1. Coordinate geometry: Vectors, line equations, circles

1.1. Introduction

In surveying calculations, we want to calculate coordinates of new, previously unknown points. For this purpose, we use the two or three-dimensional coordinates of points of objects and measurements carried out using our instruments. In many of these cases, we can use the tools provided by coordinate geometry to find the solution.

In this lesson, we cover the basics of vectors, the equations of lines, circles on the coordinate plane.

1.2. Vectors in the Cartesian coordinate system

In the simplest terms, vectors denote quantities that not only have magnitude but direction as well. Vectors are represented by arrows; the length of the arrow means the magnitude of the quantity. A typical quantity that has direction as well would be the velocity of an object.

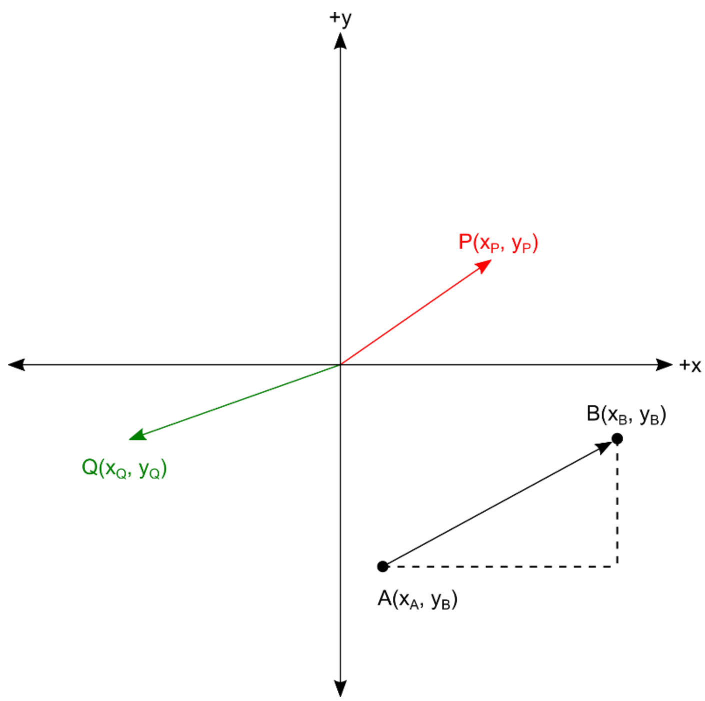


Figure 1. Vectors in the Cartesian coordinate system.

In the coordinate system, each point (coordinate pair) also represents a vector that has its starting point in the origin and endpoint at the given coordinate pair (see Figure 1, P and Q vector).

A vector between two points in the coordinate system can be written as follows (see Figure 1):

$$\overrightarrow{AB} = (x_b - x_a, y_b - y_a)$$

The norm of a vector means its length and can be calculated using the coordinates:

$$|\vec{P}| = \sqrt{x_P^2 + y_P^2}$$

We frequently use the norm to calculate the distance between two points with given coordinates, by first establishing the vector between the points.

The addition and subtraction of vectors can be done using the coordinates. If we have vectors $\vec{a}(x_a, y_a)$ and $\vec{b}(x_b, y_b)$:

$$\vec{a} + \vec{b} = (x_a + x_b, y_a + y_b) \ ,$$
$$\vec{a} - \vec{b} = (x_a - x_b, y_a - y_b).$$

The resulting vectors from addition and subtraction can be seen in Figure 2.

