Equation of the circle, ellipse, area computation

1. Equation of a circle

A circle in other words is nothing more than a group of points that are at an equal distance from a given point, which is the center of the circle. The equal distance from the center point of the circle to any point on it is the radius.

This relationship can be quantified by writing that the distance from any given point to the center of the circle must be equal to the radius. Only points that are on the circle will solve the equation (see Figure 1).

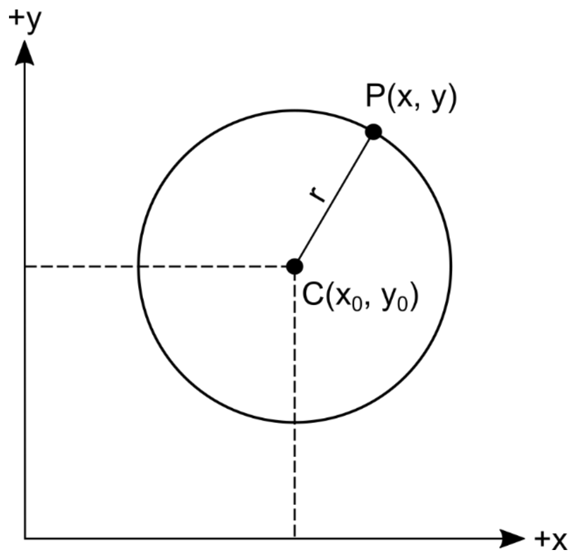

Figure 1. Equation of a circle.

The standard from of the equation of a circle with its center point at (x_0, y_0) and with a radius r:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Example 1: Find the coordinates of the points of intersection between a circle with a radius of 5 and a center point at $(14, 15)$, and a line with equation $y = \frac{1}{3}x + 6$.

Solution:

Equation of the circle:

$$(x - 14)^2 + (y - 15)^2 = 5^2$$

Substituting the equation of the line into the equation of the circle:

$$(x - 14)^2 + \left(\frac{1}{3}x - 9\right)^2 = 5^2$$

$$x^2 - 28x + 196 + \frac{1}{9}x^2 - 6x + 81 - 25 = 0$$

$$\frac{10}{9}x^2 - 34x + 252 = 0$$

$$\frac{34 \pm \sqrt{34^2 - 4 \cdot \frac{10}{9} \cdot 252}}{\frac{20}{9}} \rightarrow \begin{array}{l} x_1 = 12.6 \\ x_2 = 18 \end{array}$$

Substituting the two x values into the line equation, we find the corresponding y values:

$$y_1 = 10.2$$
$$y_2 = 12$$

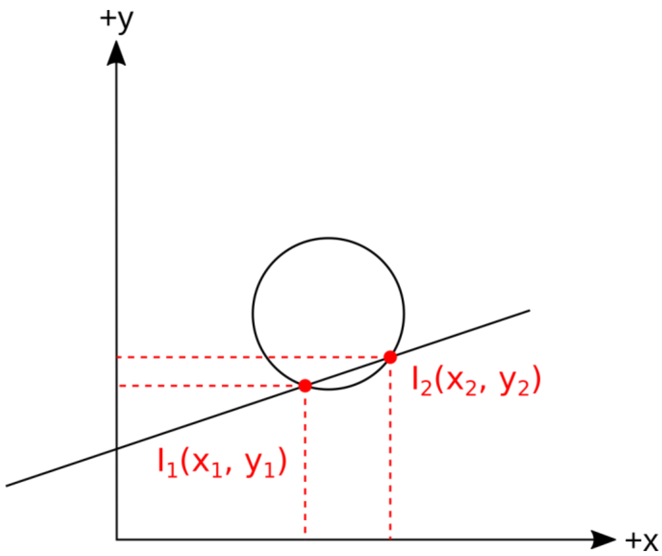

Figure 2. Solution of example 1.

2. Equation of an ellipse

An ellipse is a curve around two focal points such that the sum of the distances to the two focal points (*foci*) are the same for every point on the ellipse. On Figure 3, a denotes the semi-major, b the semi-minor axis of the ellipse, while c is the distance of the focal points from the center. The intersections of the major axis and the ellipse are called the vertices.

