Trigonometric exercises

1) A walking trail is laid out in the shape of a triangle. The lengths of the three paths that make up the trail are 2500 meters, 2000 meters and 1800 meters. Determine, **to the nearest arc second**, the measure of the greatest angle of the trail.

   Solution:

   The greatest angle of the triangle is opposite the longest side. As we know all the sides, we can use the law of cosines to determine the value of the angle:

$$2500^2 = 2000^2 + 1800^2 - 2 \cdot 2000 \cdot 1800 \cdot \cos(\gamma) \Rightarrow \gamma = \text{acos}\left(\frac{2500^2 - 2000^2 - 1800^2}{-2 \cdot 2000 \cdot 1800}\right) =$$

$$= 82\text{-}05\text{-}48.4 \approx 82\text{-}05\text{-}48$$

2) A field is bordered by two pairs of parallel roads so that the shape of the field is a parallelogram. The lengths of the two adjacent sides of the field are 2 kilometers and 3 kilometers, and the length of the shorter diagonal of the field is 3 kilometers.

   a) Find the **exact value** of the **cosine** of the acute angle of the parallelogram.
   b) Find the **exact value** of the **sine** of the acute angle of the parallelogram.
   c) Find the **exact value** of the area of the field.
   d) Find the area of the field **to the nearest integer in km^2**.

   Solution:

   a) We can use the two adjacent sides and the diagonal with the law of cosines to find the cosine of the acute angle:

$$3^2 = 3^2 + 2^2 - 2 \cdot 3 \cdot 2 \cdot \cos(\gamma) \Rightarrow \cos(\gamma) = \frac{3^2 - 3^2 - 2^2}{-2 \cdot 3 \cdot 2} = \frac{-4}{-12} = \frac{1}{3}$$

   b) From the Pythagorean Identity:

$$\sin^2(\gamma) + \cos^2(\gamma) = 1 \Rightarrow \sin(\gamma) = \sqrt{1 - \cos^2(\gamma)} = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$$

   c) The area of the field using the gamma angle and the two adjacent sides:

$$3 \cdot 2 \cdot \sin(\gamma) = 3 \cdot 2 \cdot \frac{2\sqrt{2}}{3} = 4\sqrt{2}\ \text{km}^2$$

   d) The area of the field to the nearest integer km^2:

$$4\sqrt{2} = 5.6568 \ldots \approx 6\ \text{km}^2$$

2) Two sides of a triangular lot form angles that measure α = 29-10-00 and β = 33-59-25 with the third side *c*, which is 487 meters long. To the nearest dollar, how much will it cost to fence the lot if the fencing costs $5.59 per meter?

<u>Solution:</u>

The third angle: $\gamma = 180° - \left((29\text{-}10\text{-}00) + (33\text{-}59\text{-}25)\right) = 116\text{-}50\text{-}35$

The two unknown sides from the law of sines:

$$a = 487 \cdot \frac{\sin(29\text{-}10\text{-}00)}{\sin(116\text{-}50\text{-}35)} = 266.0029 \approx 266 \text{ m}$$

$$b = 487 \cdot \frac{\sin(33\text{-}59\text{-}25)}{\sin(116\text{-}50\text{-}35)} = 305.13 \approx 305 \text{ m}$$

The perimeter of the triangle:

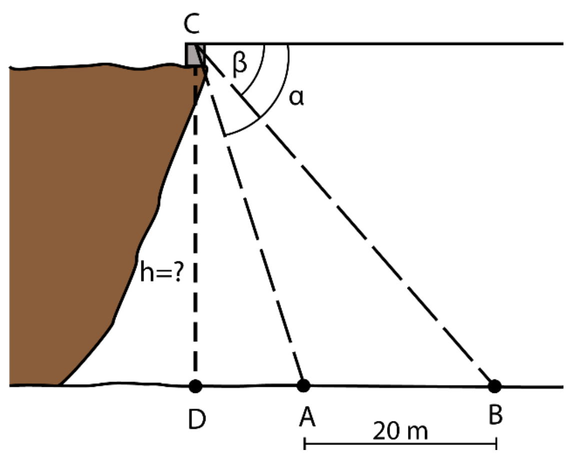
$$P = a + b + c = 1058 \text{ m}$$

The total fencing cost

$$TC = P \cdot \$5.59 = \$5914.22$$

3) From point *C* at the top of a cliff, two points, *A* and *B* are sited on the level ground. Points *A* and *B* are on a straight line with point *D*, a point directly below point *C*. The angle of depression of the nearer point, *A* is α = – 72°52'42" and the angle of depression of the farther point, *B* is β = – 48°12'09". If the points *A* and *B* are 20 meters apart, what is the height of the cliff **to the nearest meter**?

