Exercises in coordinate geometry

1. We take an angle measuring instrument to points $A(401.81, 524.78)$ and $B(1022.63, 458.96)$. On point A, we measure a direction with a whole circle bearing ($\mathrm{WCB_{AP}}$)of $56°47'12"$. On point B, we measure a direction with a whole circle bearing ($\mathrm{WCB_B}$)of $298°52'53"$. What are the coordinates of the point where the two directions intersect each other?

The whole circle bearing an angle defined in planar surveying calculations that is created by a given direction and the direction parallel with North. It is sometimes also called an azimuth or just simply bearing, however, these labels might mean different things depending on the context, so we will use the whole circle bearing (WCB) name for utmost clarity. The WCB is measured between 0° and 360°, for example, if a direction is points to the West, its WCB is 270°.

We first have to define the equations of the two lines. We know that the tangent of the WCB is equal to the $\frac{x}{y}$ ratio, so we can write the following equation for each line:

$$\frac{x - x_0}{y - y_0} = \tan(\mathrm{WCB})$$

where x_0 and y_0 are the coordinates of a point on the line. From this, the equation of a line:

$$x = y \cdot \tan(\mathrm{WCB}) - y_0 \cdot \tan(\mathrm{WCB}) + x_0$$

Equating the two lines:

$$y \cdot \tan(\mathrm{WCB_{AP}}) - y_A \cdot \tan(\mathrm{WCB_{AP}}) + x_A = y \cdot \tan(\mathrm{WCB_{BP}}) - y_B \cdot \tan(\mathrm{WCB_{BP}}) + x_B$$

$$y \cdot (\tan(\mathrm{WCB_{AP}}) - \tan(\mathrm{WCB_{BP}})) = y_A \cdot \tan(\mathrm{WCB_{AP}}) - y_B \cdot \tan(\mathrm{WCB_{BP}}) + x_B - x_A$$

$$y = \frac{y_A \cdot \tan(\mathrm{WCB_{AP}}) - y_B \cdot \tan(\mathrm{WCB_{BP}}) + x_B - x_A}{\tan(\mathrm{WCB_{AP}}) - \tan(\mathrm{WCB_{BP}})} =$$

$$= \frac{524.78 \cdot \tan(56°47'12") - 458.96 \cdot \tan(298°52'53") + 1022.63 - 401.81}{\tan(56°47'12") - \tan(298°52'53")} = 674.916\ \mathrm{m}$$

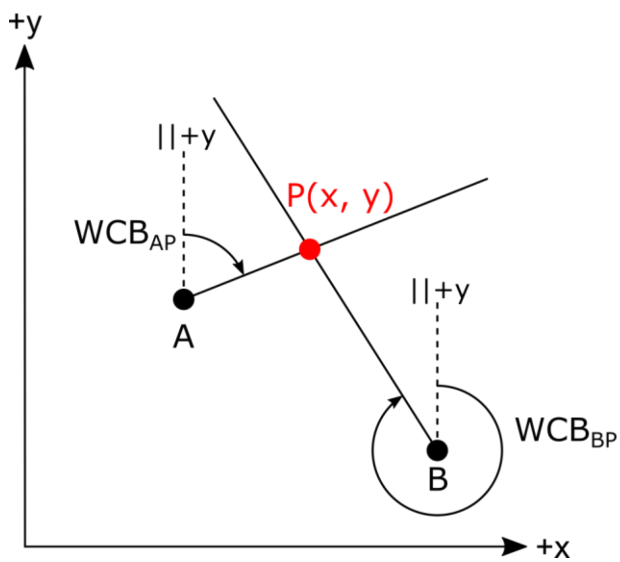


Figure 1. Equation of a circle.

Substituting into one of the line equations:

$$x = 590.317 \cdot \tan(56°47'12") - 524.78 \cdot \tan(56°47'12) + 401.81 = 631.126 \text{ m}$$

The coordinates of the intersection: $(631.126 \text{ m}, 674.916 \text{ m})$

2. Compute the area of the polygon given by the following vertices. Give the result in square kilometers with one decimal precision!

