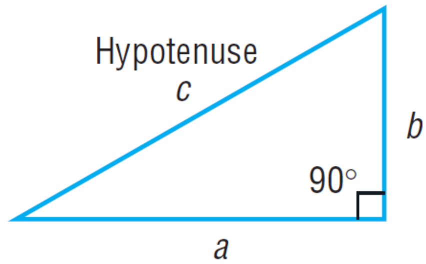
Basic geometry and trigonometry IV.

1 Right triangle trigonometry

Inside a right triangle like the one in the figure below, one of the most fundamental theorems is the Pythagorean Theorem that tells us that:

$$c^2 = a^2 + b^2$$

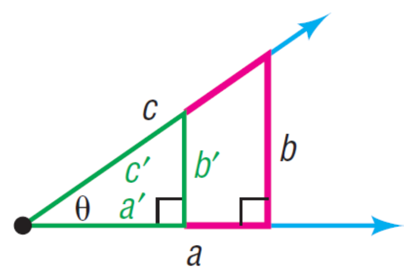
or namely, that the square of the hypotenuse is equal to the sum of the squares of the two legs of the triangle



Let the acute angle at A be θ. Using the right triangle formed by this angle, we can form exactly six ratios of the different sides:

$$\frac{b}{c}, \frac{a}{c}, \frac{b}{a}, \frac{a}{b}, \frac{c}{b}, \frac{c}{a}$$

These ratios do not depend on the size of the triangle, only on the angle θ, as the figure below shows.



As the ratios only depend on the angle θ, we can give them unique names that involve θ, respectively: sine of θ, cosine of θ, tangent of θ, cotangent of θ, cosecant of θ and the secant of θ. The table below gives an overview of the these six ratios and their names:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b} \quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a} \quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b} \tag{1}$$

1.1 Fundamental identities

Observing the definition of the trigonometric functions, we can find a couple of fundamental identities.

Reciprocal identities:

$$\csc(\theta) = \frac{c}{b} = \frac{1}{\frac{b}{c}} = \frac{1}{\sin(\theta)} \quad \sec(\theta) = \frac{c}{a} = \frac{1}{\frac{a}{c}} = \frac{1}{\cos(\theta)} \quad \cot(\theta) = \frac{a}{b} = \frac{1}{\frac{b}{a}} = \frac{1}{\tan(\theta)}$$

Quotient identities:

$$\tan(\theta) = \frac{b}{a} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\sin(\theta)}{\cos(\theta)} \quad \cot(\theta) = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\cos(\theta)}{\sin(\theta)}$$

A few more fundamental identities can be derived from the Pythagorean Theorem. These are called the Pythagorean Identites.

If we take the equation

$$b^2 + a^2 = c^2$$

and divide each side by c^2, we get

$$\frac{b^2}{c^2} + \frac{a^2}{c^2} = 1 \Rightarrow \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = 1 \Rightarrow (\sin(\theta))^2 + (\cos(\theta))^2 = 1$$

The terms $(\sin(\theta))^2$ and $(\cos(\theta))^2$ are usually written as $\sin^2(\theta)$ and $\cos^2(\Theta)$.

$$\sin^2(\Theta) + \cos^2(\Theta) = 1$$

Another identity can be obtained from the one above, if we divide each side by $\cos^2(\theta)$:

$$\frac{\sin^2(\Theta)}{\cos^2(\Theta)} + 1 = \frac{1}{\cos^2(\theta)}$$

Replacing the identities in the equation, we can arrive at:

$$\tan^2(\Theta) + 1 = \sec^2(\Theta)$$

Similarly, if we divide by $\sin^2(\theta)$ instead of $\cos^2(\theta)$, we get:

$$\cot^2(\Theta) + 1 = \csc^2(\Theta)$$

The figure below shows all the identities mentioned above:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Example 1: Find the exact values of the expressions:

(a) $\tan(20°) - \dfrac{\sin(20°)}{\cos(20°)}$ (b) $\sin^2\left(\dfrac{\pi}{12}\right) + \dfrac{1}{\sec^2\left(\frac{\pi}{12}\right)}$

Solution:

a)

$$\tan(20°) - \underbrace{\frac{\sin(20°)}{\cos(20°)}}_{\tan(20°)} = \tan(20°) - \tan(20°) = 0$$

b)

$$\sin^2\left(\frac{\pi}{12}\right) + \underbrace{\frac{1}{\sec^2\left(\frac{\pi}{12}\right)}}_{\cos^2\left(\frac{\pi}{12}\right)} = \sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right) = 1$$

Example 2: Find the exact values of the remaining trigonometric functions given that $\sin(\theta) = \frac{1}{3}$.