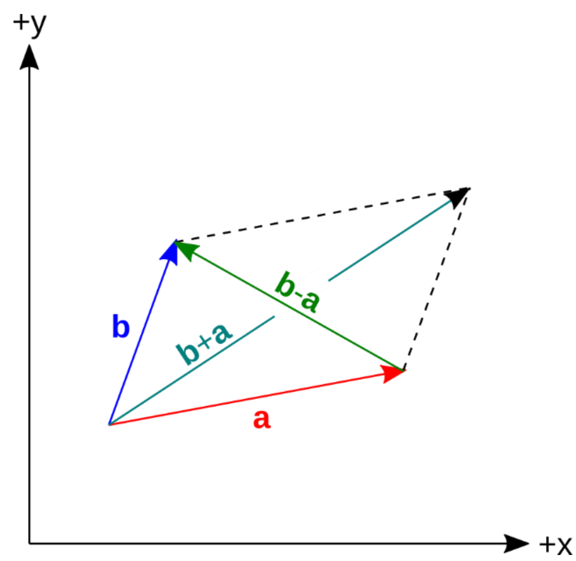


Figure 2. Addition and subtraction of vectors.

1.3. The dot product of two vectors

If we have vectors $\vec{a}(x_a, y_a)$ and $\vec{b}(x_b, y_b)$, the dot product or scalar product of two vectors means the following:

$$\vec{a} \cdot \vec{b} = x_a \cdot x_b + y_a \cdot y_b$$

The dot product can also be calculated using the following formula:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\alpha) = \sqrt{x_a^2 + y_a^2} \cdot \sqrt{x_b^2 + y_b^2} \cdot \cos(\alpha)$$

where α is the acute angle between the two vectors. The result of the dot or scalar product is a single value (scalar), hence the name. A very important note is that the dot product of any two vectors that are perpendicular to each other will be 0. This comes from the fact that for two vectors that are perpendicular, the angle between them is 90° and $\cos(90°)$ equals 0.

1.4. Equations of a line

In coordinate geometry, we often use lines in our computations by specifying their equations. The equation of a line is an algebraic equation solved only by the coordinates of the points that are on the line.

The standard form of the line equation

The standard form of a line equation can be written as the following:

$$y = m \cdot x + b$$

where x and y are coordinates, a denotes the slope of the line, and b denotes the so-called y-intercept of the line (see Figure 3). The slope of the line is equal to the tangent of the α value on Figure 3, that is, the acute angle between the line and the horizontal axis.

$$m = \tan\alpha = \frac{\Delta y}{\Delta x}$$

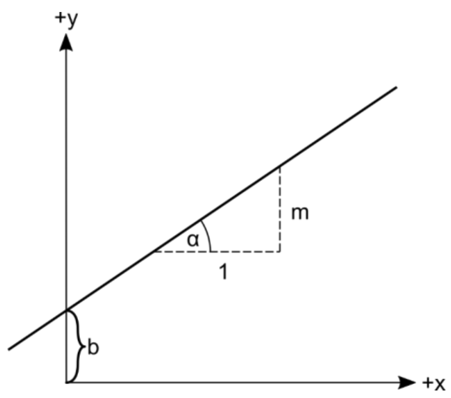


Figure 3. The standard form of the equation of a line.

In each of the different cases of writing the line equation, we can always arrive at the standard form of the equation, which is what we will do in the following sections.

Equation of line using the direction vector

We can specify any given line in the coordinate system using its direction vector $\vec{v}(x_v, y_v)$ and a point $P_0(x_{P_0}, y_{P_0})$ that happens to be on the line. The direction vector of the line is a vector that is parallel with the line (Figure 4).

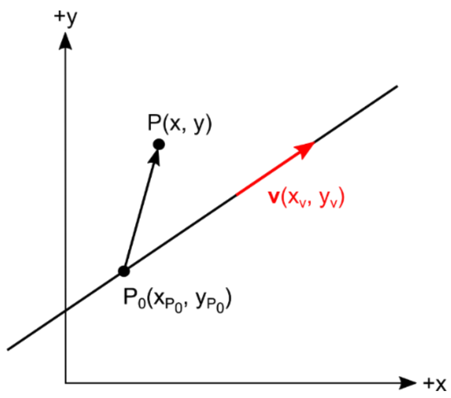


Figure 4. Equation of line using a point and the direction vector.

