Surveying II. – Propagation of the mean error

Practice exercises

1. Mean error of a function derived from measurements

If we compute the value of a function that is dependent on measurements, the mean error of the measured values will affect the function value as well. The mean error of the function value can be computed using the law of propagation of the mean error.

Let G denote some function of a single measurement L, that is $G = G(L)$. If the mean error of the measurement is m_L, the mean error of the function will be

$$m_G = \pm g \cdot m_L$$

where g is the derivative of the function G with respect to L, computed at the value of the measurement:

$$g = \frac{dG}{dL}(L)$$

If G is a function of multiple measurements, we suppose the following:

- the measurements are independent from each other, meaning that the effect of each measurement on the function's mean error can be computed separately using the function's partial derivative with respect to the measurements;

- the mean error of the function is the vector sum of the effect of the individual measurements.

$$m_G = \pm\sqrt{g_1^2 \cdot m_1^2 + g_2^2 \cdot m_2^2 + \cdots g_i^2 \cdot m_i^2 + \cdots + g_n^2 \cdot m_n^2}$$

where g_i is the partial derivative of G with respect to the i-th measurement and m_i is the mean error of the i-th measurement.

The mean error of simple functions:

- Measurement multiplied by a constant: $G = c \cdot L$

  $g = c$

  $m_G = \sqrt{g^2 \cdot m_L^2} = c \cdot m_L$

- Sum or difference of two measurements: $G = L_1 + L_2$ and $G = L_1 - L_2$

  $g_1 = +1$ in both cases

  $g_2 = +1$ in the case of the sum, and $g_2 = -1$ in the case of the difference

  $m_G = \sqrt{g_1^2 \cdot m_1^2 + g_2^2 \cdot m_2^2} = \sqrt{m_1^2 + m_2^2}$

  If both measurements have the same mean error m: $m_G = \sqrt{2} \cdot m$

- Product of two measurements: $G = L_1 \cdot L_2$

  $g_1 = L_2$

$$g_2 = L_1$$

$$m_G = \sqrt{g_1^2 \cdot m_1^2 + g_2^2 \cdot m_2^2} = \sqrt{L_2^2 \cdot m_1^2 + L_1^2 \cdot m_2^2}$$

- Mean of measurements with the same mean error m: $G = \frac{1}{n} \cdot (L_1 + L_2 + \cdots + L_n)$

$$g_1 = g_2 = \cdots = g_n = \frac{1}{n}$$

$$m_G = \sqrt{g_1^2 \cdot m_1^2 + g_2^2 \cdot m_2^2 + \cdots + g_n^2 \cdot m_n^2} = \sqrt{\left(\frac{1}{n}\right)^2 \cdot m^2 + \left(\frac{1}{n}\right)^2 \cdot m^2 + \cdots + \left(\frac{1}{n}\right)^2 \cdot m^2} =$$

$$= \sqrt{n \cdot \left(\frac{1}{n}\right)^2 \cdot m^2} = \frac{m}{\sqrt{n}}$$

2. Practice exercises

1. A distance was measured in two parts using two different instruments. The measurements are 54.872 m $\pm$ 0.02 m and 634.21 dm $\pm$ 1 dm. Compute the sum of the two measurements and its mean error. Give the results in meter, using mm precision.

$G = L_1 + L_2 = 54.872$ m $+ 63.421$ m $= 118.293$ m

The partial derivatives:

$g_1 = 1$ and $g_2 = 1$

The mean errors of the measurements:

$m_1 = 0.02$ m and $m_2 = 0.1$ m

The mean error of the function:

$$m_G = \pm\sqrt{g_1^2 \cdot m_1^2 + g_2^2 \cdot m_2^2} = \sqrt{0.02^2 + 0.1^2} = \pm 0.102 \text{ m}$$

The distance and its mean error: 118,293 m $\pm$ 0,102 m

2. In order to find the value of an angle, we measured the mean direction of the angle's two legs using the same instrument. The mean direction of the right leg is 165-12-43, the mean direction of the left leg is 91-48-40. The mean error of the instrument is $\pm$ 10". Compute the value of the angle in DMS units and its mean error in arc seconds.

The value of the angle is the mean direction of the left leg (L_2) subtracted from the right leg (L_1):

$G = L_1 - L_2 = 73\text{-}24\text{-}03$

The partial derivatives:

$g_1 = 1, \quad g_2 = -1$

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The mean error of the angle:

$$m_G = \pm\sqrt{g_1^2 \cdot m^2 + g_2^2 \cdot m^2} = \sqrt{2} \cdot m = 14.14'' \approx \pm14''$$

The value of the angle and its mean error: 73-24-03 $\pm$ 14"

3. We would like to compute the area of a parcel with a maximum mean error of $\pm$ 30 m^2. The shorter side of the parcel is measured to be 38.339 m, while the longer side is 59.165 m. The longer side can be measured with a mean error that is three times as much as the mean error of the shorter side. What is the maximum mean error of the shorter side if we don't want to exceed the limit for the mean error of the parcel's area. If we could decrease the mean error of one of the measurements, which one should be decreased to have the most effect on the mean error of the area?

