Types of mean

- Mean/Arithmetic Mean/Average: The mean is the average of the numbers. It is easy to calculate: add up all the numbers, then divide by how many numbers there are. In other words it is the sum divided by the count.

$$A = \frac{1}{n} \cdot \sum_{i=1}^{n} a_i = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

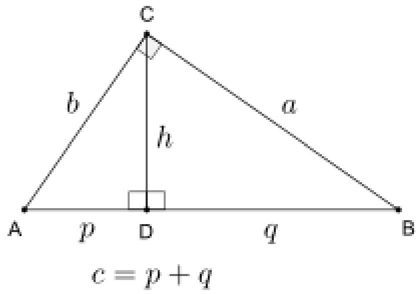
- The Geometric Mean is a special type of average where we multiply the numbers together and then take a square root (for two numbers), cube root (for three numbers) etc.

$$GM = \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} = \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n}$$

Geometric mean theorem (or Right triangle altitude theorem)

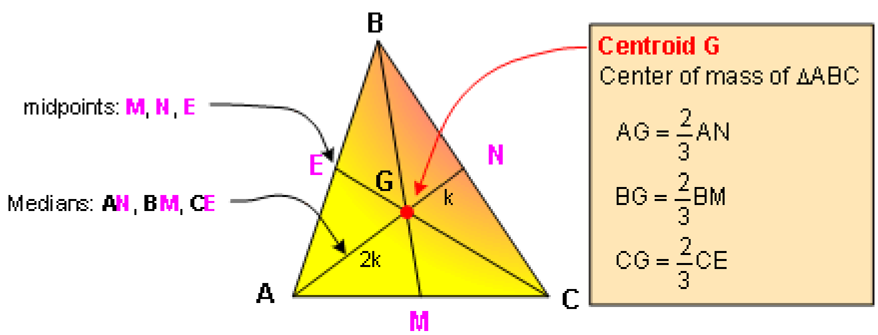
The right triangle altitude theorem or geometric mean theorem describes a relation between the lengths of the altitude on the hypotenuse in a right triangle and the two line segments it creates on the hypotenuse. It states that the geometric mean of the two segments equals the altitude. If *h* denotes the altitude in a right triangle and *p* and *q* the segments on the hypotenuse then the theorem can be stated as:

$$h = \sqrt{p \cdot q}$$



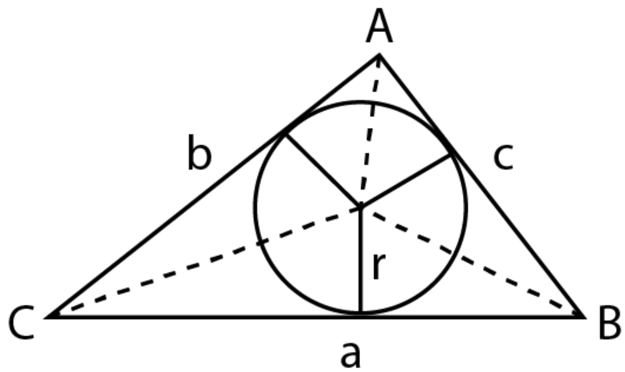
Median of triangle, centroid of triangle

Median of triangle is a line segment from a vertex (corner point) to the midpoint of the opposite side. A triangle has three medians, and they all cross over at a special point called the "centroid", which is also the "center of mass" or barycenter. The centroid divides each of the medians in the ratio 2:1, which is to say it is located ⅓ of the distance from each side to the opposite vertex.



Basic geometry and trigonometry III. (Exercises)

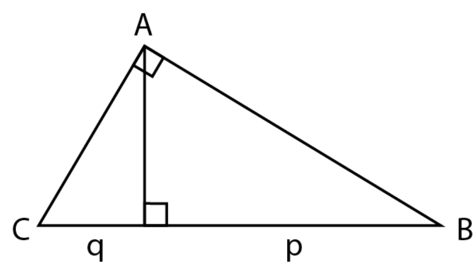
1. The radius of a triangle's incircle (inscribed circle) is 5 cm, the perimeter of the triangle is 30 cm. How much is the area of the triangle?



$$A = \frac{a \cdot r}{2} + \frac{b \cdot r}{2} + \frac{c \cdot r}{2} = (a + b + c) \cdot \frac{r}{2} = p \cdot \frac{r}{2} = 30 \cdot \frac{5}{2} = 75 \text{ cm}$$

2. The legs of a right triangle are 28 cm and 45 cm.

    2.1. Determine the proportion of the lengths of the segments (q and p on the figure), up to 4 decimals of precision, that are created by drawing the altitude for the hypotenuse.



