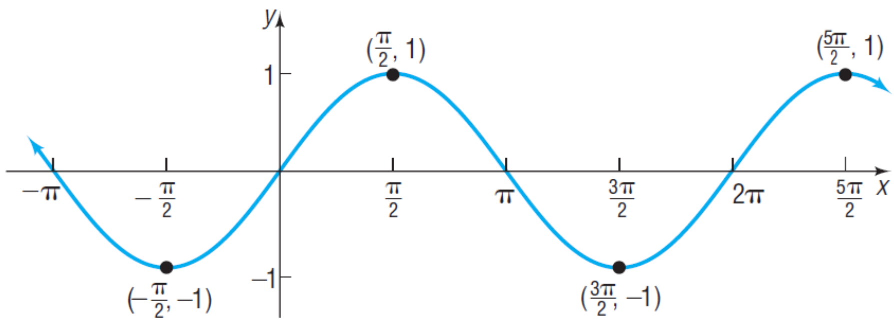
Graph of the trigonometric functions, identities

1. Graph of the trigonometric functions

1.1. Graph of the sine function

As the sine function has a period of 2π, it is enough to graph only the part between $[0, 2\pi]$ or one cycle.



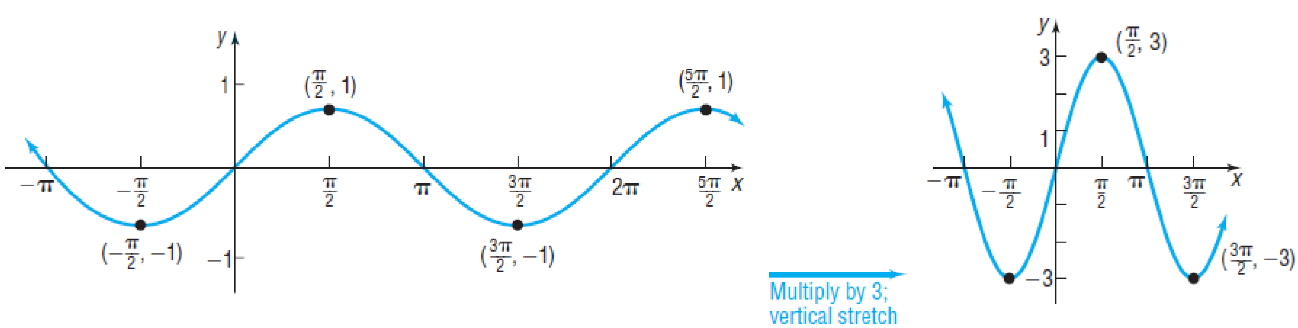
Properties of the sine function:

1. The domain is set of all real numbers.
2. The range consists of all real numbers from -1 to +1, inclusive.
3. The sine function is an odd function, as shown by the symmetry of the graph with respect to the origin.
4. The sine function is periodic with period 2π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$, the y-intercept is 0.
6. The maximum value occurs at $x = \frac{\pi}{2} + n \cdot 2\pi$, where n is an integer.

   The minimum value occurs at $x = \frac{3\pi}{2} + n \cdot 2\pi$, where n is an integer.

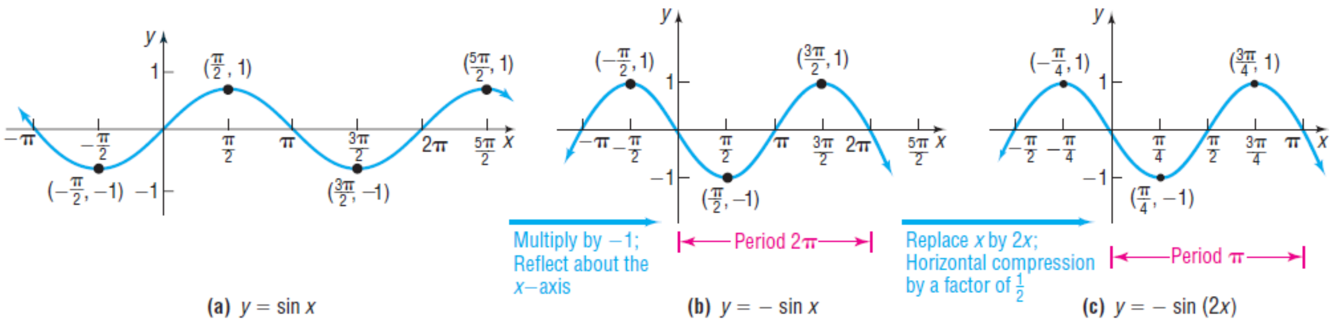
1.2. Graps of functions in the form $y = Asin(\omega x)$ using transformations

Graphing the $y = 3\sin(x)$ function can be done by stretching the $\sin(x)$ function vertically by a factor of 3.



