# Surveying II – Week 2
# Intersections

Intersections are the group of planar surveying calculations where we use two control points (three in the case of resection) with known coordinates and certain angle/distance measurements to compute the coordinates of an unknown point.

## 1. Intersections

### 1.1. *Intersection using inner angles*

Given information: coordinates of points $A$ and $B$.

Measured quantities: $\alpha$ and $\beta$ inner angles.

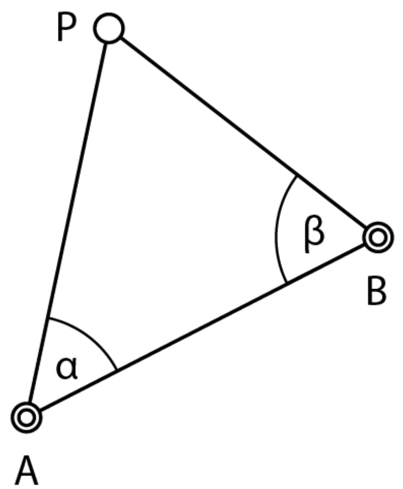


Figure 1. Intersection using inner angles.

Computing the coordinates of the unknown point $P$:

1. Using the II. fundamental task of surveying (abbreviated further as II. FTS), we compute the distance $d_{AB}$ and the whole circle bearings (WCB) $\mathrm{WCB}_{AB}$ and $\mathrm{WCB}_{BA}$.
2. Using the measured inner angles, we compute the WCBs between the control points and the unknown point $\mathrm{WCB}_{AP}$ and $\mathrm{WCB}_{BP}$. According to the figure:

$$\mathrm{WCB}_{AP} = \mathrm{WCB}_{AB} - \alpha$$

$$\mathrm{WCB}_{BP} = \mathrm{WCB}_{BA} + \beta$$

3. Using the sine theorem, we can compute the distances between the control points and point $P$:

$$\frac{\sin(\alpha)}{\sin(\alpha+\beta)} = \frac{d_{BP}}{d_{AB}} \Rightarrow d_{BP} = \frac{\sin(\alpha)}{\sin(\alpha+\beta)} \cdot d_{AB}$$

$$\frac{\sin(\beta)}{\sin(\alpha+\beta)} = \frac{d_{BP}}{d_{AB}} \Rightarrow d_{BP} = \frac{\sin(\beta)}{\sin(\alpha+\beta)} \cdot d_{AB}$$

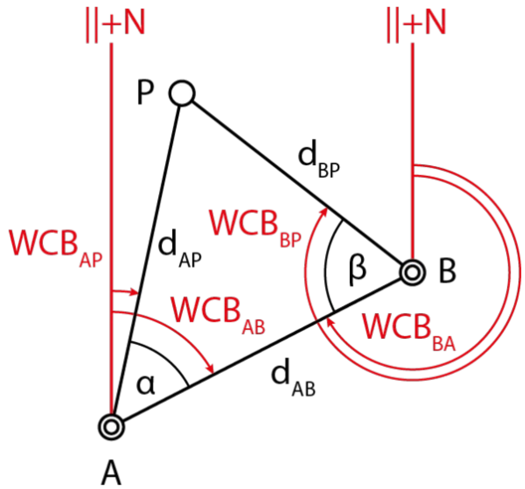


Figure 2. Computation of the WCB's.

4. We use the I. fundamental task of surveying (abbreviated further as I. FTS) to find the coordinates of the unknown point $P$. We can do this computation from using both $A$ and $B$, so we can check our results:

$$E_P = E_A + d_{AP} \cdot \sin(\mathrm{WCB}_{AP})$$

$$N_P = N_A + d_{AP} \cdot \cos(\mathrm{WCB}_{AP})$$

or

$$E_P = E_B + d_{BP} \cdot \sin(\mathrm{WCB}_{BP})$$

$$N_P = N_B + d_{BP} \cdot \cos(\mathrm{WCB}_{BP})$$

<u>Example 1:</u>

Coordinates of the control points:

| Point | E [m] | N [m] |
|---|---|---|
| A | 658 077.70 | 247 431.38 |
| B | 657 310.23 | 247 123.54 |

Measured angles:

$$\alpha = 81\text{-}34\text{-}45$$

$$\beta = 66\text{-}45\text{-}57$$

The clockwise order of the points is $B$, $P$, $A$.