First, we can establish a vector from the given point on the line to an arbitrary point $P(x, y)$ on the plane. We want to find those (x, y) coordinate pairs for which our established vector is parallel with the direction vector of the line. In other words, their slope is the same.

The vector from the given point on the line to any $P(x, y)$ point can be written in the following form:

$$\overrightarrow{PP_0} = (x - x_{P_0}, y - y_{P_0})$$

The slope of our vector and the direction vector of the line have to be equal to each other. They can be written as such:

$$m_{\overrightarrow{PP_0}} = m_{\vec{v}}$$

$$\frac{y - y_{P_0}}{x - x_{P_0}} = \frac{y_v}{x_v}$$

We can rewrite the equation and find the equation of the line in the standard form:

$$y = \underbrace{\frac{y_v}{x_v}}_{m} \cdot x + \underbrace{\left(y_{P_0} - \frac{y_v}{x_v} \cdot x_{P_0}\right)}_{b}$$

Example 1: Find the equation of the line that has the direction vector $v(6, 11)$ and contains the point $P_0(14, 36)$. What are the slope and the y-intercept of the line?

Solution:

The slope of the vector from P_0 to an arbitrary point $P(x, y)$ has to be equal to the slope of the direction vector of the line:

$$\frac{y - 36}{x - 14} = \frac{11}{6} \Rightarrow y = \frac{11}{6}x - \frac{11}{6} \cdot 14 + 36 \Rightarrow y = \frac{11}{6}x + \frac{62}{6}$$

The slope of the line is $\frac{11}{6}$ and the y-intercept is $\frac{62}{6}$.

Equation of line using the normal vector

If we know the normal vector $\vec{n}(x_n, y_n)$ of a line, that is, the vector which is perpendicular to the line (and its direction vector) and a point $P_0(x_{P_0}, y_{P_0})$ that the line contains (see Figure 5), we can approach the problem in two ways.

One way would be to find the direction vector $\vec{v}(x_v, y_v)$ of the line. As we know the normal vector, we can easily do this by swapping the coordinates and changing the sign of one them. In our case, a possible direction vector would be $\vec{v}(y_n, -x_n)$. Now that we know the direction vector of the line and a point on it, we can go back to the previous section to find the equation of the line.

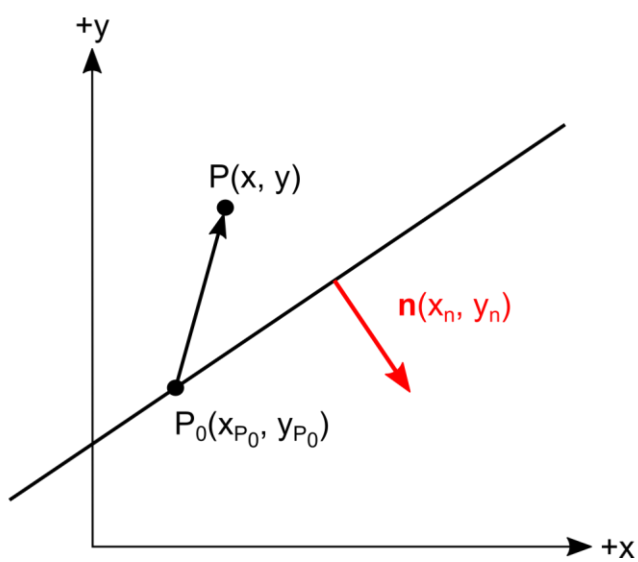


Figure 5. Equation of a line using a point and the normal vector.