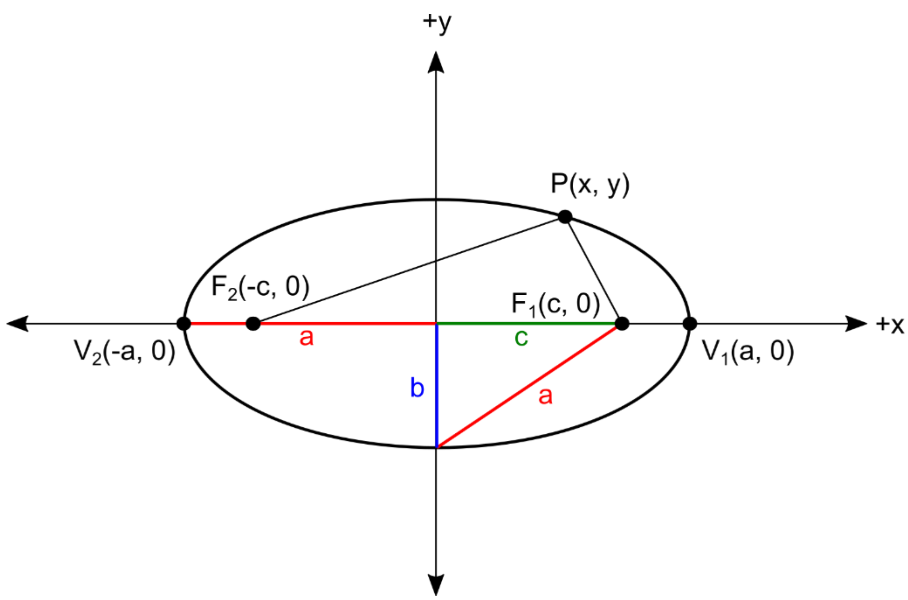

Figure 3. Properties of an ellipse.

What Figure 3 also shows is that the hypotenuse of the right triangle created by c and b ha length a, in other words:

$$a^2 = b^2 + c^2$$

In order to find the equation of the ellipse, we have to specify the sum of the distances from any $P(x, y)$ point to the two focal points:

$$d_1 + d_2 = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}$$

We know that this sum is equal to some constant value which we can find by taking a look at the vertex $V_1(a, 0)$. The sum of the distances from point A to F_1 and F_2 is:

$$(a + c) + (a - c) = 2a$$

We can now write:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2xc + c^2 + y^2$$

$$4xc = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$xc = a^2 - a\sqrt{(x-c)^2 + y^2}$$

$$\left(a\sqrt{(x-c)^2 + y^2}\right)^2 = (a^2 - xc)^2$$

$$a^2((x-c)^2 + y^2) = a^4 - 2a^2xc + x^2c^2$$

$$a^2(x^2 - 2xc + c^2 + y^2) = a^4 - 2a^2xc + x^2c^2$$

$$a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2 = a^2 - 2a^2xc + x^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Using the properties of the ellipse, we can write that $a^2 - c^2 = b^2$ (from the equation above):

$$x^2b^2 + a^2y^2 = a^2b^2$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

This formula is of course true for an ellipse that is centered on the origin. If we have an ellipse that is not centered on the origin, we can use the following formula:

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

where (x_0, y_0) are the coordinates of the center of the ellipse.

Example 2: Find the length of the semi-major and the semi-minor axes of the ellipse defined by the following equation. What are the coordinates of the center, the foci and the vertices?

$$16x^2 - 672x + 14756 + 25y^2 - 900y = 0$$

Solution:

$$16x^2 - 672x + 14756 + 25y^2 - 900y = 0$$
$$16(x^2 - 42x) + 25(y^2 - 36y) + 14756 = 0$$
$$16(x^2 - 42x + 441) - 7056 + 25(y^2 - 36y + 324) - 8100 + 14756 = 0$$
$$16(x - 21)^2 + 25(y - 18)^2 - 400 = 0$$
$$\frac{(x-21)^2}{25} + \frac{(y-18)^2}{16} = 1$$

Length of the semi-major axis: 5

Length of the semi-minor axis: 4

Coordinates of the center: $C(21, 18)$

Coordinates of the foci:

$$c^2 = a^2 - b^2 = 25 - 16 = 9 \Rightarrow c = 3$$
$$F = (C_x \pm c, C_y) \Rightarrow F_1(24, 18) \text{ and } F_2(18, 18)$$