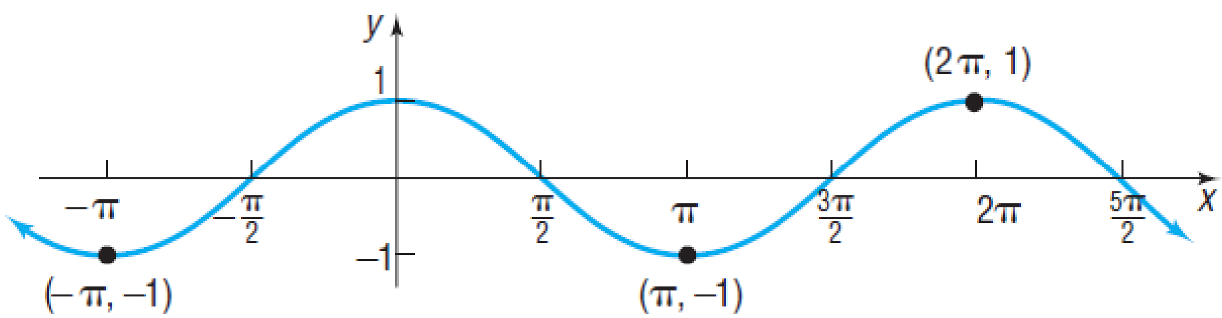
(a) $y = \sin x$ **(b)** $y = 3\sin x$

Graphing the $y = -\sin(2x)$ can be done by flipping the $\sin(x)$ function about the x-axis and compressing it by a factor of 1/2.



1.3. Graph of the cosine function

Similarly to the sine function, it is enough to graph only one cycle of the function, between $[0, 2\pi]$.



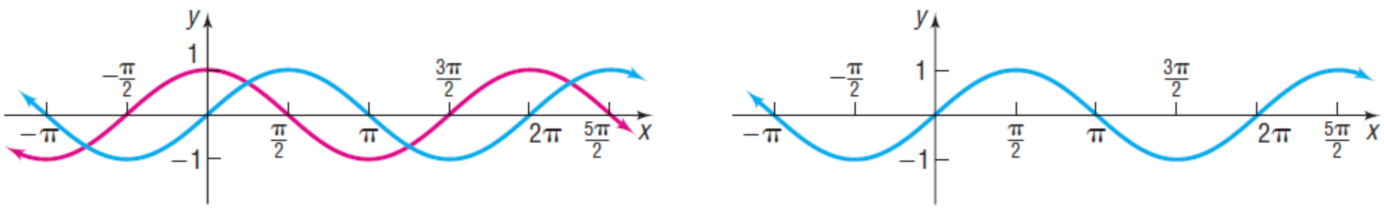
Properties of the cosine function:

1. The domain is set of all real numbers.
2. The range consists of all real numbers from -1 to +1, inclusive.
3. The sine function is an even function, as shown by the symmetry of the graph with respect to the y-axis.
4. The cosine function is periodic with period 2π.
5. The x-intercepts are $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, the y-intercept is 1.
6. The maximum value occurs at $x = 0 + n \cdot 2\pi$, where n is an integer.

   The minimum value occurs at $x = \pi + n \cdot 2\pi$, where n is an integer.

1.4. *Shifting sinusoidal graphs*

Shifting the graph means changing the argument of the functions by addition or subtraction. If we take the graph of the cosine function and shift it to the right by subtracting $\frac{\pi}{2}$ from the argument, we get the graph of the sine function.



(a) $y = \cos x$ $y = \cos\left(x - \frac{\pi}{2}\right)$ **(b)** $y = \sin x$

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

1.5. *Amplitude and period of sinusoidal functions*

In general, the values of the functions $y = A \cdot sin(x)$ and $y = A \cdot cos(x)$ where $A \neq 0$, will always satisfy the inequalities

$$-|A| \leq A \cdot \sin(x) \leq |A| \text{ and } -|A| \leq A \cdot \cos(x) \leq |A|$$

respectively. The number $|A|$ is called the amplitude of the function.

