangle are concurrent

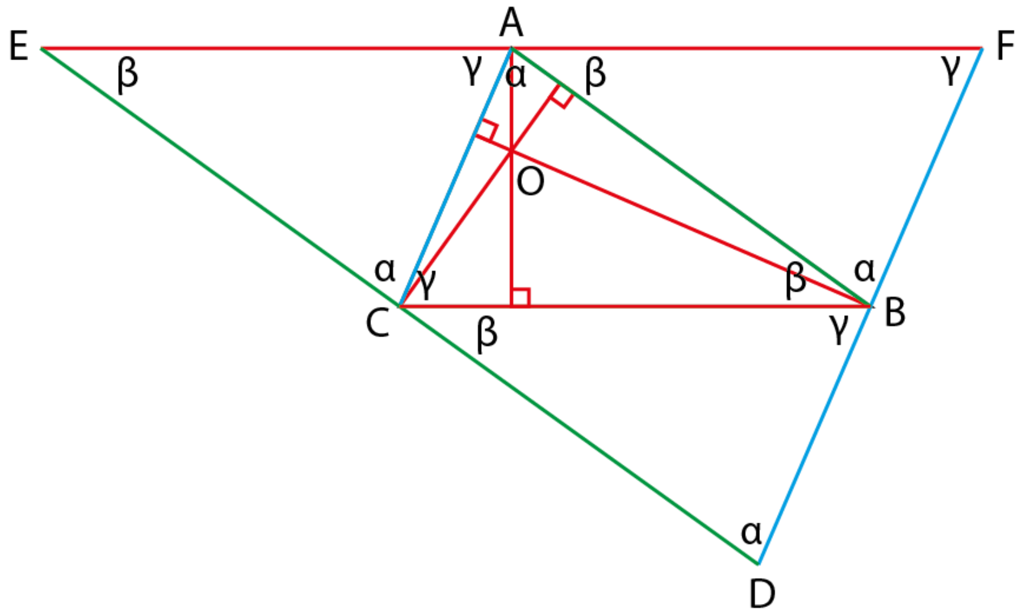
This means that all three altitudes (lines starting from a vertex and perpendicular to the opposing side) meet in one point, which is called the orthocenter of triangle.



Proof:

An easy proof can be done by establishing a Cartesian coordinate-system in one of the vertices and defining the equations of the lines, checking if they intersect in the same point. However, as we are not dealing with coordinate geometry just yet, another method has to be used.

First, we create a new triangle which has our original triangle as its so-called medial triangle. This means that the vertices of our original triangle are the midpoints of the new triangle. We can do this by taking a vertex and drawing a line through the vertex that is parallel to the opposite side (see figure below). The lines that are parallel to each other are drawn using the same color.



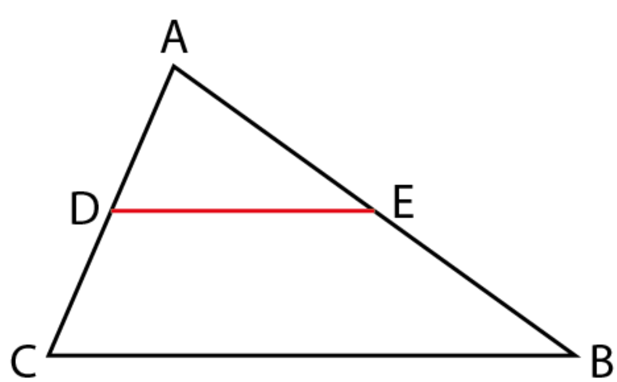
If we look at the angles, we can find that we have the same α, β and γ angles of the original triangle repeated everywhere. This is due to the fact, that the sides composing the larger triangle are all parallel to the sides of the original one. The smaller triangles $\Delta EAC, \Delta AFB, \Delta CBD$ are all congruent to the original one as they all share the same angles and one side with the original triangle.