Solution: If $\sin(\theta)$ is 1/3, the opposite side is 1 unit long, while the hypotenuse is 3 units long. Using the Pythagorean Theorem, we can find the length of the adjacent side:

$$a^2 + 1^2 = 3^2 \Rightarrow a = \sqrt{8}$$

$$a = 2\sqrt{2}$$

From here, the values of the other five trigonometric functions:

$$\cos(\Theta) = \frac{a}{c} = \frac{2\sqrt{2}}{3} \qquad \tan(\theta) = \frac{b}{a} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\cot(\theta) = \frac{a}{b} = 2\sqrt{2} \qquad \csc(\theta) = \frac{c}{b} = 3$$

$$\sec(\theta) = \frac{c}{a} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

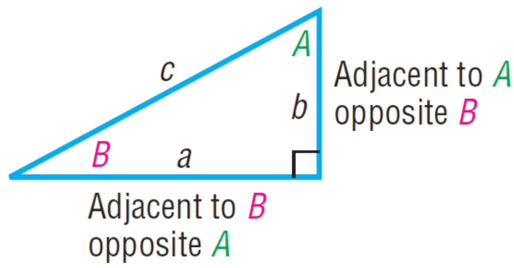
Another solution could be to compute the value of $\cos(\theta)$ using

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

and then use the identities to find the exact values of the functions.

1.2 Complementary angle theorem

Two angles are called complementary if their sum is 90°. As the sum of the angles in any triangle is 180°, it follows that in a right triangle the sum of the acute angles is always 90°, that is, they are complementary.



As shown in the figure above, if we are talking about the sine of angle B, we mean the ratio given by the opposite side divided by the hypotenuse. The same ratio is given by taking the adjacent side to angle A and dividing it by the hypotenuse. It follows from this that the sine of B is equal to the cosine of A. This is true for all the trigonometric functions:

$$\sin(B) = \frac{b}{c} = \cos(A) \qquad \cos(B) = \frac{a}{c} = \sin(A)$$

$$\tan(B) = \frac{b}{a} = \cot(A) \qquad \cot(B) = \frac{a}{b} = \tan(A)$$

$$\csc(B) = \frac{c}{b} = \sec(A) \qquad \sec(B) = \frac{c}{a} = \csc(A)$$

Because of this property, these trigonometric functions are called the cofunctions of each other.

The theorem in words: **Cofunctions of complementary angles are equal.**

Example 3: Find the exact values of the following expressions.

a) $\cos(35°)\sin(55°) + \cos(55°)\sin(35°)$

c) $\cot(40°) - \dfrac{\sin(50°)}{\sin(40°)}$

b) $\tan(35°) \cdot \sec(55°) \cdot \cos(35°)$

d) $1 - \cos^2(20°) - \cos^2(70°)$

Solution:

a)

$$\cos(35°)\sin(55°) + \cos(55°)\sin(35°) = \sin(55°)\sin(55°) + \cos(55°)\cos(55°) =$$

$$= \sin^2(55°) + \cos^2(55°) = 1$$

b)

$$\tan(35°) \cdot \sec(55°) \cdot \cos(35°) = \frac{\sin(35°)}{\cos(35°)} \cdot \cos(35°) \cdot \csc(35°) = \sin(35°) \cdot \frac{1}{\sin(35°)} = 1$$

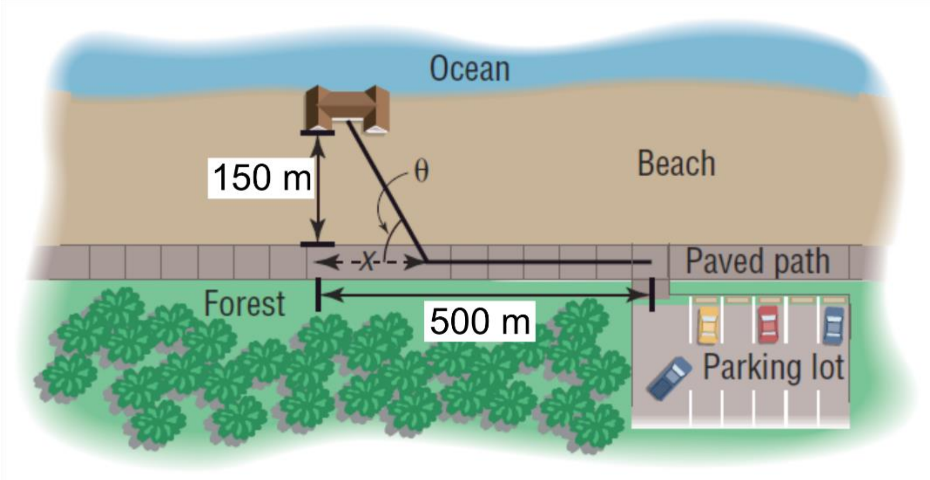
c)