In the graph of $y = -\sin(2x)$ above, we can notice that the period of the function is $\frac{2\pi}{2} = \pi$, due to the horizontal compression of the original period 2π by a factor of 2.

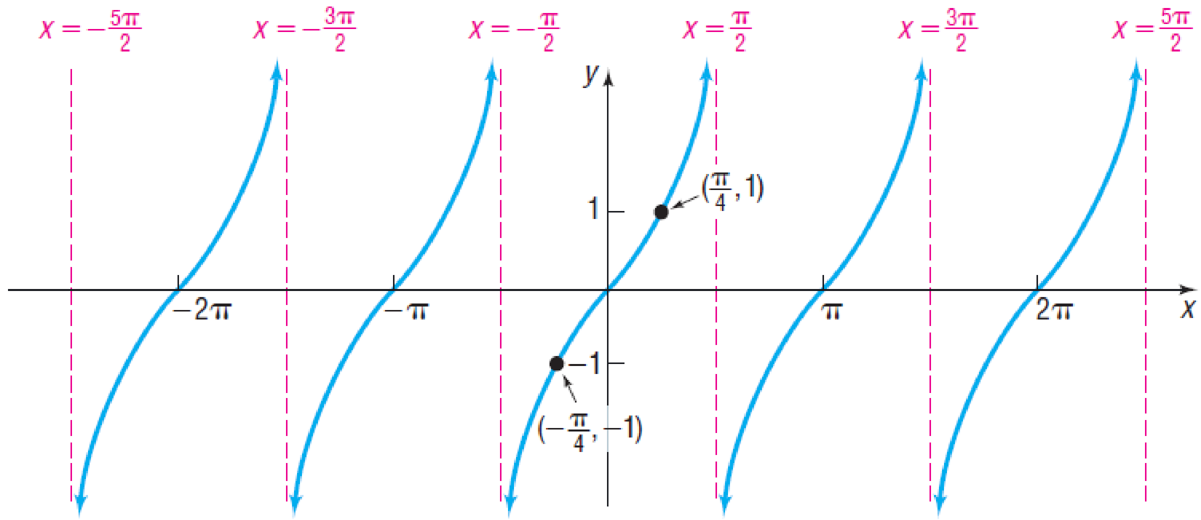
In general, if $\omega > 0$, the functions $y = \sin(\omega x)$ and $y = \cos(\omega x)$ will have period $T = \frac{2\pi}{\omega}$. To see why, we can recall the fact that the graph of $y = \sin(\omega x)$ is obtained from the graph of $y = \sin(x)$ by performing a horizontal compression or stretch by a factor of $\frac{1}{\omega}$. This horizontal compression replaces the original interval $[0, 2\pi]$ by the interval $[0, \frac{2\pi}{\omega}]$, which contains one period of the graph of $y = \sin(\omega x)$. One period of the graph is also called a cycle.

1.6. *Graph of the tangent function*

Similarly to the sine and cosine functions, it is enough to graph only one cycle of the tangent function, however, the period of the tangent function is π instead of 2π. If we use the identity

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

we can see that as the function approaches $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, where the $\cos(x)$ becomes 0, the function will tend towards negative infinity and positive infinity respectively. The magenta lines in the figure below represent the asymptotes of the function.



Properties of the tangent function:

