1. Thin lenses

Lenses are transparent objects with two refracting surfaces whose central axes coincide. This common central axis is the central axis of the lens. If a lens causes light rays that are initially parallel to the central axis to converge to a focal point, it is called a converging lens and the point where the rays converge is the focal point of the lens. If however, rays parallel to the central axis diverge after refraction, the lens is called a diverging lens. The diverging rays seem to come from a common source, this is the (virtual) focal point of the diverging lens.

1.1. Ray tracing with thin lenses

The ray tracing approach can be used in the case of thin lenses as well to find the image of an object placed in from of the lens. The rules for tracing rays in the case of thin lenses

- Rays that are parallel to the central axis of the lens are refracted in such a way that they converge in the focal point of the lens.
- Rays that pass through the focal point on one side of the lens become parallel to the central axis after refraction.
- Rays that are directed toward the center of the lens will suffer no change to their direction.

According to the figures below, when an object is placed in from of a converging lens, outside the focal length, the lens produces an inverted, real image on the opposite side of the object. If the object is inside the focal length of the lens, a virtual image of the same orientation is produced on the object's side.

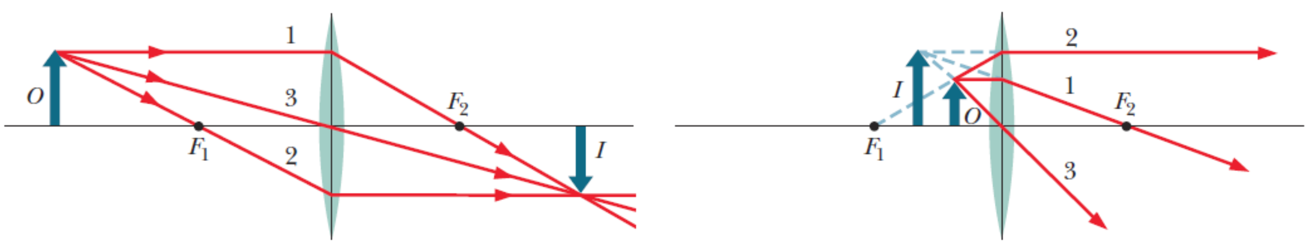


Figure 1. Ray tracing with a converging lens.

In case of the diverging lens, the image produced by the lens is always virtual, has the same orientation as the object and is located on the object's side.

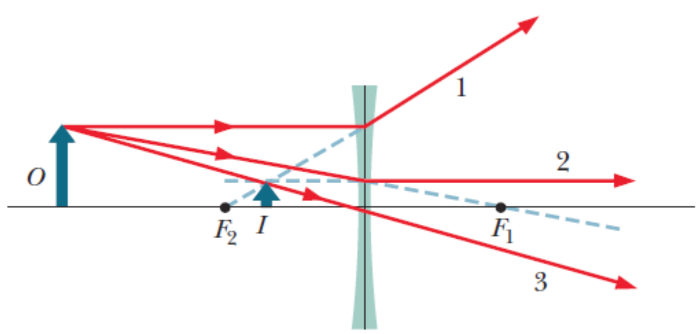


Figure 2. Ray tracing with a diverging lens.

1.2. Analytical image formation of thin lenses

If we consider a special case of lenses, the so-called thin lens, that is, a lens in which the thickest part is thin relative to the object distance p, the image distance i and the radii of curvature r_1 and r_2 of the two surfaces of the lens. Moreover, we only consider light rays that make small angels with the central axis. For such a lens and light rays, the focal length, the object distance and the image distance is related by

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q},$$

which is the same equation as in the case of spherical mirrors, however, in this case we call it the lens equation. The focal length can also be related to the radii of curvature:

$$\frac{1}{f} = (n-1) \cdot \left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

where r_1 is the radius of the surface close to the object, r_2 is the radius of the surface that is farther from the object and n is the index of refractivity of the lens. The equation is also called the lens maker's equation. To derive the equation, the following assumptions have to be made beforehand:

- We are dealing with light rays that are close to the principal axis (so called paraxial rays) and make a very small angle with it, resulting in small angles of incidence. Therefore, we can approximate the angles with their sine or tangent values:

  $$\sin(\alpha) = \alpha \quad \text{and} \quad \tan(\alpha) = \alpha$$

- The lens is assumed to be constructed of a single substance with refractive index of n and the medium in which the lens is immersed is air, with refractive index of 1.
- Because of the thin nature of the lens, we can approximate that the refractions inside the lens happen at the same distance from the principal axis.