According to the geometric mean theorem: the leg of a right triangle is the geometric mean of the hypotenuse and the perpendicularly projected length of the leg onto the hypotenuse.

The length of the hypotenuse: $c = \sqrt{28^2 + 45^2} = 53$ cm

The lengths of the segments:

$$\overline{AC} = \sqrt{q \cdot \overline{CB}} \rightarrow q = \frac{\overline{AC}^2}{\overline{CB}} = \frac{28^2}{53} = 14.79245 \text{ cm}$$

$$\overline{AB} = \sqrt{p \cdot \overline{CB}} \rightarrow p = \frac{\overline{AB}^2}{\overline{CB}} = \frac{45^2}{53} = 38.20755 \text{ cm}$$

The ratio of the segments:

$$r = \frac{p}{q} = \frac{38.20755}{14.79245} = 2.58291 \approx 2.5829$$

2.2. Determine the lengths of the segments with mm precision that are created by the bisector of the right angle.

According to the angle bisector theorem, we have to divide the hypotenuse using the proportion of the adjacent sides, that is, 28:45.

The lengths of the segments:

$$s_1 = \frac{\overline{CB}}{(28+45)} \cdot 45 = \frac{53}{73} \cdot 45 = 32.671 \approx 32.7 \text{ cm}$$

$$s_2 = \frac{\overline{CB}}{(28+45)} \cdot 28 = \frac{53}{73} \cdot 28 = 20.328 \approx 20.3 \text{ cm}$$

Check: $32.7 + 20.3 = 53$

2.3. Determine the radius of the triangle's incircle!

The area of the triangle using the two legs:

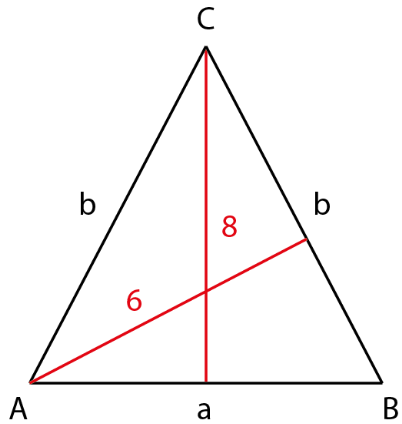
$$A = \frac{\overline{AB} \cdot \overline{BC}}{2} = \frac{28 \cdot 45}{2} = 630 \text{ cm}^2$$

The area of the triangle using the radius of the incircle (see exercise 1):

$$A = \frac{P \cdot r}{2} = \frac{(28+45+53) \cdot r}{2} = 63 \cdot r$$

$$630 = 63 \cdot r \to r = 10 \text{ cm}^2$$

3. We have an isosceles triangle. Its altitude for the base is 8 cm, one its other altitudes is 6 cm. Determine the lengths of the sides with mm precision!



The area of the triangle from the sides:

$$A = \frac{1}{2} \cdot a \cdot 8 = \frac{1}{2} \cdot b \cdot 6 \to 4a = 3b \to a = \frac{3}{4} b$$

The Pythagorean using the triangle created by the altitude:

$$b^2 = \left(\frac{a}{2}\right)^2 + 8^2$$

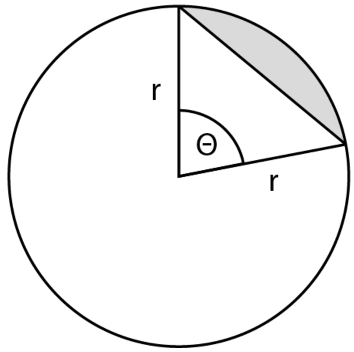
Combining the two equations and solving them:

$$b^2 = \left(\frac{3b}{8}\right)^2 + 8^2 = \frac{9b^2}{64} + 8^2 \to b = \sqrt{\frac{8^2}{\left(1 - \frac{9}{64}\right)}} = 8.630 \approx 8.6 \text{ cm}$$

Finding a from the Pythagorean:

$$a = 2 \cdot \sqrt{b^2 - 8^2} = 2 \cdot \sqrt{8.63^2 - 8^2} = 6.474 \approx 6.5 \text{ cm}$$

4. Determine the area of the segment in cm^2 with the precision of three decimals if the central angle is $\Theta = 80°$ and the radius is 3 cm.