(Angle of depression: an angle measure from a horizontal line. If the angle is negative, the direction it represents is below the horizontal line, if it's positive, the direction is above the horizontal line.)



<u>Solution:</u>

The angle in ΔABC at point C is $\gamma = \alpha - \beta = 24\text{-}40\text{-}33$

The angle in ΔABC at point B is β due to being alternate angles.

Law of sines in ΔABC:

$$\frac{\overline{CA}}{\sin(\beta)} = \frac{20}{\sin(\gamma)} \Rightarrow \overline{CA} = \frac{\sin(\beta)}{\sin(\gamma)} \cdot 20 = 35.71 \text{ m}$$

The angle in ΔACD at point A is α due to being alternate angles.

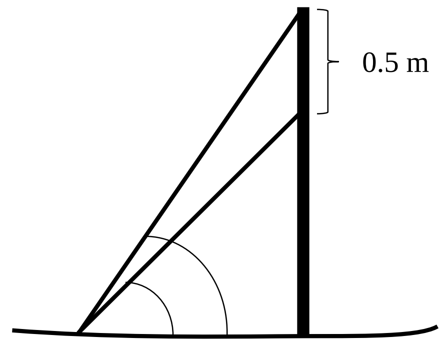
In ΔACD:

$$\frac{h}{\overline{CA}} = \sin(\alpha) \Rightarrow h = \overline{CA} \cdot \sin(\alpha) = 34.13 \text{ m}$$

4) A vertical pole is braced by two wires that extend from different points on the pole to the same point on the level ground. One wire is fastened to the pole 0.5 meters from the top of the pole and makes an angle of 61 degrees with the ground. The second wire is fastened to the top of the pole and makes an angle of 66 degrees with the ground. Find the height of the pole!

Solution:

The total height of the pole can be written as $h = 0.5 + y$, where y is the distance from the bottom of the pole to the lower wire.



Let x denote the horizontal distance between point on the ground where the wires are fixed and the pole. In the triangle created by the pole, the ground and the lower wire:

$$\tan(61°) = \frac{y}{x} \Rightarrow x = \frac{y}{\tan(61°)}$$

In the triangle created by the pole, the ground and the upper wire:

$$\tan(66°) = \frac{0.5 + y}{x}$$

Combining the two equations:

$$\tan(66°) = \frac{0.5 + y}{\frac{y}{\tan(61°)}} = (0.5 + y) \cdot \frac{\tan(61°)}{y}$$

$$\tan(66°) \cdot y = 0.5 \cdot \tan(61°) + y \cdot \tan(61°)$$

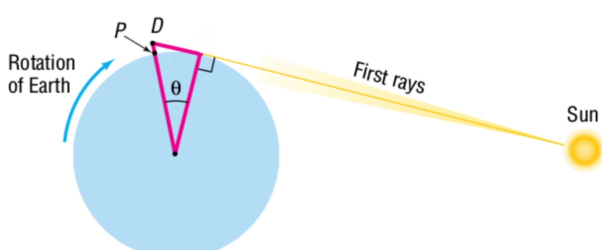
$$y \cdot (\tan(66°) - \tan(61°)) = 0.5 \cdot \tan(61°)$$

$$y = 0.5 \cdot \frac{\tan(61°)}{\tan(66°) - \tan(61°)} = 2.04 \text{ m}$$

The height of the pole

$$h = 0.5 + y = 2.54 \text{ m}$$

5) Imagine a 500 m high mountain on the Equator of the Earth. If a person is standing on top of the mountain (point D on the figure), he will see the first rays of the sun a little bit earlier than as if he were standing at the foot of the mountain (point P). How much is this time difference? (The radius of the Earth is 6378 km.)