If all of these triangles are congruent, the sides between the same angles have to be equal everywhere. For example, $\overline{EC}$ is between α and β in triangle ΔEAC, so it has to be equal to $\overline{CD}$ in triangle ΔCBD, which is also between α and β. Ultimately, all of this means that the points A, B, C, that are the vertices of the original triangle are the midpoints of the sides that make up the larger triangle.

Now, let's look at the altitudes of the original triangle. The altitude starting at point C is perpendicular to both green lines (as they are parallel). This means that the altitude from point C in the original triangle is a perpendicular bisector of the larger triangle. This is true for the other two altitudes as well. As we have already proven above that the perpendicular bisectors are concurrent for any triangle (including the larger triangle in the figure above), we have also proven the altitudes of the smaller triangle are concurrent as well.

Midsegment theorem

In any triangle, if we connect the midpoints of two sides, the resulting line will be parallel to the third side and half as long.



In other words:

$$\overline{DE} \parallel \overline{CB} \text{ and } \overline{DE} = \frac{1}{2} \cdot \overline{CB}$$

Proof:

In the proof, we have to see that the triangles ΔADE and ΔACB are similar. Because D and E are midpoints of the sides:

$$\frac{\overline{AD}}{\overline{AE}} = \frac{\frac{1}{2} \cdot \overline{AC}}{\frac{1}{2} \cdot \overline{AB}} = \frac{\overline{AC}}{\overline{AB}}$$

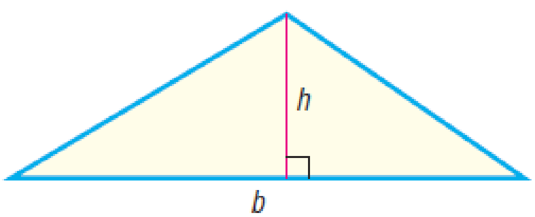
that is, the ratio between two corresponding sides is the same, in other words, they are similar triangles. As $\overline{DE}$ and $\overline{CB}$ are corresponding sides of these triangles:

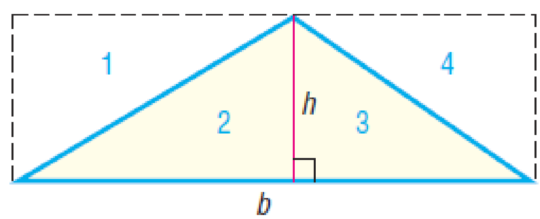
$$\frac{\overline{DE}}{\overline{CB}} = \frac{\overline{DA}}{\overline{CA}} = \frac{\frac{1}{2} \cdot \overline{CA}}{\overline{CA}} = \frac{1}{2}$$

Moreover, as the two similar triangles share an angle (namely $\angle CAB = \angle DAE$), the other two corresponding pairs of angles have to be equal as well. This concludes that $\overline{DE}$ is parallel to $\overline{CB}$.

2. Area of the triangle

Given a scalene triangle, the area can be computed using the following formulae:





- If the base (b) and height (h) of the triangle are known:

$$A = \frac{1}{2} \cdot b \cdot h$$

- If two sides (a and b) and the angle between them (γ) are known:

$$A = \frac{1}{2} \cdot a \cdot b \cdot \sin(\gamma)$$

- If all three sides (a, b and c) are known (this is also called Heron's formula):

$$s = \frac{1}{2} \cdot (a + b + c)$$

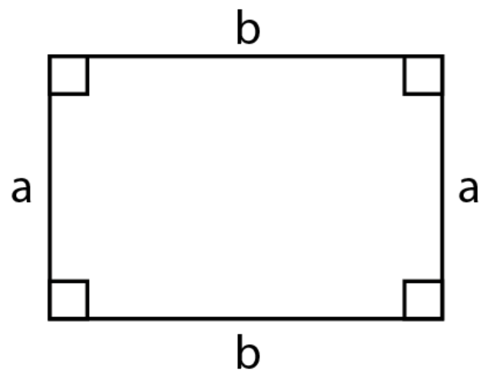
$$A = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)}$$

3. Quadrilaterals and their areas

Quadrilaterals are geometrical shapes that have four sides (as the name states). Their common trait is that the sum of their inner angles are 360°. There are certain special types of quadrilaterals, such as the following:

Rectangle

The rectangle is a quadrilateral that has only right angles and its opposite sides are parallel and equal.