According to the coordinates and the clockwise order, we can create the following figure:

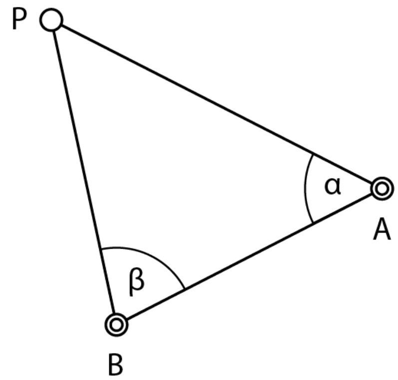


Figure 3. The layout of the points in Example 1.

Distance and WCB between the control points from II: FTS:

$$d_{AB} = 826.907 \text{ m}$$

$$\text{WCB}_{AB} = 248\text{-}08\text{-}38$$

$$\text{WCB}_{BA} = 68\text{-}08\text{-}38$$

The WCBs from the control points to point $P$:

$$\text{WCB}_{AP} = \text{WCB}_{AB} + \alpha = 329\text{-}43\text{-}23$$

$$\text{WCB}_{BP} = \text{WCB}_{BA} - \beta = 1\text{-}22\text{-}41$$

The distances between the control points and point $P$:

$$d_{AP} = \frac{\sin(\beta)}{\sin(\alpha + \beta)} \cdot d_{AB} = 1447.866 \text{ m}$$

$$d_{BP} = \frac{\sin(\alpha)}{\sin(\alpha + \beta)} \cdot d_{AB} = 1558.655 \text{ m}$$

The coordinates of $P$ computed from point $A$:

$$E_P = E_A + d_{AP} \cdot \sin(\text{WCB}_{\text{AP}}) = 657\,347.714 \text{ m} \approx 657\,347.71 \text{ m}$$

$$N_p = N_A + d_{AP} \cdot \cos(\text{WCB}_{\text{AP}}) = 248\,681.754 \text{ m} \approx 248\,681.75 \text{ m}$$

The coordinates of $P$ computed from point $B$:

$$E_P = E_B + d_{BP} \cdot \sin(\text{WCB}_{\text{BP}}) = 657\,347.714 \text{ m} \approx 657\,347.71 \text{ m}$$

$$N_p = N_B + d_{BP} \cdot \cos(\text{WCB}_{\text{BP}}) = 248\,681.754 \text{ m} \approx 248\,681.75 \text{ m}$$

The two sets of coordinates match at least up to centimeter precision.

## *1.2. Intersection using whole circle bearings*

If there is no line of sight between the control points, we cannot directly measure the inner angles mentioned in section 1.1. In such a case, we instead measure the angles $\varphi_A$ and $\varphi_B$ in the figure between the unknown point $P$ and two control points $C$ and $D$.

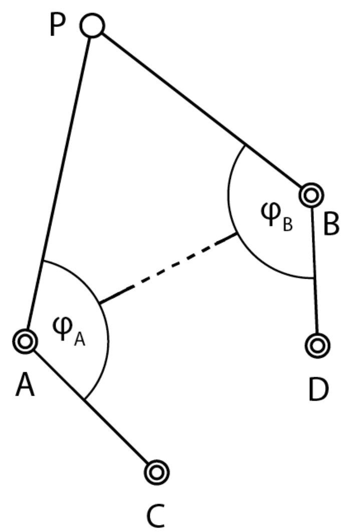


Figure 4. Intersection using whole circle bearings.

Given information: coordinates of points $A, B, C, D$

Measured quantites: angles $\varphi_A$ and $\varphi_B$

Computing the coordinates of point $P$:

1. Using the II. FTS, we can compute the $d_{AB}$ and $\mathrm{WCB}_{AB}, \mathrm{WCB}_{BA}, \mathrm{WCB}_{AC}$ and $\mathrm{WCB}_{BD}$.
2. Using the WCBs and the measured angles, we can compute the WCBs from $A$ and $B$ to the unknown point:

$$\mathrm{WCB}_{AP} = \mathrm{WCB}_{AC} - \varphi_A$$
$$\mathrm{WCB}_{BP} = \mathrm{WCB}_{BD} + \varphi_B$$

3. We can compute the inner angles at $A$ ($\alpha$) and $B$ ($\beta$) from the calculated WCBs:

$$\alpha = \mathrm{WCB}_{AB} - \mathrm{WCB}_{AP}$$
$$\beta = \mathrm{WCB}_{BP} - \mathrm{WCB}_{BA}$$

4. The computation is now a simple intersection with inner angles (see section 1.1.)

# 2. Side section

In case of the side section, one of the control points is inaccessible, meaning that we cannot set up our station on that point. To compensate for this, we set up the instrument on the unknown point $P$ and measure the inner angle here.