Coordinates of the vertices: $V_1(26, 18)$ and $V_2(16, 18)$

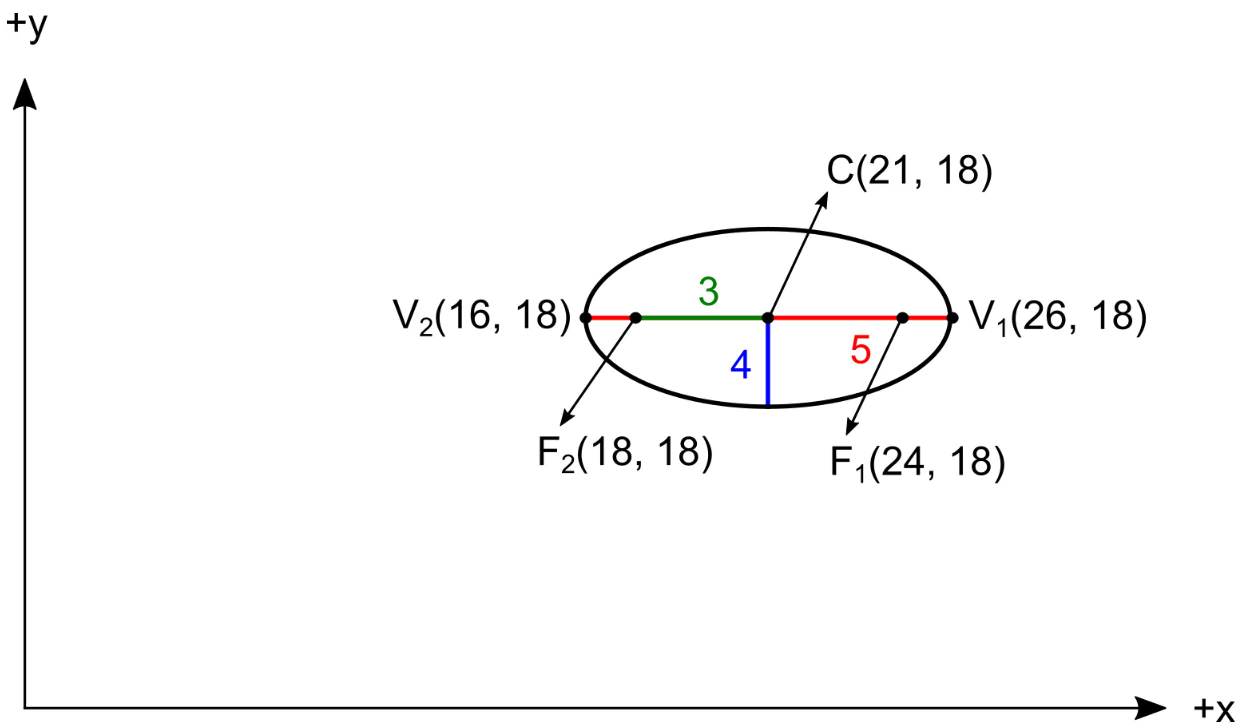

Figure 4. Solution of example 2.

3. Computing the area of a polygon from coordinates

If we have any polygon given with the coordinates of its vertices (corner points), we can use a simple procedure to calculate its area. First, we need to establish a direction to go around the polygon. Then, we choose a vertex a start from and create a trapezoid using the starting vertex, the subsequent vertex and two lines that are perpendicular to the x-axis. We create a trapezoid using each vertex pair and sum their areas (see Figure 5). Depending on our direction, we may end up with a negative value, in this case the area is the absolute value of the result.

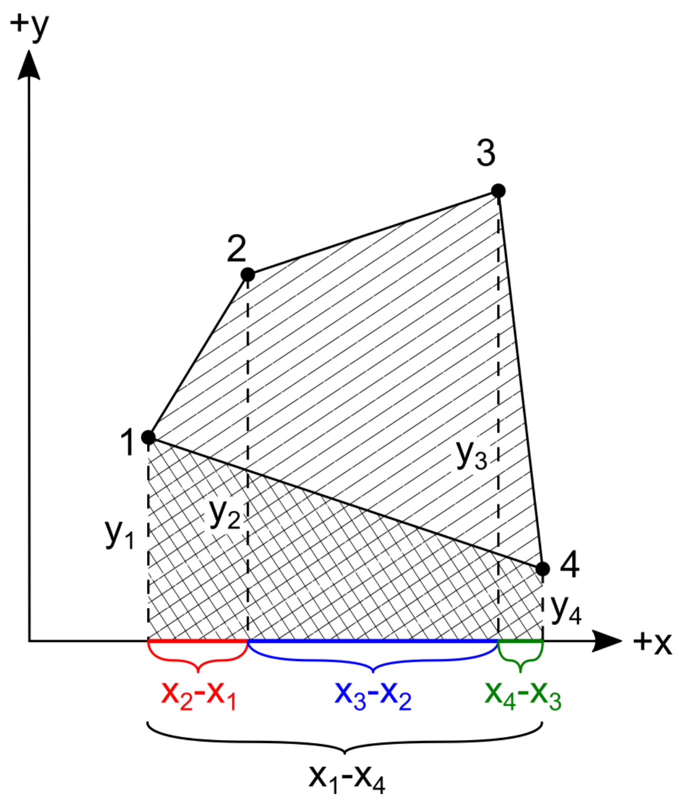

Figure 5. Computing the area of a polygon using trapezoids.