The second way would be to use the normal vector directly. First, we establish a vector from the point on the line to any arbitrary $P(x, y)$ point. If our point is on the line, then the established vector has to be parallel to the direction vector of the line or perpendicular to its normal vector. We can use the dot product to find all the vectors that are perpendicular to the normal vector of our line (their dot product has to be 0):

$$\overrightarrow{PP_0} \cdot \vec{n} = \left(x - x_{P_0}\right) \cdot x_n + \left(y - y_{P_0}\right) \cdot y_n = 0$$

$$y = -\frac{x_n}{y_n} \cdot x + \left(y_{P_0} + \frac{x_n}{y_n} \cdot x_{P_0}\right)$$

Example 2: Find the equation of the line that has a normal vector $\vec{n}(-2, 5)$ and contains the point $P_0(25, 32)$! What is the slope and y-intercept of the line?

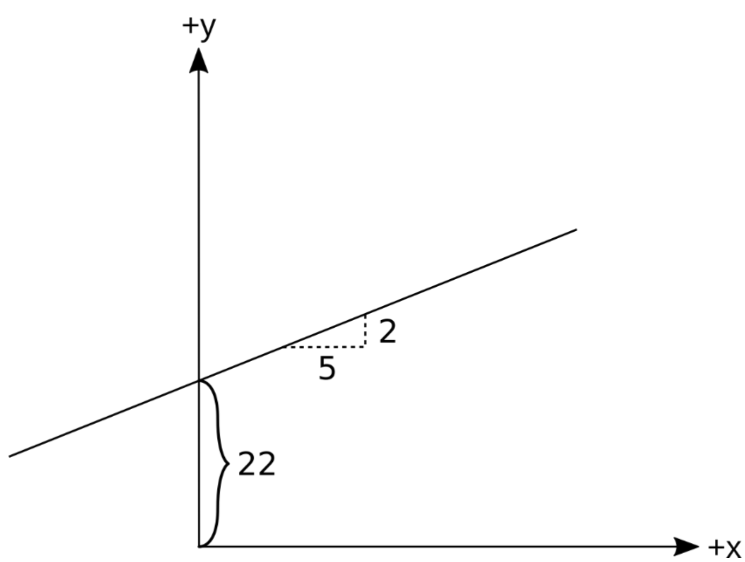
Solution:

$$\overrightarrow{P_0P} = (x - 25, y - 32)$$

$$P_0P \cdot n = 0 \rightarrow -2(x - 25) + 5(y - 32) = 0$$

$$-2x + 50 + 5y - 160 = 0$$

$$y = \frac{2}{5}x + 22$$



Equation of line using two points on the line

In this case, we have two points on the line with known coordinates: $P1(x_1, y_1)$ and $P2(x_2, y_2)$. We can either establish the direction vector of the line or the normal vector of the line using the two points (see Figure 6). Afterwards we can use the methods in section 0 and 0 to find the equation of the line.

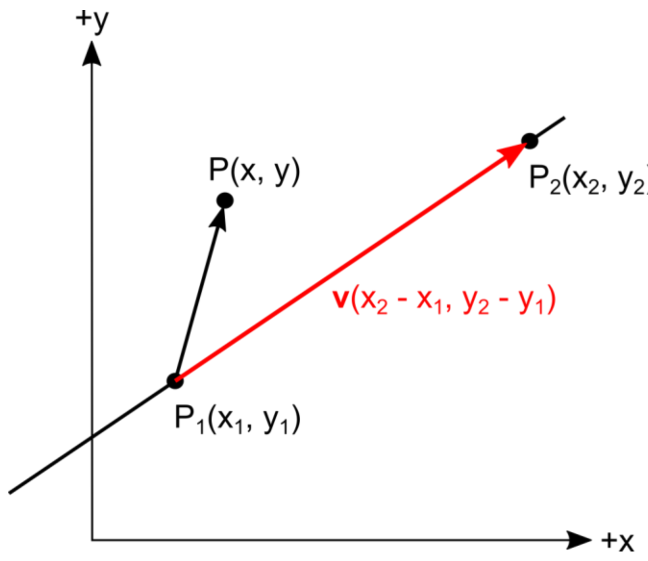


Figure 6. Equation of a line using two points.