Let L_1 denote the shorter side and L_2 denote the longer side:

$G = A = L_1 \cdot L_2$

The partial derivatives:

$g_1 = L_2 = 59.165$ m

$g_2 = L_1 = 38.339$ m

The mean errors of the measurements:

$m_1 = m_1$

$m_2 = 3 \cdot m_1$

The mean error of the area:

$$m_G = \pm30 = \sqrt{g_1^2 \cdot m_1^2 + g_2^2 \cdot (3 \cdot m_1)^2} = \sqrt{m_1^2 \cdot (g_1^2 + 9g_2^2)} \Rightarrow m_1 = \sqrt{\frac{30^2}{g_1^2 + 9g_2^2}} =$$

$$= \sqrt{\frac{900}{59.165^2 + 9 \cdot 38.339^2}} = \pm0.231 \text{ m}$$

The maximum mean error of the shorter side can be ±0.231 m.

If we can decrease the mean error of the one of the measurements, we should decrease the one that has the larger coefficient in the formula as this will have the most effect on the mean error of the function The coefficient of the mean error of the shorter side is greater than that of the longer side ($g_1 > g_2$), which means that we should decrease the mean error of the shorter side.

4. In order to determine the coordinates of point P, we oriented our theodolite on station A, then measured the unknown point's mean direction and its horizontal distance from the station. Compute the easting coordinate of the unknown point and specify it in meter units with cm precision. Compute the mean error of the coordinate and give it in millimeters.

$E_A = 2643.53 \pm 6$ mm

$z = 95\text{-}14\text{-}30 \pm 15"$

$MD = 320\text{-}52\text{-}37 \pm 10"$

$d_H = 68.963$ m ± 6.5 cm

Our function is the easing coordinate of point P:

$$G = E_P = E_A + d_H \cdot \cos(MD + z) = 2643.53 + 68.963 \cdot \cos\big((320\text{-}52\text{-}37) + (95\text{-}14\text{-}30)\big) = 2681.975$$
$$\approx 2681.98 \text{ m}$$

$g_{E_A} = 1 \, [-]$

$g_z = -d_H \cdot \sin(MD + z) = 68.963 \cdot \sin\big((320\text{-}52\text{-}37) + (95\text{-}14\text{-}30)\big) = -57.252628 \, [\text{m}]$

$g_{MD} = -d_H \cdot \sin(MD + z) = g_z = -57.252628 \, [\text{m}]$

$g_{d_H} = \cos(MD + z) = \cos\big((320\text{-}52\text{-}37) + (95\text{-}14\text{-}30)\big) = 0.557475 \, [-]$

We have to convert the mean errors given in arc seconds to radians:

$$m_z = \frac{15"}{\rho"} = \frac{15}{2 \cdot 10^5} \quad \left(\rho" = \frac{180}{\pi} \cdot 3600 = 206\,264.8 \approx 2 \cdot 10^5 \right)$$

$$m_{MD} = \frac{10"}{\rho"} = \frac{10}{2 \cdot 10^5}$$

$$m_G = \sqrt{g_{E_A}^2 \cdot m_{E_A}^2 + g_z^2 \cdot m_z^2 + g_{MD}^2 \cdot m_{MD}^2 + g_{d_H}^2 \cdot m_{d_H}^2} =$$

$$= \sqrt{1^2 \cdot 0.006^2 + (-57.252628)^2 \cdot \left(\frac{15}{2 \cdot 10^5}\right)^2 + (-57.252628)^2 \cdot \left(\frac{10}{2 \cdot 10^5}\right)^2 + 0.557475^2 \cdot 0.065^2} = 0.037 \text{ m}$$
$$\approx \pm 37 \text{ mm}$$

The final value of the easting of P: 2681.98 ± 37mm.

5. We have to measure the area of a triangular parcel with a maximum of ± 10 m². We measured two adjacent sides (a and b) of the triangle and the angle between the two sides (γ). We could measure the longer side half as accurately then the shorter side. What can the maximum error of the shorter side be? What if error of the area cannot exceed ± 10 m² with 99.7% confidence?