Let's suppose a converging lens as shown below in Fig. 3(a). Point P is the object and Q is its image. A ray from object P reaches a lens in point A, it is refracted and passes to point D where it is refracted again and reaches the principal axis in point Q. C1 is the center of curvature for the right side of the lens and C2 is the same for the left side of the lens. Fig. 3(b) shows the first refraction at point A. Using Snell's law at point A gives

$$\sin(i_1) \cdot 1 = \sin(r_1) \cdot n \tag{1}$$

Applying the small angle approximation, (1) becomes

$$\mathrm{i}_1 = \mathrm{r}_1 \cdot n \tag{2}$$

In the triangle PAC_1 i_1 is an outer angle and can be written as the sum of the two inner angles:

$$\mathrm{i}_1 = \alpha_1 + \beta_1 \tag{3}$$

In the triangle ABC_1 r_1 is an inner angle and can be written as:

$$\mathrm{r}_1 = \beta_1 - \gamma \tag{4}$$

Substituting (3) and (4) into (2) gives

$$\alpha_1 + \beta_1 = n(\beta_1 - \gamma) \tag{5}$$

The ray that is refracted at point A naturally never reaches the principal axis as it will be refracted on the right side of the lens at point D. Looking at the refraction at point D in Fig. 3(c), Snell's law becomes

$$\sin(i_2) \cdot 1 = \sin(r_2) \cdot n \tag{6}$$

Applying the small angle approximation, this becomes

$$i_2 = r_2 \cdot n \tag{7}$$

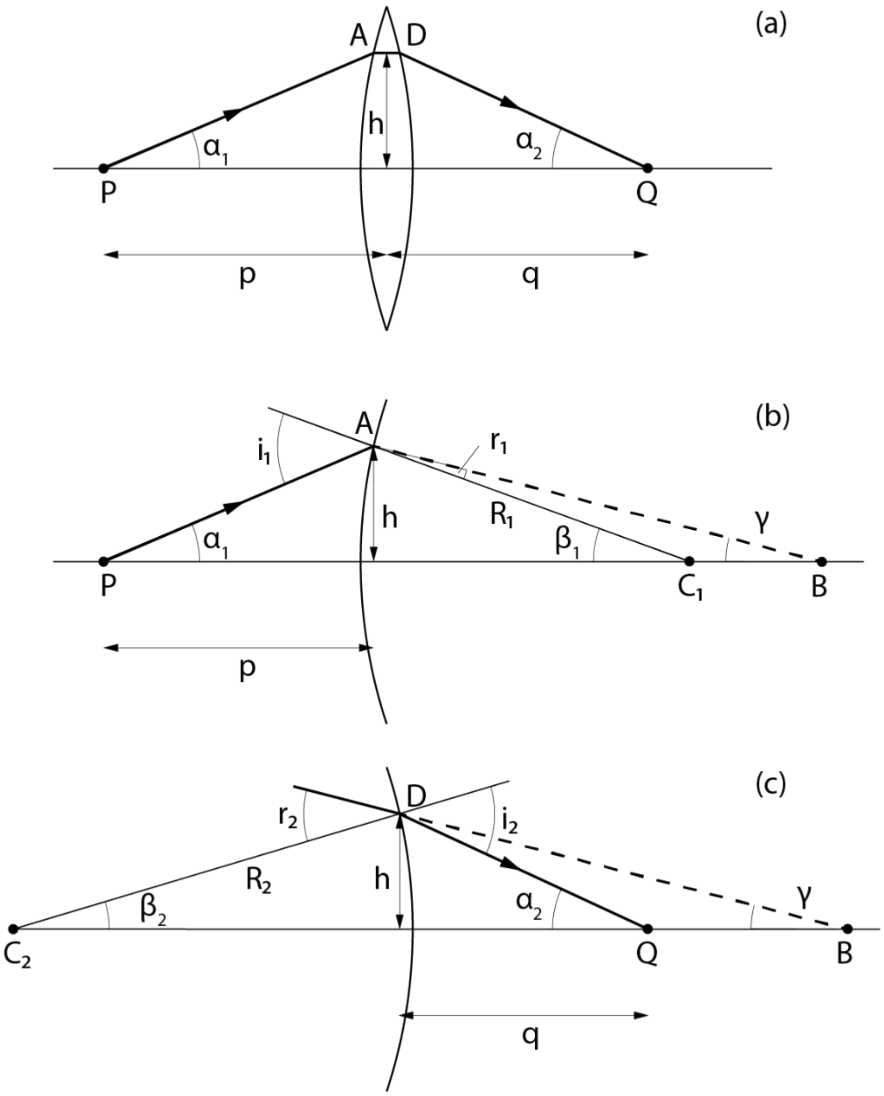


Figure 3. Derivation of the lens maker's equation.