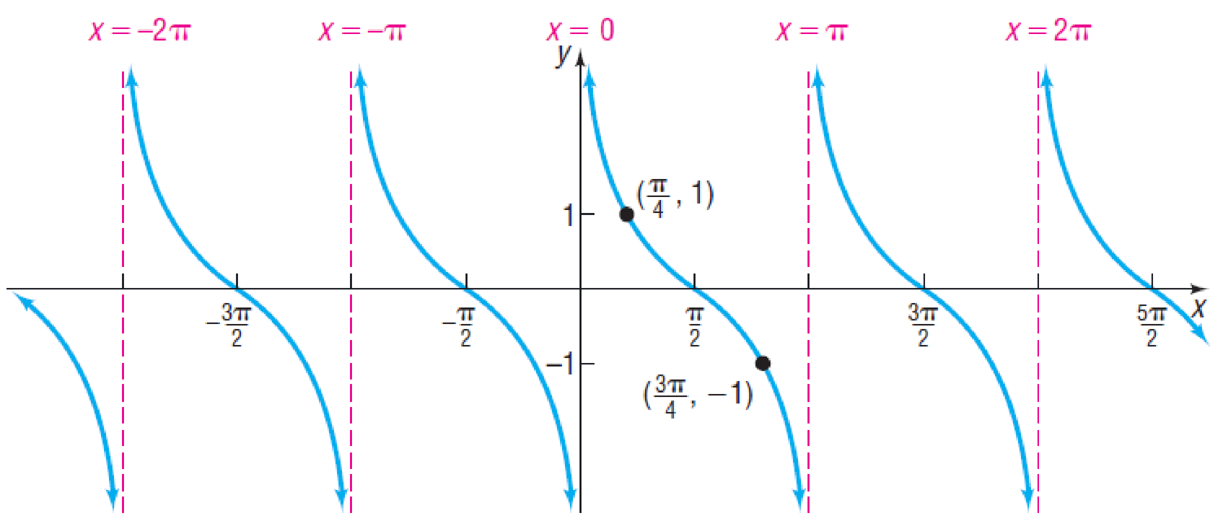
1. The domain is set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
2. The range is set of all real numbers.
3. The tangent function is an odd function, as shown by the symmetry of the graph with respect to the origin.
4. The tangent function is periodic with period π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$, the y-intercept is 0.

   Vertical asymptotes occur at $x = \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

1.7. Graph of the cotangent function

The graph of the cotangent function can be obtained similarly to the tangent function, but now, the function is the reciprocal of the tangent function

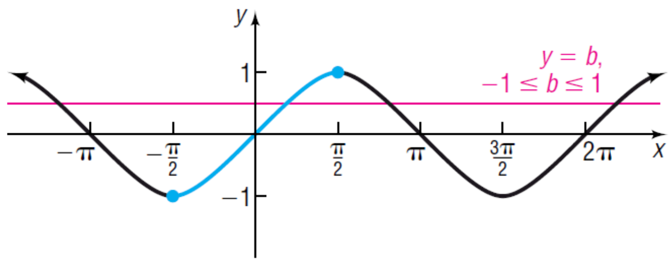
$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$



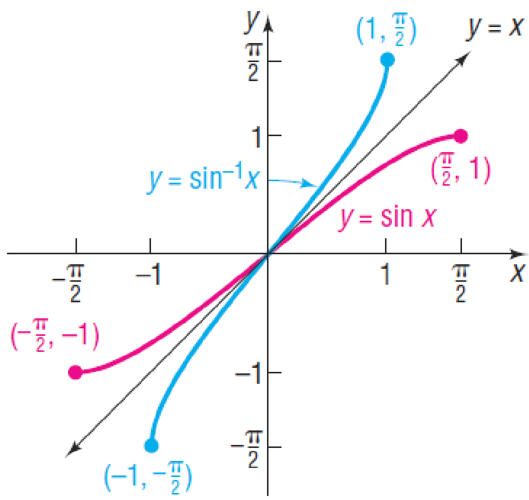
The asymptotes occur at $\ldots - \pi, 0, \pi, 2\pi, \ldots$

2. The inverse trigonometric functions

As shown in the figure below, the horizontal line $y = b$, where b is between -1 and 1, inclusive, intersects the graph of the function of $y = \sin(x)$ infinitely many times, so it follows that the function $y = \sin(x)$ is not one-to-one.

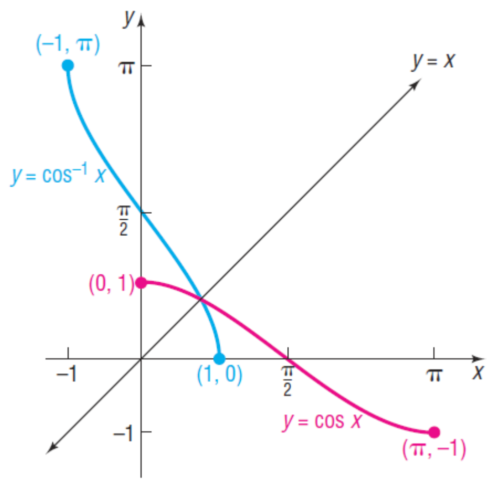


However, if we restrict the domain of the function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the restricted funtion is one-to-one.