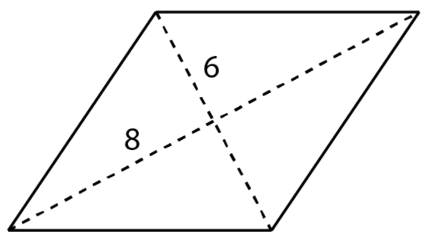


The area of the segment is the difference between the area of the section and the corresponding triangular shape:

$$A_{\text{seg}} = A_{\text{sec}} - A_{\text{tri}} = r^2 \cdot \pi \cdot \frac{\Theta}{360°} - \frac{1}{2} \cdot r^2 \cdot \sin(\Theta) = r^2 \left(\frac{\Theta \cdot \pi}{360°} - \frac{1}{2} \cdot \sin(\Theta) \right) =$$

$$= 3^2 \cdot \left(\frac{80}{360} \cdot \pi - \frac{1}{2} \cdot \sin(80°) \right) = 1.85155 \approx 1.852 \text{ cm}^2$$

5. Determine the perimeter and the area of a rhombus in cm and cm^2 units with two decimals precision, if its diagonals are 6 cm and 8 cm.



The diagonals of the rhombus intersect each other under 90° and divide each other in half, so we can use the Pythagorean theorem to find the side:

$$a^2 = 3^2 + 4^2 \rightarrow a = 5$$

The perimeter: $P = a \cdot 4 = 5 \cdot 4 = 20.00$ cm

The area can be computed using the diagonals:

$$A = \frac{d_1 \cdot d_2}{2} = \frac{6 \cdot 8}{2} = 24.00 \text{ cm}^2$$

6. The proportion of a parallelograms two inner angles is 3:7. Determine the inner angles of the parallelogram.

The sum of the two inner angles of the parallelogram is 180°.

$$\alpha + \beta = 180°$$

The proportion of the angles is given:

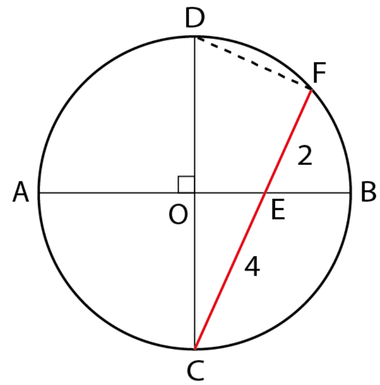
$$\frac{\alpha}{\beta} = \frac{3}{7} \rightarrow \alpha = \frac{3}{7}\beta$$

Combining the two equations:

$$\frac{3}{7}\beta + \beta = 180° \to \frac{10}{7}\beta = 180° \to \beta = 126°$$

$$\alpha = 180° - 126° = 54°$$

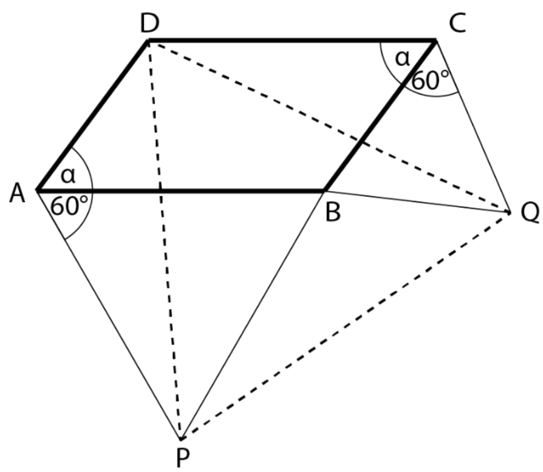
7. In a circle, we are given two perpendicular diameters. A chord is drawn from one of the endpoints of the diameters. The other diameter divides the chord into 2 and 4 units. Determine the radius of the circle.



The triangle ΔCDE is a right triangle according to Thales' theorem. The ΔCOE and the ΔCDF triangles are similar as they have one common acute angle and both are right triangles. Due to the similarity, a proportion can be found between the corresponding sides:

$$\frac{\overline{CO}}{\overline{CF}} = \frac{\overline{CE}}{\overline{CD}} \to \frac{r}{6} = \frac{4}{2r} \to r^2 = 12 \to r = 2\sqrt{3}$$

8. We have an $ABCD$ parallelogram and create the equilateral triangles ΔABP and ΔBCQ from the $\overline{AB}$ and $\overline{BC}$ sides. prove that the triangle ΔPQD is also equilateral.