Point ID	x [m]	y [m]
1	1452.14	1011.58
2	2649.36	3874.25
3	2933.41	1982.36
4	568.13	1536.14
5	1531.58	3178.19

To determine the correct order of the points, we can create a sketch. The correct order in this case will be 1-4-5-2-3. The formula for computing the area of the shape given its vertices is the following:

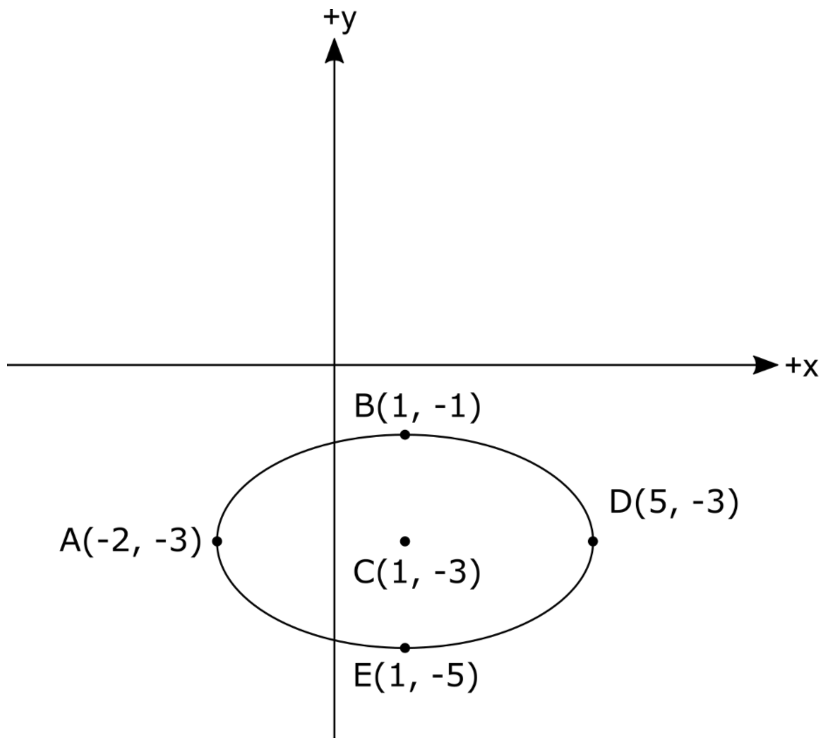
$$A = \left| \frac{1}{2} \cdot \sum_{i=1}^{n} (y_{i+1} + y_i) \cdot (x_{i+1} - x_i) \right|$$

where n is the number of vertices. Remember that when we are at the last vertex, the next point becomes the first vertex, that is, we close the polygon. We can create a table to make the computation more straightforward.

Point no.	x [m]	y [m]	$\frac{1}{2}(y_{i+1} + y_i)$	$x_{i+1} - x_i$	A_i
1	1452.14	1011.58	1010.14	524.56	529879.04
4	568.13	1536.14	1049.86	1642.05	1723922.61
5	1531.58	3178.19	2090.47	696.06	1455092.55
2	2649.36	3874.25	2791.39	-1891.89	-5281002.83
3	2933.41	1982.36	2192.78	-970.78	-2128706.97
1	1452.14	1011.58			$\sum \approx -3700815.60 \text{ m}^2$

As the sum is negative, we have to take its absolute value, making the final result 3.7 km^2.

3. Find the equation of the ellipse given in the figure below. How long are the semi-major and semi-minor axes? What are the coordinates of the foci?



Length of semi-major axis: $a = \left|\overrightarrow{CD}\right| = 5 - 1 = 4$

Length of the semi-minor axis: $b = \left|\overrightarrow{CB}\right| = -1 - (-3) = 2$

Coordinates of the foci: $c^2 = a^2 - b^2 = 4^2 - 2^2 \Rightarrow c = \sqrt{12} = 2\sqrt{3}$

$F_1 = (x_C + c, y_c) = (1 + 2\sqrt{3}, -3)$

$F_2 = (x_c - c, y_c) = (1 - 2\sqrt{3}, -3)$

4. An arch for a bridge over a highway is in the form of half an ellipse. The top arch is 6 meters above the ground level (the length of the minor axis). The highway has four lanes, each 4 meters wide, a center safety strip that is 2 meters wide and two side strips, each 1 meter wide. What should the span of the bridge be (the length of the major axis) given in integer meters, if the height 9 meters from the center is to be 4 meters?

The length of the minor axis is 6 meters: $b = 6$

If we define a Cartesian coordinate-system that has the same center as the half ellipse with its x-axis pointing in the direction of the span of the bridge and the y-axis pointing in the direction of the minor axis, the equation of the ellipse representing the brindge:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The height is 4 meters above the ground ($y = 4$) 9 meters from the center ($x = 9$). Substituting these and the length of the minor axis into the equation:

$$\frac{9^2}{a^2} + \frac{4^2}{6^2} = 1 \Rightarrow a = \sqrt{\frac{81}{1 - \frac{16}{36}}} = 12.07 \approx 12 \text{ m}$$

The span of the bridge is $2a$, that is, 24 meters.