From the triangle DQC_2:

$$i_2 = \alpha_2 + \beta_2 \tag{8}$$

From the triangle QBC_2:

$$r_1 = \beta_2 + \gamma \tag{9}$$

Substituting (8) and (9) into (7) gives

$$\alpha_2 + \beta_2 = n(\beta_2 + \gamma) \tag{10}$$

Using (5) and (10), we can eliminate γ:

$$\alpha_1 + \alpha_2 = (n-1)(\beta_1 + \beta_2) \tag{11}$$

Given the thin lens approximation, $\alpha_1 = h/p$, $\alpha_2 = h/q$, $\beta_1 = h/R_1$ and $\beta_2 = h/R_2$. Substituting these into (11):

$$\frac{1}{p} + \frac{1}{q} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{12}$$

Combining this we the lens equation, we get

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{13}$$

Equation (13) is also true for diverging lenses using the following sign conventions:

- The radius of curvature is positive for a convex surface and negative for a concave one (from the object's point of view).
- Light is assumed to come from left side.
- The object distance p is positive when the object is to the left of the lens.
- The image distance q is positive if the object is to the right of the lens.
- The focal length f is positive for a converging lens and negative for a diverging lense.

If the lens is surrounded by a medium other than air, instead of n, we use n/n_{medium} in the equation.

Example 1:

An object is placed in front of a thin symmetric lens, 20 cm from it. The lateral magnification of the lens is $m = -0.25$ and the index of refraction of the lens is 1.65. (a) Determine the type of image produced by the lens, the type of the lens, whether the object is inside or outside the focal point, the side of the lens on which the image appears and whether the image is inverted. (b) What are the two radii of curvature of the lens?

(a) Using the equation for the lateral magnification we can find that

$$q = -mp = 0.25p$$

As p is positive, i has to be positive as well, which means, that we have a real image and a converging lens (only it can produce a real image). In case of a converging lens, the object has to be outside the focal length and the image is inverted, as the sign of the lateral magnification is negative.

(b) First, we calculate the value of the image distance from the equation above:

$$q = -mp = 0.25 \cdot 20 = 5 \text{ cm}$$

The lens equation gives us the focal length:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i} = \frac{1}{20} + \frac{1}{5} = 4 \text{ cm}$$

Now we can use the lens maker's equation to find the radii of curvature. As the lens is symmetric, the two radii of curvature are the same.

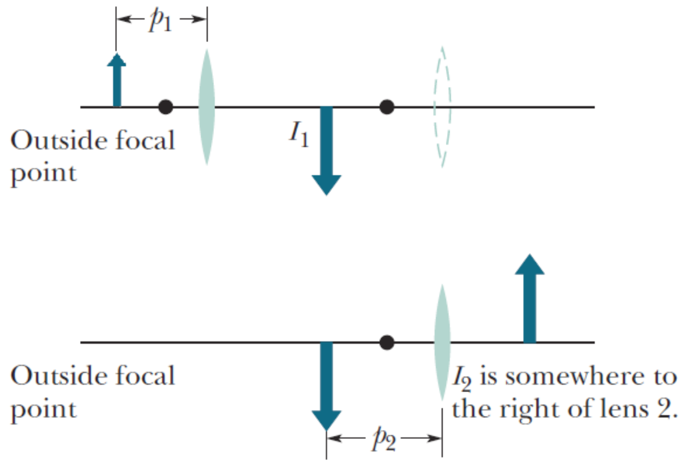
$$\frac{1}{f} = (n-1) \cdot \left(\frac{1}{r} + \frac{1}{r}\right) = (1.65 - 1) \cdot \frac{2}{r} \rightarrow r = 4 \cdot 2 \cdot (1.65 - 1) = 5.2 \text{ cm}$$

2. Two lens systems

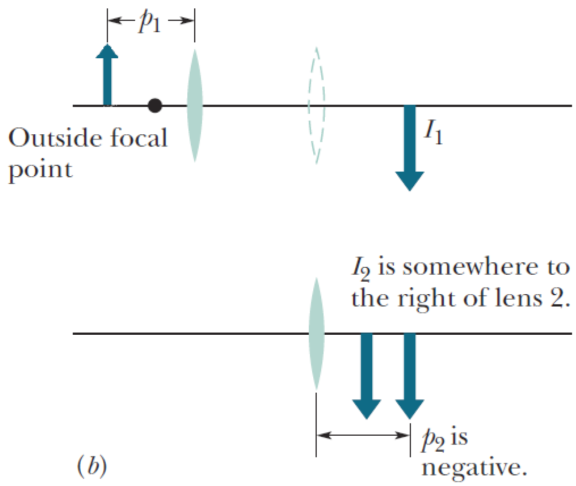
Consider an object sitting in from of two lenses whose central axes coincide. In this case, the ray tracing method can be a bit more challenging, but we can follow a simple two-step solution, demonstrated on the figures below.