If the inner angle of the parallelogram at point A is denoted α, then $\angle DAP = \angle DCQ = \alpha + 60°$.

From the common sides of the parallelogram and the triangles:

$$\overline{AD} = \overline{BC} = \overline{CQ}$$

$$\overline{AB} = \overline{DC} = \overline{AP}$$

$$\overline{PB} = \overline{AB}$$

$$\overline{BQ} = \overline{CB}$$

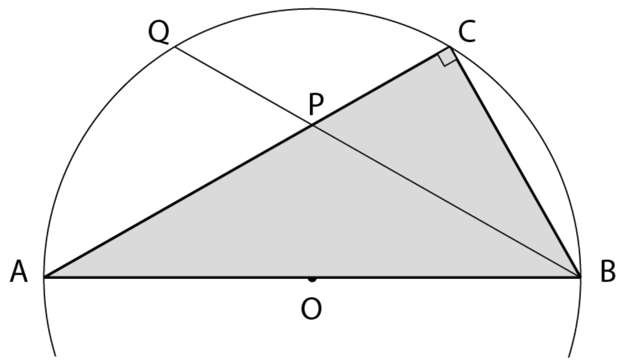
We can find that $\angle PBQ = 360° - (180° - \alpha) - 60° - 60° = \alpha + 60°$

The triangles ΔAPD, ΔPBQ and ΔDCQ are congruent as two of their corresponding sides and the angle between them are the same. From the congruency comes the fact that their third sides are equal as well:

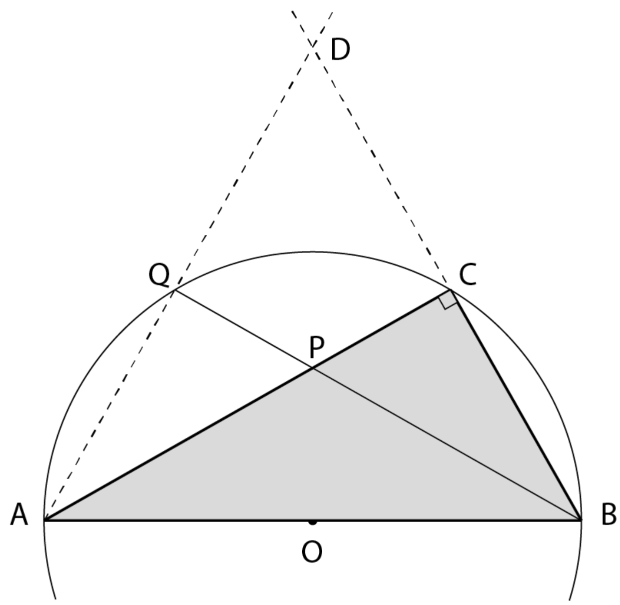
$$\overline{PD} = \overline{DQ} = \overline{PQ}$$

Which means that the triangle ΔPDQ is equilateral.

9. The triangle ΔABC has a right angle at point C. An angular bisector drawn from point B intersects the $\overline{AC}$ line in point P and the circle in point Q. Determine the angles of the triangle if $\overline{BP} = 2 \cdot \overline{PQ}$



First, we extend the lines $\overline{AQ}$ and $\overline{BC}$. These meet in point D.



In the triangle ΔABQ, we have a right angle at point Q according to Thales' theorem. In the triangle ΔABD, this makes the line $\overline{BQ}$ an angular bisector and an altitude as well. From this, we can conclude that this triangle is an isosceles triangle, $\overline{AB} = \overline{BD}$. This makes the point Q a midpoint of the $\overline{AD}$ segment.

If Q is the midpoint, the $\overline{PQ}$ line is a perpendicular bisector and because of the $\overline{BP} = 2 \cdot \overline{PQ}$ fact, point P is the barycenter (or centroid) of ΔABD. From this, we can conclude, that P is the centroid, then $\overline{AC}$ is also a median, which is perpendicular to $\overline{BD}$. This makes the line $\overline{AC}$ a perpendicular bisector and an altitude as well. From this, we can conclude that this triangle is also an isosceles triangle, $\overline{AB} = \overline{AD}$. As $\overline{AB} = \overline{BD} = \overline{AD}$ which makes the triangle equilateral. If the triangle ΔABD is equilateral, the angle $\angle DBA$ is 60° and the angle $\angle CAB$ has to be 30°.