$$\cot(40°) - \frac{\sin(50°)}{\sin(40°)} = \cot(40°) - \frac{\cos(40°)}{\sin(40°)} = \cot(40°) - \cot(40°) = 0$$

d)

$$1 - \cos^2(20°) - \cos^2(70°) = 1 - (\cos^2(20°) + \sin^2(20°)) = 1 - 1 = 0$$

Example 4: From a parking lot we want to walk to a house on the ocean. The house is located 500 meters down a paved path that parallels the beach, which is 150 meter wide. Along the path you can walk 100 meters per minute, but in the sand on the beach you can only walk 30 meters per minute. See the illustration.



a) Calculate the time T if we walk 500 meters along the paved path then walk 150 meters in the sand to the house.

b) Calculate the time T if we walk in the sand directly towards the ocean for 150 meters and then turn left and walk along the beach for 500 to the house.

c) Express the time T to get from the parking lot to the beachhouse as a function of the angle θ shown in the illustration.

d) Calculate the time T if we walk directly from the parking lot to the house.

e) Graph $T = T(\theta)$. For what angle θ is T the least? What is the value of x for this angle. What is the minimum time?

Solution:

a) $T = \frac{500}{100} + \frac{150}{30} = 5 + 5 = 10 \text{ min}$

b) $T = \frac{150}{30} + \frac{500}{30} = 5 + \frac{50}{3} = \frac{65}{3} \approx 21.67 \text{ min}$

c) The value of x as a function of θ: $x = 150 \cdot \cot(\theta)$

The length of the diagonal path P in the sand as a function of θ:

$$\sin(\theta) = \frac{150}{P} \Rightarrow P = \frac{150}{\sin(\theta)}$$

The total time T:

$$T = \frac{500 - x}{100} + \frac{P}{30} = \frac{500 - 150 \cdot \cot(\theta)}{100} + \frac{\frac{150}{\sin(\theta)}}{30} = 5 - 1.5 \cdot \cot(\theta) + \frac{5}{\sin(\theta)} = 5\left(1 - \frac{0.3}{\tan(\theta)} + \frac{1}{\sin(\theta)}\right)$$

a) Using the Pythagorean Theorem:

$$P^2 = 150^2 + 500^2 \Rightarrow P \approx 522 \text{ m}$$

$$T = \frac{522}{30} = 17.4 \text{ min}$$

b) The maximum value of θ is 90°, that is the case we computed in a). If consider the case in d), where the value of θ is minimal, its tangent is

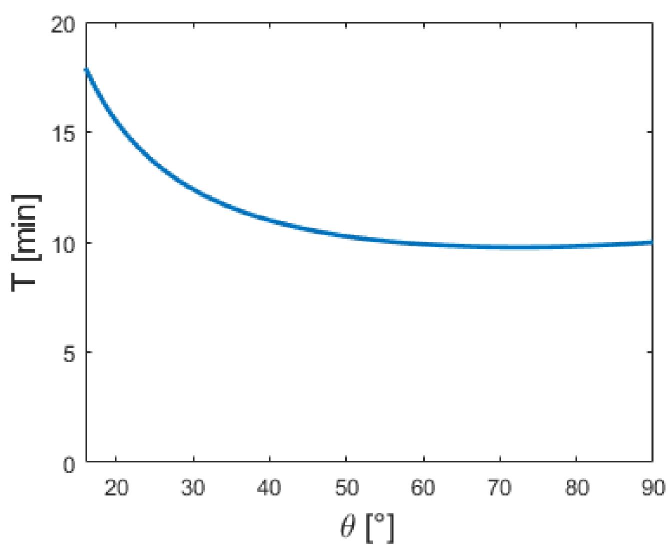
$$\tan(\theta) = \frac{150}{500} = \frac{3}{10}$$

from here $\theta = \arctan\left(\frac{3}{10}\right) \approx 16.7°$.

Below is a table of the function values $T(\theta)$ for every 5° of θ between the maximum and minimum value:

$\boldsymbol{\theta}$	$\boldsymbol{T(\theta)}$	$\boldsymbol{\theta}$	$\boldsymbol{T(\theta)}$
16.7	17.40	55	10.05
20	15.50	60	9.91
25	13.61	65	9.82
30	12.40	70	9.77
35	11.58	75	9.77
40	10.99	80	9.81
45	10.57	85	9.89
50	10.27	90	10.00

The graph of the function:



From the table above, we can see that the minimum value of T corresponds to a θ value between 70° and 75°. We wouldn't be far off, if we use average of these two values: 72.5°.