## *2.1. Side section using inner angles*

Given information: coordinates of points $A, B$

Measured quantities: angles: $\alpha$ and $\gamma$ inner angles

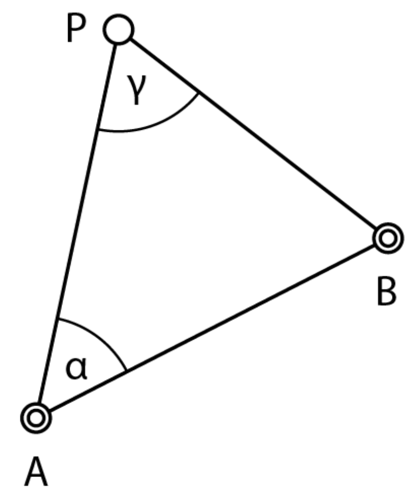


Figure 5. Side section using inner angles.

Computing the coordinates of point $P$:

1. Using the II. FTS, we compute the distance $d_{AB}$, $\mathrm{WCB}_{AB}$ and $\mathrm{WCB}_{BA}$.
2. Using the formula for the sum of the inner angles of a triangle, we can find the value of $\beta$ at point $B$:

$$\beta = 180° - \alpha - \gamma$$

3. Compute the intersection with inner angles (see section 1.1.).

## *2.2. Side section using a distance measurement*

Given information: coordinates of points $A, B$

Measured quantities: angles: $\gamma$ inner angle and distance $d_{AP}$

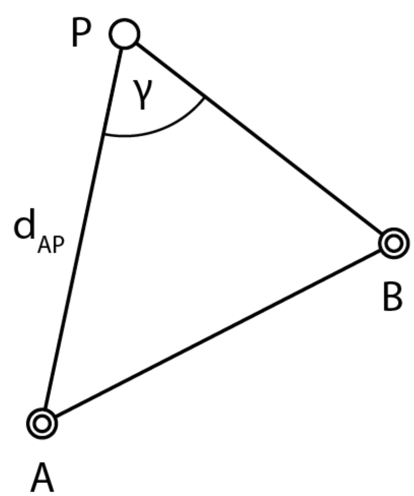


Figure 6. Side section using a distance measurement.

Computing the coordinates of point $P$:

1. Using the II. FTS, we compute the distance $d_{AB}$, $\mathrm{WCB}_{AB}$ and $\mathrm{WCB}_{BA}$.
2. Using the sine theorem, we can compute the inner angle at point $B$:

$$\frac{\sin(\beta)}{\sin(\gamma)} = \frac{d_{AP}}{d_{AB}} \Rightarrow \beta = \arcsin\left(\frac{d_{AP}}{d_{AB}} \cdot sin(\gamma)\right)$$

3. We can compute the third inner angle, at point $A$, using the other two inner angles:

$$\alpha = 180° - \beta - \gamma$$

4. Compute the intersection with inner angles (see section 1.1.).

## 3. Arc section

In case of the arc section, instead of angles, we measure distances between the control points and the unknown point. We also have to specify the clockwise order of the points, as only using the two distances results in two possible locations of point $P$.

Given information: coordinates of points $A, B$, the clockwise order of the points

Measured quantites: distances $d_{AP}$ and $d_{BP}$

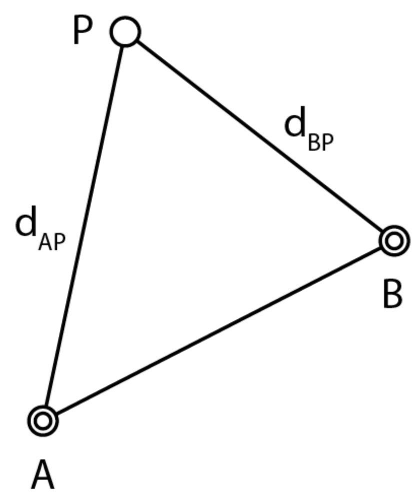


Figure 7. Arc section.