Step 1. We neglect lens 2, use the lens equation to locate image I_1 produced by lens 1. We determine whether the image is on the left or right side of lens 1, whether it is real of virtual and whether it has the same orientation as the object.

Step 2. Neglecting lens 1, we treat I_1 as the object for lens 2. We use the lens equation to locate the image I_2 produced by lens 2 which will be the final image of the system. We determine whether it is located on the left or right side of the lens, whether it is virtual or real and whether it has the same orientation as the object.



This basically means that we treat the two-lens system as two single-lens calculations using the normal rules for a single lens (for deciding the signs of the different quantities). There is one exception though: whenever the image I_1 of the first lens is located on the right side of lens 2, the object distance p_2 becomes negative (see example below).



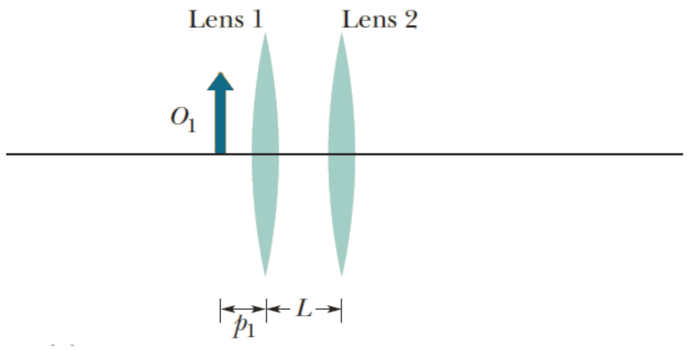
The same step-by-step approach can be used for any number of lenses. It can also be applied when a mirror is substituted for any of the lenses. The overall lateral magnification M of a system of lenses is product of the individual lateral magnifications of the lenses given by $m = -q/p$. Thus, for a two-lens system, we have

$$M = m_1 \cdot m_2$$

If M is positive, the orientation of the final image is the same as the (original) object, if M is negative, the image is inverted.

Example 2:

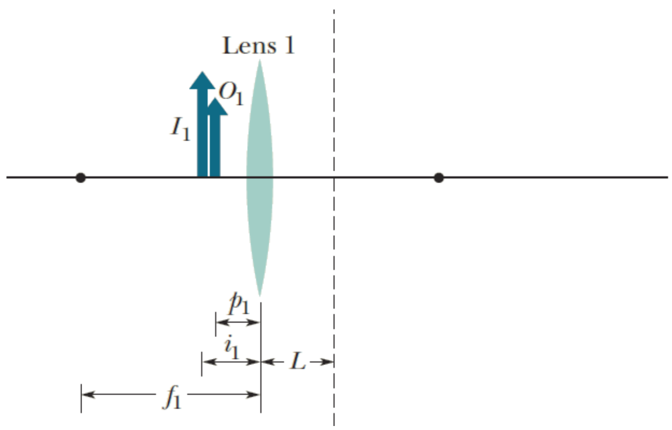
The figure below shows an object O_1 that is placed in front of two thin symmetrical lenses 1 and 2 with focal lengths $f_1 = +24$ cm and $f_2 = +9.0$ cm, respectively. The separation between the lenses is $L = 10$ cm. The object is $p_1 = 6.0$ cm from the first lens. Where does the system of lenses produce the image of the object?



First, we ignore lens 2 and deal with lens 1 only. The image I_1 of lens 1 can be located using the lens equation:

$$\frac{1}{f_1} = \frac{1}{p_1} + \frac{1}{i_1} \rightarrow \frac{1}{i_1} = \frac{1}{24} - \frac{1}{6} = -8.0\text{ cm}$$

This shows, that the image is virtual and is located 8.0 cm from lens 1. As the lens is a converging lens, the image is on the same side as the object.



Now we use the image I_1 as the object of the second lens. The object distance in this case is $p_2 = L + |i| =$ 18 cm. The lens equation:

$$\frac{1}{f_2} = \frac{1}{p_2} + \frac{1}{i_2} \rightarrow \frac{1}{i_2} = \frac{1}{9} - \frac{1}{18} = 18\text{ cm}$$

We can see, that the final, resulting image is a real image, located on the right side of lens 2, 18 cm from it.