A more precise approximation is 72.54° (acquired using numerical derivation), which is pretty close to our initial guess.

The value of x corresponding to this T:

$$x = 150 \cdot \cot(72.54°) \approx 47.18 \text{ m}$$

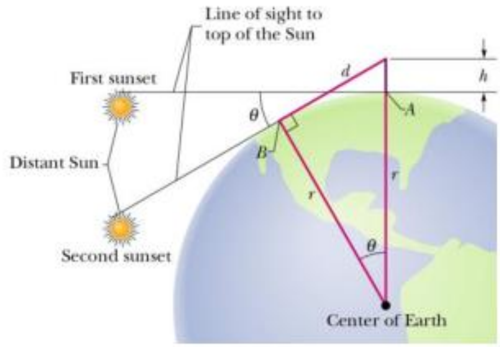
The minimum time $T_{\min}$ can be found by substituting the found value of θ into the function:

$$T_{\min} = 5\left(1 - \frac{0.3}{\tan(72.54°)} + \frac{1}{\sin(72.54°)}\right) \approx 9.8 \text{ min}$$

Example: David Halliday: Fundamentals of Physics, I/19.

Suppose that, while lying on a beach near the equator watching the Sun set over a calm ocean, you start a stopwatch just as the top of the Sun disappears. You then stand, elevating your eyes by a height H = 1.70 m, and stop the watch when the top of the Sun again disappears. If the elapsed time is t = 11.1 s, what is the radius r of Earth?

19. When the Sun first disappears while lying down, your line of sight to the top of the Sun is tangent to the Earth's surface at point A shown in the figure. As you stand, elevating your eyes by a height h, the line of sight to the Sun is tangent to the Earth's surface at point B.



Let d be the distance from point B to your eyes. From the Pythagorean theorem, we have

$$d^2 + r^2 = (r+h)^2 = r^2 + 2rh + h^2$$

or $d^2 = 2rh + h^2$, where r is the radius of the Earth. Since $r \gg h$, the second term can be dropped, leading to $d^2 \approx 2rh$. Now the angle between the two radii to the two tangent points A and B is θ, which is also the angle through which the Sun moves about Earth during the time interval $t = 11.1$ s. The value of θ can be obtained by using

$$\frac{\theta}{360°} = \frac{t}{24\text{ h}}.$$

This yields

$$\theta = \frac{(360°)(11.1\text{ s})}{(24\text{ h})(60\text{ min/h})(60\text{ s/min})} = 0.04625°.$$

Using $d = r\tan\theta$, we have $d^2 = r^2\tan^2\theta = 2rh$, or

$$r = \frac{2h}{\tan^2\theta}$$

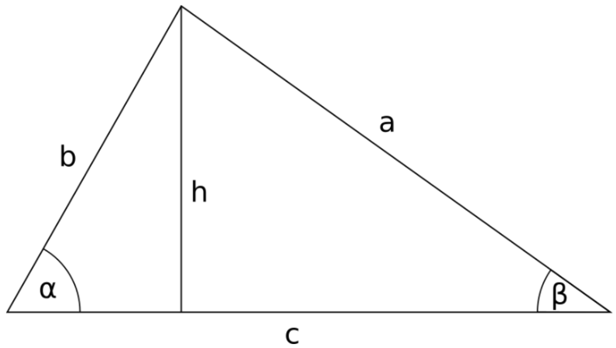
Using the above value for θ and $h = 1.7$ m, we have $r = 5.2\times10^6$ m.

2 Law of sines

In any triangle (scalene) the quotient of the sine of an angle and the opposing side is constant.

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Proof:



1. $h = a\sin(\beta)$ and $h = b\sin(\alpha)$
2. $a\sin(\beta) = b\sin(\alpha)$
3. $\sin(\beta)/b = \sin(\gamma)/c$

The same formula can be given for another height in the triangle. Combining them gives the law of sines.

3 Law of cosines

The law of cosines states the following (using the notation in the figure above):

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha)$$

Proof, using the figure above:

1. $h = b\sin(\alpha)$
2. The side adjacent to β is $c' = c - b \cdot \cos(\alpha)$
3. Using Pythagoras's theorem:

$$a^2 = h^2 + (c')^2 = \big(b \cdot sin(\alpha)\big)^2 + \big(c - b \cdot cos(\alpha)\big)^2 =$$
$$= b^2\sin^2(\alpha) + (c^2 - 2 \cdot b \cdot c \cdot cos(\alpha) + b^2\cos^2(\alpha)) =$$
$$= b^2 \cdot (\sin^2(\alpha) + \cos^2(\alpha)) + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha)$$
$$= b^2 + c^2 - 2 \cdot b \cdot c \cdot cos(\alpha)$$