Computing the coordinates of point $P$:

1. Using the II. FTS, we compute the distance $d_{AB}$, $\mathrm{WCB}_{AB}$ and $\mathrm{WCB}_{BA}$.
2. Using the cosine theorem, we can find the two inner angles $\alpha$ and $\beta$:

$$d_{BP}^2 = d_{AP}^2 + d_{AB}^2 - 2 \cdot d_{AB} \cdot d_{AP} \cdot \cos(\alpha) \Rightarrow \alpha = \arccos\left(\frac{d_{BP}^2 - d_{AP}^2 - d_{AB}^2}{-2 \cdot d_{AB} \cdot d_{AP}}\right)$$

$$\beta = \arccos\left(\frac{d_{AP}^2 - d_{AB}^2 - d_{BP}^2}{-2 \cdot d_{AB} \cdot d_{BP}}\right)$$

3. Compute the intersection with inner angles (see section 1.1.).

Example 2.

Coordinates of the control points:

| Point | E [m] | N [m] |
|---|---|---|
| A | 654 653.23 | 232 456.39 |
| B | 654 234.92 | 232 167.47 |

Measured distances:

$$d_{AP} = 967.34 \text{ m}$$

$$d_{BP} = 846.45 \text{ m}$$

The clockwise order of the points is $B$, $P$, $A$.

According to the coordinates and the clockwise order, we can create the following figure:

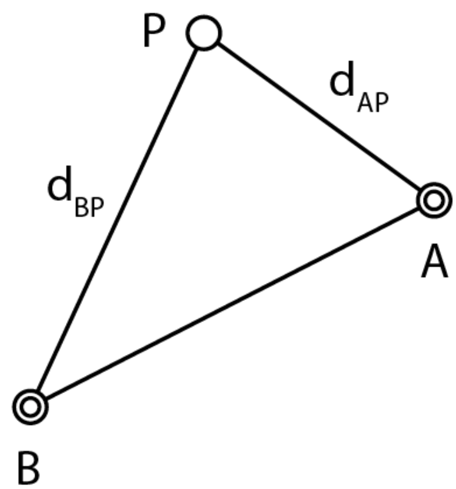


Figure 8. The layout of the points in Example 2.

Distance and WCB between the control points from II: FTS:

$$d_{AB} = 508.388 \text{ m}$$

$$\text{WCB}_{AB} = 235\text{-}22\text{-}04$$

$$\text{WCB}_{BA} = 55\text{-}22\text{-}04$$

The inner angles from the cosine theorem:

$$\alpha = \arccos\left(\frac{d_{BP}^2 - d_{AP}^2 - d_{AB}^2}{-2 \cdot d_{AB} \cdot d_{AP}}\right) = 60\text{-}56\text{-}28$$

$$\beta = \arccos\left(\frac{d_{AP}^2 - d_{AB}^2 - d_{BP}^2}{-2 \cdot d_{AB} \cdot d_{BP}}\right) = 87\text{-}23\text{-}35$$

The WCBs from the control points to point $P$:

$$\text{WCB}_{AP} = \text{WCB}_{AB} + \alpha = 296\text{-}18\text{-}32$$

$$\text{WCB}_{BP} = \text{WCB}_{BA} - \beta = 327\text{-}58\text{-}39$$

The coordinates of $P$ computed from point $A$:

$$E_P = E_A + d_{AP} \cdot \sin(\mathrm{WCB_{AP}}) = 653\,786.089 \text{ m} \approx 653\,786.09 \text{ m}$$

$$N_p = N_A + d_{AP} \cdot \cos(\mathrm{WCB_{AP}}) = 232\,885.125 \text{ m} \approx 232\,885.12 \text{ m}$$

The coordinates of $P$ computed from point $B$:

$$E_P = E_B + d_{BP} \cdot \sin(\mathrm{WCB_{BP}}) = 653\,786.087 \text{ m} \approx 653\,786.09 \text{ m}$$

$$N_p = N_B + d_{BP} \cdot \cos(\mathrm{WCB_{BP}}) = 232\,885.124 \text{ m} \approx 232\,885.12 \text{ m}$$

The two sets of coordinates match to at least centimeter precision.