Solution:

From the magenta triangle in the figure:

$$\cos(\theta) = \frac{R}{R + 0.5} \Rightarrow \theta = \operatorname{acos}\left(\frac{R}{R + h}\right) = \text{0-43-03}$$

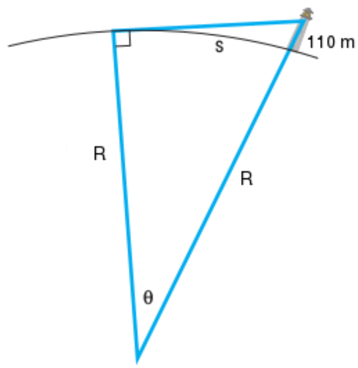
The rotational velocity of the earth is:

$$\omega = \frac{360°}{24\text{h}} = \frac{15°}{\text{h}} = \frac{0.25°}{\text{min}}$$

The time it takes for the Earth to rotate by θ:

$$t = \frac{\theta}{\omega} = \frac{0\text{-}43\text{-}03}{0.25°} = 2.87 \text{ min} \approx 3 \text{ min}$$

6) The top of a lighthouse is 110 meters above sea level. What is the maximum distance where a ship on sea level can still see the light emitted by the lighthouse? Refraction of the light can be omitted. (The radius of the Earth is 6378 km.)



Solution:

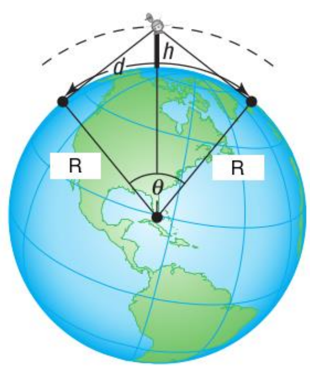
In the blue triangle in the figure:

$$\cos(\theta) = \frac{R}{R+h} \Rightarrow \theta = \text{acos}\left(\frac{6378}{6378+0.11}\right) = 0\text{-}20\text{-}11$$

The length of the arc corresponding to the central angle θ:

$$s = R \cdot \theta[\text{deg}] \cdot \frac{\pi}{180} = 22.39 \text{ km}$$

7) A surveillance satellite circles the Earth at a height of h kilometers above the surface. Suppose that d is the distance, in kilometers, on the surface of the Earth that can be observed from the satellite. (The radius of the Earth is 6378 km.)



a) Find an equation that relates the central angle Θ (in radians) to the height h!

The length of the line connecting the center of the Earth and satellite is $R + h$.

In one of the triangles that make up the kite in the figure:

$$\cos\left(\frac{\theta}{2}\right) = \frac{R}{R+h} \Rightarrow \theta = 2 \cdot \operatorname{acos}\left(\frac{R}{R+h}\right)$$

b) Find an equation that relates the observable distance *d* and Θ!

$$d = R \cdot \theta[\mathrm{rad}]$$

c) Find an equation that relates *d* and *h*!

$$d = 2 \cdot R \cdot \operatorname{acos}\left(\frac{R}{R+h}\right)$$

d) If *d* is to be 4500 kilometers, how high must the satellite orbit above the Earth?

From the equation in c):

$$d = 2 \cdot R \cdot \operatorname{acos}\left(\frac{R}{R+h}\right) \Rightarrow \frac{d}{2R} = \operatorname{acos}\left(\frac{R}{R+h}\right)$$

$$\cos\left(\frac{d}{2R}\right) = \frac{R}{R+h}$$

$$h = R \cdot \frac{1 - \cos\left(\frac{d}{2R}\right)}{\cos\left(\frac{d}{2R}\right)} = 418.55 \approx 419\ \mathrm{km}$$

e) If the satellite orbits at a height of 1500 kilometers, what distance *d* on the surface can be observed?

From the equation in c):

$$d = 2 \cdot 6378 \cdot \operatorname{acos}\left(\frac{6378}{6378 + 1500}\right) = 8002.25\ \mathrm{km} \approx 8002\ \mathrm{km}$$