$a = 58.192$ m

$b = 67.618$ m

$\gamma = 34\text{-}25\text{-}10 \pm 0.3'$ (18")

The longer side is half as accurate as the shorter side (its mean error is double): $m_b = 2 \cdot m_a$

Our function is the area of the triangle:

$$G = \frac{a \cdot b \cdot \sin(\gamma)}{2}$$

$$g_a = \frac{b \cdot \sin(\gamma)}{2} = \frac{67.618 \cdot \sin(34\text{-}25\text{-}10)}{2} = 19.110435$$

$$g_b = \frac{a \cdot \sin(\gamma)}{2} = \frac{58.192 \cdot \sin(34\text{-}25\text{-}10)}{2} = 32.892853$$

$$g_\gamma = \frac{a \cdot b \cdot \cos(\gamma)}{2} = \frac{58.192 \cdot 67.618 \cdot \cos(34\text{-}25\text{-}10)}{2} = 3245.923966$$

The mean error of the angle has to be converted into radians: $m_\gamma = \frac{18}{2 \cdot 10^5}$

$$m_G = \sqrt{g_a^2 \cdot m_a^2 + g_b^2 \cdot (2m_a)^2 + g_\gamma^2 \cdot m_\gamma^2} \leq \pm 10 \text{ m}^2 \Rightarrow \sqrt{m_a^2(g_a^2 + 4g_b^2) + g_\gamma^2 \cdot m_\gamma^2} \Rightarrow m_a^2 \leq \frac{m_G^2 - g_\gamma^2 \cdot m_\gamma^2}{g_a^2 + 4g_b^2}$$

$$m_a^2 \leq \frac{10^2 - 3245.923966^2 \cdot \dfrac{18}{2 \cdot 10^5}}{19.110435^2 + 4 \cdot 32.892853^2} = 0.021290 \Rightarrow m_a \leq 0.146 \text{ m} = 14.6 \text{ cm}$$

The maximum mean error of a can be 14.6 cm.

The maximum mean error of b is therefore $m_b = 2 \cdot m_a = 0.292$ m $= 29$ cm.

Check:

$$m_G = \sqrt{g_a^2 \cdot m_a^2 + g_b^2 \cdot m_b^2 + g_\gamma^2 \cdot m_\gamma^2} = \pm 9{,}93 \text{ m}^2 < 10 \text{ m}^2$$

What if error of the area cannot exceed ± 10 m² with 99.7% confidence?

99.7% confidence refers to 3σ, which means that the mean error of the area can be maximum a third of the specified threshold.

$$m_{G,1\sigma} = \pm 10 \text{ m}^2 \Rightarrow m_{G,3\sigma} = \pm \frac{10}{3} = \pm 3.33 \text{ m}^2$$

We have to recompute our example with this new value as the threshold. The maximum mean error of a in this case would be $m_a = \pm 0{,}048$ m.

6. We need to find the volume of a pyramid with a square base. In order to manage this, we measured one side of the base ($a = 25.032$ m $\pm$ 3.5 cm), the length of the edge ($b = 3765.9$ cm $\pm$ 2 dm) and zenith angle of the edge ($z = $ 42-11-33 $\pm$ 0.2′). What is the volume of the pyramid and its mean error? Give the volume in integer cubic meters and the mean error in cubic meters with one decimal precision.

$a = 25.032$ m $\qquad\qquad$ $m_a = 3.5$ cm $= 0.035$ m

$b = 37.659$ m $\pm$ 0.200 m $\qquad$ $m_b = 2$ dm $= 0.2$ m

$z = $ 42-11-33 $\pm$ 12" $\qquad\qquad$ $m_z = 12" = \dfrac{12}{2 \cdot 10^5}$ rad

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$$G = V = \frac{1}{3} \cdot a^2 \cdot b \cdot \cos(z) = \frac{1}{3} \cdot 25.032^2 \cdot 37.659 \cdot \cos(\text{42-11-33}) = 5827.66 \approx 5828 \text{ m}^3$$

$$g_a = \frac{1}{3} \cdot 2 \cdot a \cdot b \cdot \cos(z) = \frac{2}{3} \cdot 25.032 \cdot 37.659 \cdot \cos(\text{42-11-33}) = 465.616416$$

$$g_b = \frac{1}{3} \cdot a^2 \cdot \cos(z) = \frac{1}{3} \cdot 25.032^2 \cdot \cos(\text{42-11-33}) = 154.748003$$

$$g_z = -\frac{1}{3} \cdot a^2 \cdot b \cdot \sin(z) = -\frac{1}{3} \cdot 25.032^2 \cdot 37.659 \cdot \sin(\text{42-11-33}) = -5282.805064$$

$$m_G = \sqrt{g_a^2 \cdot m_a^2 + g_b^2 \cdot m_b^2 + g_z^2 \cdot m_z^2} =$$

$$= \sqrt{465.616416^2 \cdot 0.035^2 + 154.748003^2 \cdot 0.2^2 + (-5282.805064)^2 \cdot \left(\frac{12}{2 \cdot 10^5}\right)^2} = 34.98 \approx \pm 35.0 \text{ m}^3$$

The volume and its mean error: 5828 m² $\pm$ 35.0 m².