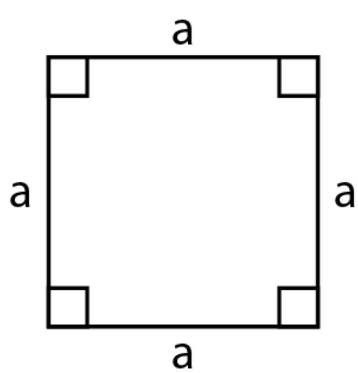


Its area can be computed by multiplying the two opposite sides:

$$A = a \cdot b$$

Square

The square has four equal sides (a) and its opposite sides are parallel. It can be considered a special type of rectangle, where all sides are equal (see above) or a special type of rhombus, where all angles are right angles (see below).

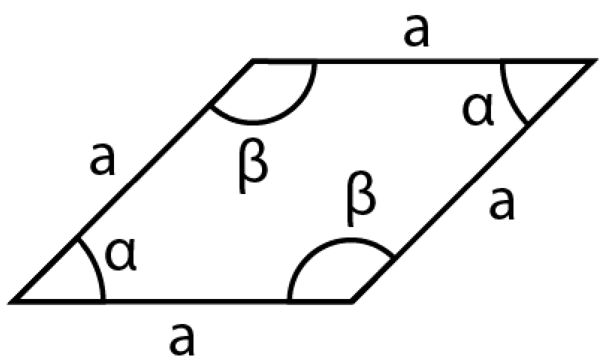


Its area is simply the square of its side:

$$A = a^2$$

Rhombus

The rhombus has equal sides, its opposite sides are parallel and its opposite angles are equal.



The area of the rhombus can be computed in multiple ways:

- If we know the length of its base (a) and height (h):

$$A = a \cdot h$$

- If we know the length of its diagonals (d_1 and d_2):

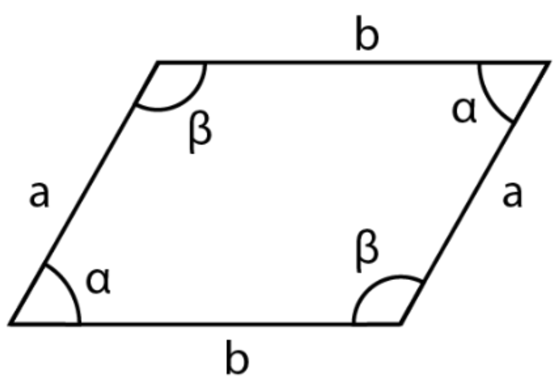
$$A = \frac{d_1 \cdot d_2}{2}$$

- If we know its base and one of its angles:

$$A = a^2 \cdot \sin(\alpha) = a^2 \cdot \sin(\beta)$$

Parallelogram

The parallelogram is a mix of the rhombus and the rectangle. Its opposite sides are equal and parallel and its opposite angles are equal as well.

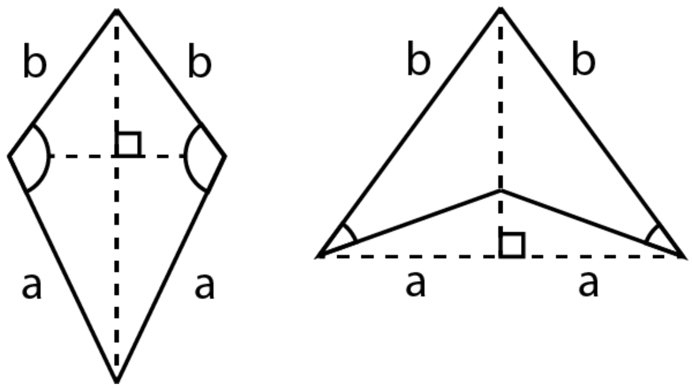


Its area can be computed similarly to the rectangle:

$$A = a \cdot b$$

Kite

The kite has two pairs of equal sides and the sides of each pair are adjacent. The angles where the two pairs meet are equal and the diagonals meet in a right angle.