For the polygon in Figure 5, the areas of the trapezoids can be written like so:

$$A = A_1 + A_2 + A_3 + A_4 =$$

$$= \frac{(y_2 + y_1) \cdot (x_2 - x_1)}{2} + \frac{(y_3 + y_2) \cdot (x_3 - x_2)}{2} + \frac{(y_4 + y_3) \cdot (x_4 - x_3)}{2} + \frac{(y_1 + y_4) \cdot (x_1 - x_4)}{2} =$$

$$= \frac{\sum_i (y_{i+1} + y_i) \cdot (x_{i+1} - x_i)}{2}$$

For the last trapezoid, $x_1 < x_4$, so its area will be negative, which is just what we need, as it means that the area between our polygon and the x-axis (which is of course not part of the polygon) will be subtracted. When using the formula we have to make a complete circle around the polygon, so when we are at the last point, our next point is the vertex where we started the calculation.

Example 3: Calculate the area of the polygon defined by the following vertices.

Point no.	x [m]	y [m]
1	251.54	526.41
2	202.13	350.11
3	1050.56	614.66
4	670.92	195.98
5	1221.28	280.82

We first create a sketch using the coordinates of the vertices to find the correct order.

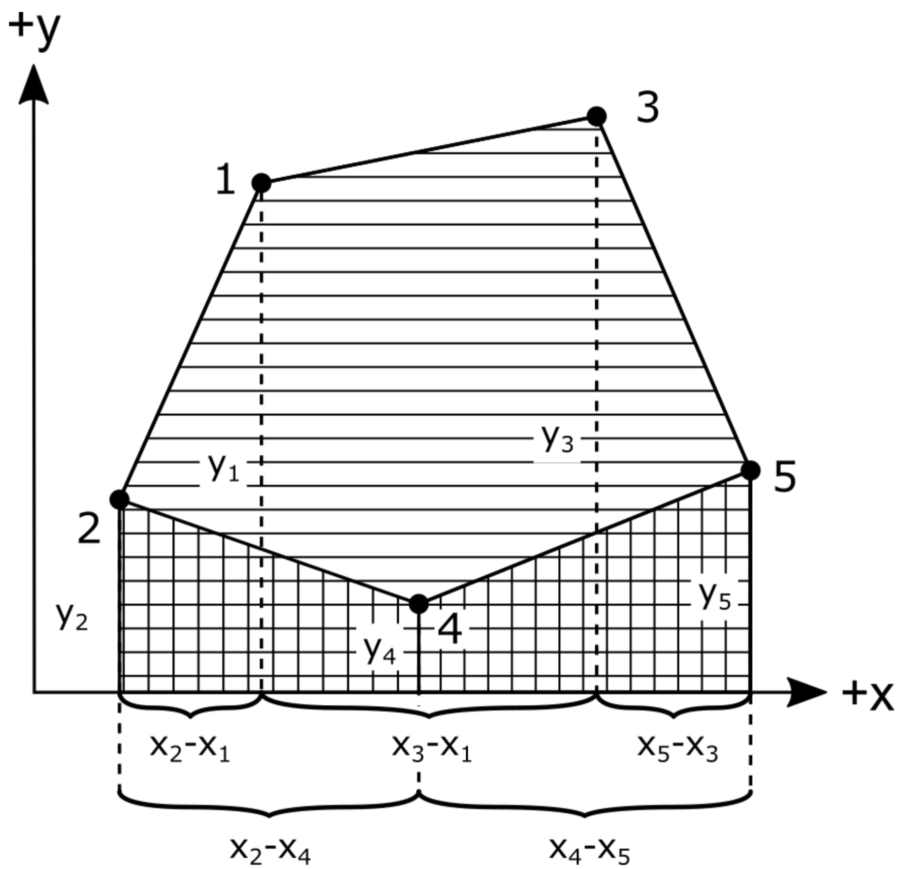

Figure 6. The polygon in example 3.

The order is: 2-1-3-5-4. It is best to create a new table with the vertices in the correct order.

Point no.	x [m]	y [m]
2	202.13	350.11
1	251.54	526.41
3	1050.56	614.66
5	1221.28	280.82
4	670.92	195.98

The formula for calculating the area:

$$A = \left| \frac{\sum_i (y_{i+1} + y_i) \cdot (x_{i+1} - x_i)}{2} \right|$$

We can modify the table to make the calculation easier:

Point no.	x [m]	y [m]	$y_{i+1} + y_i$	$x_{i+1} - x_i$	A_i
2	202.13	350.11	876.52	49.41	21654.43
1	251.54	526.41	1141.07	799.02	455868.88
3	1050.56	614.66	895.48	170.72	76438.17
5	1221.28	280.82	476.80	–550.36	–131205.82
4	670.92	195.98	546.09	–468.79	–128000.77
2	202.13	350.11			$\sum \approx 294755\ m^2$