The direction vector of the line is the vector between the two points:

$$\vec{v} = \overrightarrow{P1P2} = (x_2 - x_1, y_2 - y_1)$$

The normal vector can be found using the coordinates of the direction vector:

$$v(x_v, y_v) \to n(-y_v, x_v) \text{ or } n(y_v, -x_v)$$

<u>Example 3:</u> Find the equation of the line that contains the following two points: $A(10, 18)$, $B(-12, -21)$. What is the slope and the y-intercept of the line?

<u>Solution:</u>

The $\overrightarrow{BA}$ vector, which is the direction vector of the line: $(10 + 12, 18 + 21) = (22, 39)$

The vector from A to an arbitrary point $P(x, y)$: $(x - 10, y - 18)$

The slope of the $\overrightarrow{AP}$ vector has to be equal to the slope of the BA vector:

$$\frac{y - 18}{x - 10} = \frac{39}{22}$$

$$22(y - 180) = 39(x - 10)$$

$$22y - 396 = 39x - 390$$

$$y = \frac{39}{22}x + \frac{6}{22}$$

1.5. Calculating the point of intersection using the line equations

Whenever we have two lines given with their respective equations, we can use them to find whether the two lines intersect (so they are not parallel) and if so, the coordinates of the intersection point. Given these lines with their equations:

$$y_1 = a_1 \cdot x + b_1$$

$$y_2 = a_2 \cdot x + b_2$$

we can say that the lines don't intersect when $a_1 = a_2$, that is, their slope is the same. This of course means, that the two lines are parallel with each other. If this is not the case, then point of intersection has to be contained by both lines (Figure 7), so if we make the two equations equal to each other, we can first find the x coordinate of the intersection and then we just have to substitute the x coordinate into any one of the two equations to find the y coordinate.

$$a_1 \cdot x_i + b_1 = a_2 \cdot x_i + b_2$$

$$x_i = \frac{b_2 - b_1}{a_1 - a_2}$$

The y coordinate of the point of intersection from substituting the x coordinate into the equations above:

$$y_i = a_1 \cdot x_i + b_1 \quad \text{or} \quad y_i = a_2 \cdot x_i + b_2$$

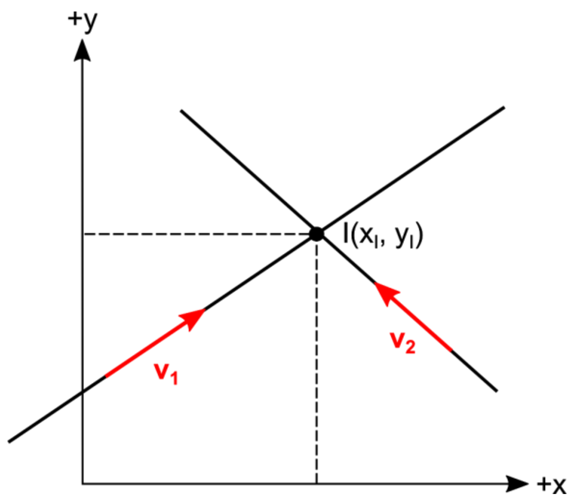


Figure 7. Intersection of two lines.

Example 4: Find the intersection point of two lines. The first line is given by its direction vector $\vec{v}(5, -3)$ and a point $A(-15, 24)$. The second line is defined by two points $B(-18, -8)$ and $C(13, 14)$.