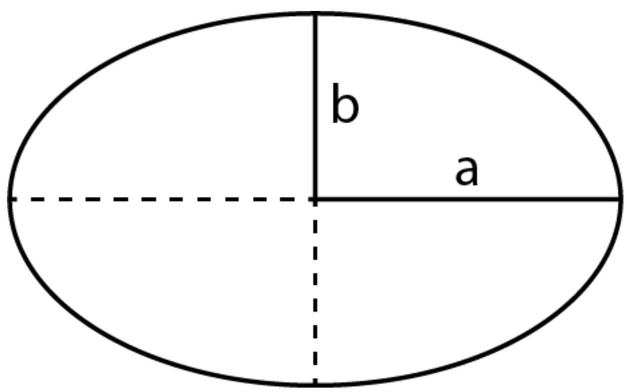
The area of a kite can be computed using its diagonals (d_1 and d_2):

$$A = \frac{d_1 \cdot d_2}{2}$$

4. Other shapes and their area

The ellipse, its area and its circumference

We can imagine the ellipse as a circle that is squashed on the top and the bottom. This results in the fact that the ellipse has not one radius, but two so called semi axes, the semi-minor axis (b) and the semi-major axis (a).



The area of the ellipse can be analytically computed using the following formula:

$$A = \pi \cdot a \cdot b$$

The circumference of the ellipse unfortunately cannot be computed analytically, however, many formulae exist that approximate this property. One of these is the Ramanujan approximation:

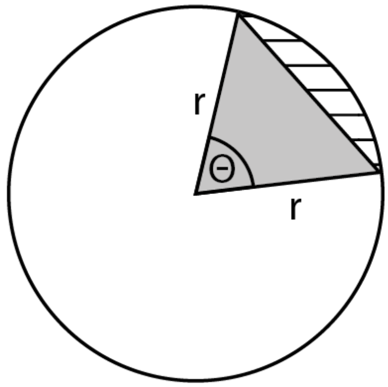
$$C \approx \pi \left[3(a+b) - \sqrt{(3a+b)(a+3b)} \right]$$

Area of a sector and segment of a circle

The area of a sector can be computed by dividing the area of the full circle by the proportion of the sector's central angle to 360°. If the central angle of the sector is Θ (measured in degrees) and the radius of the circle is r, its area is:

$$A = \frac{\Theta}{360°} \cdot r^2 \cdot \pi$$

The area of segment (filled with diagonal lines in the figure) can be calculated by computing the area of the corresponding segment and subtracting triangular area shown in the figure below:



$$A = \frac{\Theta}{360°} \cdot r^2 \cdot \pi - \frac{1}{2} \cdot r^2 \cdot \sin(\Theta) = r^2 \left(\frac{\Theta \cdot \pi}{360°} - \frac{\sin(\Theta)}{2} \right)$$

5. Arc length

The length of an arc can be found similarly to the area of a segment of a circle. If we know the central angle corresponding to the arc, we can use its proportion to 360° and the circumference of the circle.

Let the central angle of the arc be Θ degrees and the radius of the circle be r:

$$L = \frac{\Theta}{360°} \cdot 2 \cdot r \cdot \pi = \frac{\Theta}{180°} \cdot r \cdot \pi$$

If we have Θ in degrees (which we do), then the $\Theta/180°$ part of the equation means that we are converting our angle from degrees to radians. If we already have our angle in radians, we can simply write:

$$L = \Theta \text{ [rad]} \cdot r$$