## 4. Resection

In case of the resection, we only set up the instrument on the unknown point $P$ and measure the angles subtended by the directions from the unknown point to exactly 3 control points.

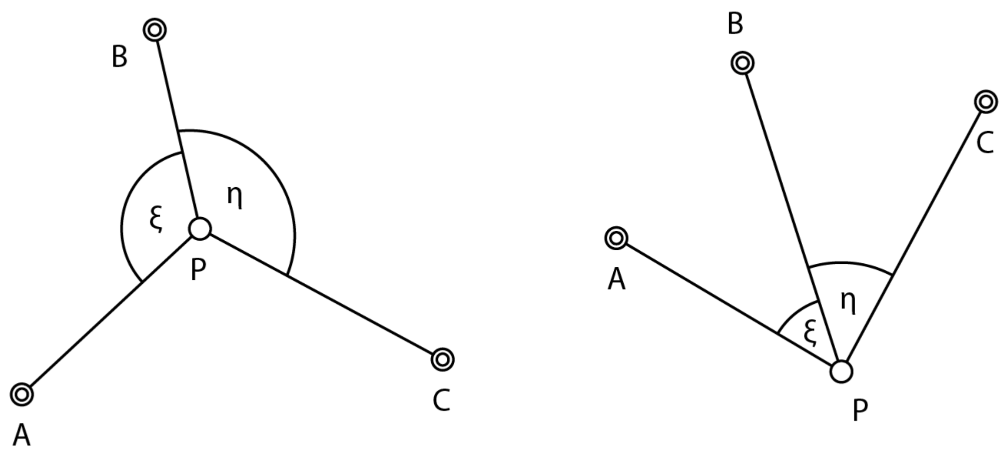


Figure 9. Two possible layouts of the points in the resection problem.

In the figure above, the angles $\xi$ and $\eta$ are measured. There are multiple solutions to the resection problem, graphical ones include the solutions by Collins and Sossna, examples for analytical solutions would be Tienstra's method or Runge's method and there are numerical solutions as well. In the following, Collins' method is briefly introduced.

Suppose the layout given in Figure 10 below. First, we draw a circle around the points $A$, $C$ and $P$. The line connecting $P$ and $B$ intersect the circle at point $S$.

According to the inscribed angle theorem, the angle at $A$ in the triangle $ASC$ is $\eta$ and the angle at $C$ in the triangle $ASC$ is $\xi$. We can compute the $\mathrm{WCB}_{AS}$ and the $\mathrm{WCB}_{CS}$

$$\mathrm{WCB}_{AS} = \mathrm{WCB}_{AC} - \eta$$

$$\mathrm{WCB}_{CS} = \mathrm{WCB}_{CA} + \xi$$

and using the control points $A$ and $C$, we can find the coordinates of $S$. As $S$ is a known point now, we can compute the $\mathrm{WCB}_{BS}$ which is equal to $\mathrm{WCB}_{BP}$ as $S$ and $P$ are on the same line. The $\mathrm{WCB}_{AP}$ and $\mathrm{WCB}_{CP}$ can be found:

$$\mathrm{WCB}_{AP} = \mathrm{WCB}_{BP} - \xi$$

$$\mathrm{WCB}_{CP} = \mathrm{WCB}_{BP} + \eta$$

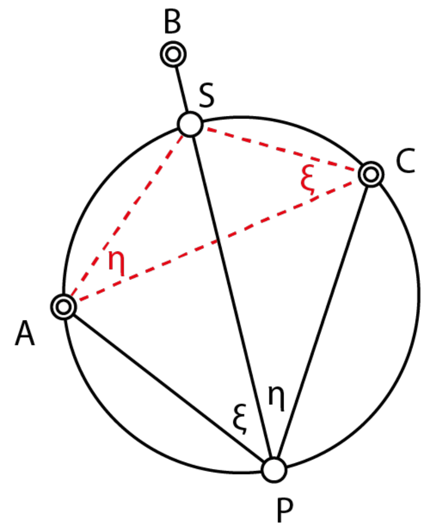


Figure 10. Computing the resection using Collins' method.

The coordinates of $P$ can now be found as the intersection of any combination of the lines $AP$, $BP$ and $CP$ and therefore we can check our computation.