Solution:

Check if the two lines are parallel with each other:

$$-\frac{3}{5} \neq \frac{22}{31} \rightarrow \text{The two lines are not parallel.}$$

The equation of the first line:

$$\frac{y - 24}{x + 15} = \frac{-3}{5}$$

$$y = -\frac{3}{5}x + 15$$

The equation of the second line:

$$\frac{y - 14}{x - 13} = \frac{22}{31} \qquad \text{or} \qquad \frac{y + 8}{x + 18} = \frac{22}{31}$$

$$31y - 434 = 22x - 286$$

$$y = \frac{22}{31}x + \frac{148}{31}$$

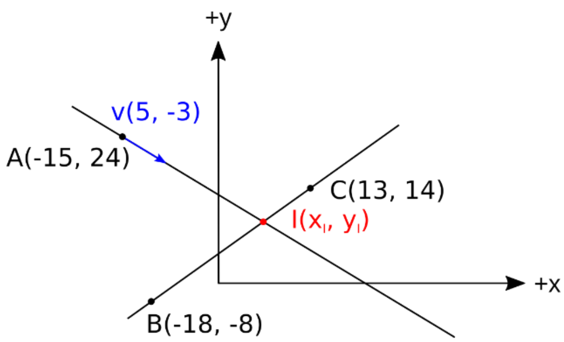
Equating the two line equations:

$$-\frac{3}{5}x + 15 = \frac{22}{31}x + \frac{148}{31}$$

$$x = \frac{\frac{148}{31} - 15}{-\frac{3}{5} - \frac{22}{31}} = 7.81$$

Substituting into one of the line equations:

$$y = -\frac{3}{5} \cdot 7.81 + 15 = 10.31$$



2. Equation of a circle

A circle in other words is nothing more than a group of points that are at an equal distance from a given point, which is the center of the circle. The equal distance from the center point of the circle to any point on it is the radius.

This relationship can be quantified by writing that the distance from any given point to the center of the circle must be equal to the radius. Only points that are on the circle will solve the equation (see Figure 8).

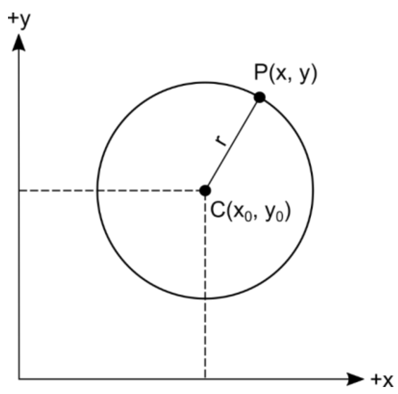


Figure 8. Equation of a circle.

The standard from of the equation of a circle with its center point at (x_0, y_0) and with a radius r:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Example 5: Find the coordinates of the points of intersection between a circle with a radius of 5 and a center point at $(14, 15)$, and a line with equation $y = \frac{1}{3}x + 6$.

Solution:

Equation of the circle:

$$(x-14)^2 + (y-15)^2 = 5^2$$

Substituting the equation of the line into the equation of the circle:

$$(x-14)^2 + \left(\frac{1}{3}x - 9\right)^2 = 5^2$$

$$x^2 - 28x + 196 + \frac{1}{9}x^2 - 6x + 81 - 25 = 0$$

$$\frac{10}{9}x^2 - 34x + 252 = 0$$

$$\frac{34 \pm \sqrt{34^2 - 4 \cdot \frac{10}{9} \cdot 252}}{\frac{20}{9}} \rightarrow \begin{matrix} x_1 = 12.6 \\ x_2 = 18 \end{matrix}$$

Substituting the two x values into the line equation, we find the corresponding y values:

$$y_1 = 10.2$$
$$y_2 = 12$$

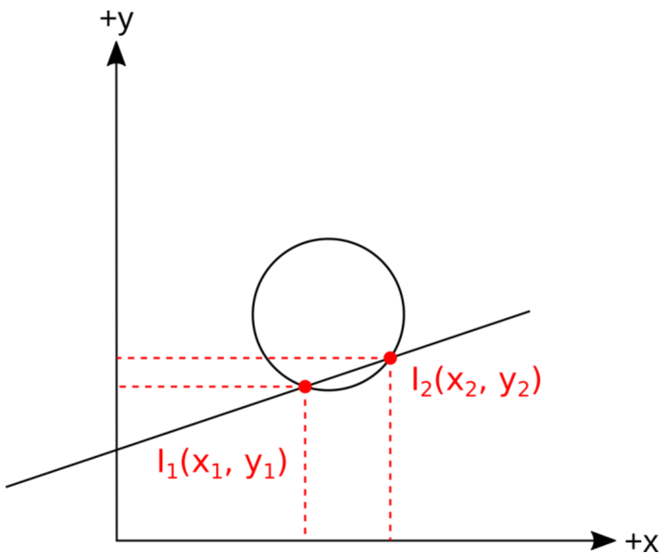


Figure 9. Solution of example 1.