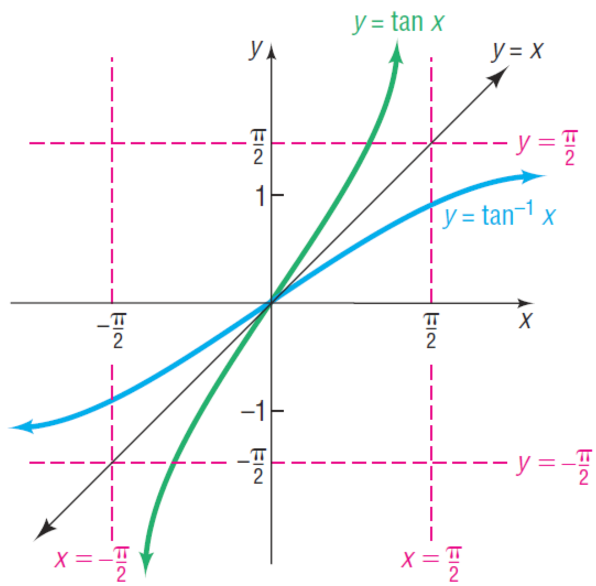
The explicit form of the inverse sine function is $y = \sin^{-1}(x)$ or also written as $y = \text{asin}(x)$ (the arcus sine function), where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse cosine function is obtained the same way, but now we restrain x to the interval $[0, \pi]$.

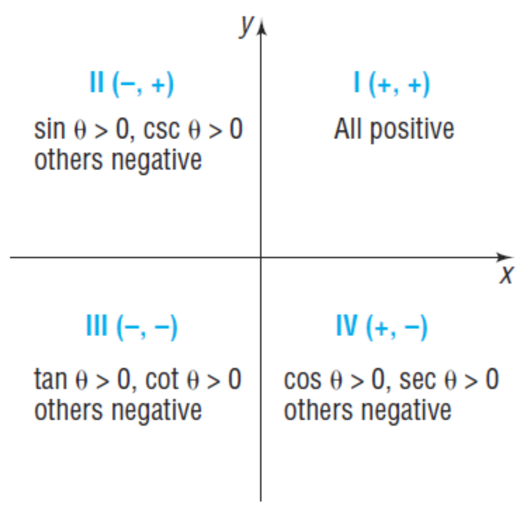


The explicit form of the inverse cosine function is $y = \cos^{-1}(x)$ or $y = \text{acos}(x)$ (the arcus cosine function), where $x \in [-1,1]$ and $y \in [0, \pi]$.

In case of the inverse tangent, we use one period of the function, to define the inverse.



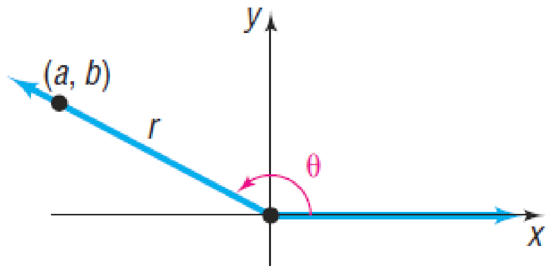
In case of the inverse tangent, the value of x can go from $-\infty$ to $+\infty$, while y is restricted to between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. or $-90°$ and $+90°$ The problem with this, is that if we are measuring angles between $0°$ and $360°$, the results of the inverse tangent function has to be changed according to the quadrant our direction is in.



The quadrant can be determined by looking at the sign of the sine and the cosine of a given angle θ. For example, in the figure below a and b represent a points distance from the origin along the x and the y axis respectively. The value of r can be computed as $\sqrt{a^2 + b^2}$, which is clearly positive, and the sine and cosine can be given as

$$\sin(\theta) = \frac{b}{r} > 0$$

$$\cos(\theta) = \frac{a}{r} < 0$$



Example: determine the quadrant of θ if:

a) $\sin(\theta) > 0, \cos(\theta) < 0$ Solution: quadrant II

b) $\cos(\theta) > 0, \tan(\theta) > 0$ Solution: quadrant I

c) $\sin(\theta) < 0, \cos(\theta) > 0$ Solution: quadrant IV

d) $\sec(\theta) < 0, \tan(\theta) > 0$ Solution: quadrant III