3. Geodetic example using coordinate geometry

We take an angle measuring instrument to points $A(401.81, 524.78)$ and $B(1022.63, 458.96)$. On point A, we measure a direction with a whole circle bearing (WCB_{AP})of $56°47'12"$. On point B, we measure a direction with a whole circle bearing (WCB_{B})of $298°52'53"$. What are the coordinates of the point where the two directions intersect each other?

The whole circle bearing an angle defined in planar surveying calculations that is created by a given direction and the direction parallel with North. It is sometimes also called an azimuth or just simply bearing, however, these labels might mean different things depending on the context, so we will use the whole circle bearing (WCB) name for utmost clarity. The WCB is measured between 0° and 360°, for example, if a direction is points to the West, its WCB is 270°.

We first have to define the equations of the two lines. We know that the tangent of the WCB is equal to the $\frac{x}{y}$ ratio, so we can write the following equation for each line:

$$\frac{x - x_0}{y - y_0} = \tan(\mathrm{WCB})$$

where x_0 and y_0 are the coordinates of a point on the line. From this, the equation of a line:

$$x = y \cdot \tan(\mathrm{WCB}) - y_0 \cdot \tan(\mathrm{WCB}) + x_0$$

Equating the two lines:

$$y \cdot \tan(\mathrm{WCB_{AP}}) - y_A \cdot \tan(\mathrm{WCB_{AP}}) + x_A = y \cdot \tan(\mathrm{WCB_{BP}}) - y_B \cdot \tan(\mathrm{WCB_{BP}}) + x_B$$

$$y \cdot (\tan(\mathrm{WCB_{AP}}) - \tan(\mathrm{WCB_{BP}})) = y_A \cdot \tan(\mathrm{WCB_{AP}}) - y_B \cdot \tan(\mathrm{WCB_{BP}}) + x_B - x_A$$

$$y = \frac{y_A \cdot \tan(\mathrm{WCB_{AP}}) - y_B \cdot \tan(\mathrm{WCB_{BP}}) + x_B - x_A}{\tan(\mathrm{WCB_{AP}}) - \tan(\mathrm{WCB_{BP}})} =$$

$$= \frac{524.78 \cdot \tan(56°47'12") - 458.96 \cdot \tan(298°52'53") + 1022.63 - 401.81}{\tan(56°47'12") - \tan(298°52'53")} = 674.916\ \mathrm{m}$$

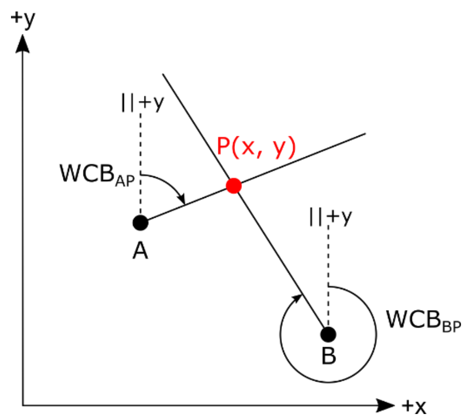


Figure 10. Equation of a circle.

Substituting into one of the line equations:

$$x = 590.317 \cdot \tan(56°47'12") - 524.78 \cdot \tan(56°47'12) + 401.81 = 631.126\ \mathrm{m}$$

The coordinates of the intersection: $(631.126\ \mathrm{m}, 674.916\ \mathrm{m})$