Surveying II. – Adjustment of measurements and propagation of the mean error

Practice exercises

1. The coordinates of the vertices of a parcel were measured. Find the area of the parcel and round the result to integer square meters.

Point No.	E [m]	N [m]
2050	656 734.03	224 150.97
2051	656 695.47	223 964.68
2052	656 888.45	224 113.58
2053	656 627.37	224 076.18
2054	656 794.53	223 920.25
2055	656 943.37	223 923.44

First, we have to check if the points are numbered in the correct order. We can do this by drawing a small sketch of the layout of the points using their coordinates. It is recommended to create a new table with the points in the correct order. Please note that the row of the first point is added after the last row as it becomes the “next” point when we are computing the area of the last trapezoid.

The formula for finding the area by dividing it into trapezoids is the following:

$$A = \left| \frac{1}{2} \cdot \sum (E_{i+1} - E_i) \cdot (N_{i+1} + N_i) \right|$$

To simplify the computation, we can create three extra columns to the right of the coordinates and compute the area of each trapezoid:

Point No.	E [m]	N [m]	$E_{i+1} - E_i$	$\frac{1}{2}(N_{i+1} + N_i)$	A_i
2051	656 695.47	223 964.68	-68.09	224 020.43	-15254447.43
2053	656 627.37	224 076.18	106.66	224 113.58	23904178.08
2050	656 734.03	224 150.97	154.42	224 132.27	34610281.16
2052	656 888.45	224 113.58	54.92	224 018.51	12303768.35
2055	656 943.37	223 923.44	-148.85	223 921.84	-33330318.64
2054	656 794.53	223 920.25	-99.06	223 942.47	-22183964.87
2051	656 695.47	223 964.68	–	–	

The area of the parcel is the sum of the signed areas of the trapezoids: 49496.64 ≈ 49497 m^2.

2. In order to determine a distance, we took measurements with two different instruments. What is the adjusted value of the distance and its mean value in meter units with millimeter precision? What is the *a posteriori* mean error of the measurements in millimeter units with one decimal precision?

Measurement (L_i) [m]	**Mean error (μ_i) [mm]**	**Weights (w_i) [mm^{-2}]**	**Correction (v_i) [mm]**	**$w_i v_i$**	**$w_i v_i^2$**	**A posteriori mean error (m_i) [mm]**
82.176	6	*1*	*-6.30*	*-6.30*	*39.69*	*5,7*
82.163	6	*1*	*6.70*	*6.70*	*44.89*	*5,7*
82.162	6	*1*	*7.70*	*7.70*	*59.29*	*5,7*
82.171	2	*9*	*-1.30*	*-11.70*	*15.21*	*1,9*
82.170	2	*9*	*-0.30*	*-2.70*	*0.81*	*1,9*
82.169	2	*9*	*0.70*	*6.30*	*4.41*	*1,9*
$\sum$		*30*		*0,00*	*164,30*	

(The computed values are in italic in the table above.)

The first step is compute the weights of the measurements using the *a priori* mean errors of the instruments. We can choose the measurements with the higher mean error to have unit weight (1) and compute the weights of the measurements with the lower mean error:

$$w_i = w_0 \cdot \frac{\mu_0^2}{\mu_i^2}$$

where w_0 is the unit weight ($w_0 = 1$) and μ_0^2 is the variance of the measurements with the unit weight ($\mu_0^2 = 6^2 = 36$).

The adjusted value of the distance is the weighted average of the measurements:

$$\bar{L} = \frac{\sum w_i \cdot L_i}{\sum w_i} = 82.1697 \text{ m}$$

It is recommended to use at least one extra decimal during the computation to avoid rounding errors. The final rounded value of the adjusted measurement is of course 82.170 m. In further computations, we use the value which is not rounded.

The corrections are the differences between the adjusted value and the measurements:

$$v_i = \bar{L} - L_i$$

As a check, we compute the sum of the weighted corrections, which should be zero or close to zero.

$$\sum w_i \cdot v_i \approx 0$$

The mean error of the unit weight is the corrected standard deviation of the weighted corrections:

$$m_0 = \sqrt{\frac{1}{n-1} \cdot \sum (w_i \cdot v_i)^2} = 5.73 \text{ mm}$$

The mean error of the adjusted value and the a posteriori mean errors of the measurements can be computed using the mean error of the unit weight and the weights of the corresponding values. The weight of the adjusted value is the sum of all weights:

$$\bar{w} = \sum w_i = 30$$

The mean error of the adjusted value:

$$w_i = \frac{m_0^2}{m_i^2} \Rightarrow m_i = \frac{m_0}{\sqrt{w_i}} = \frac{5.73}{\sqrt{30}} = 1.04 \approx 1.0 \text{ mm}$$

The a posteriori mean errors of the measurements using the formula above:

$$w_i = 1 \Rightarrow m = 5.73 \approx 5.7 \text{ mm}$$

$$w_i = 9 \Rightarrow m = 1.91 \approx 1.9 \text{ mm}$$

3. We measured the length of the side of a cube shaped water tank ($a = 6.50$ m $\pm$ 2 cm). What is the least amount of water in cubic meters the tank can hold on a 99.7% or 3σ confidence level?

The volume of the tank:

$$V = a^3 = 274.625 \text{ m}^3$$

The partial derivative of the volume function with respect to a:

$$v_a = \frac{\partial V}{\partial a} = 3a^2 = 126.8 \text{ m}^2$$

The mean error of the volume:

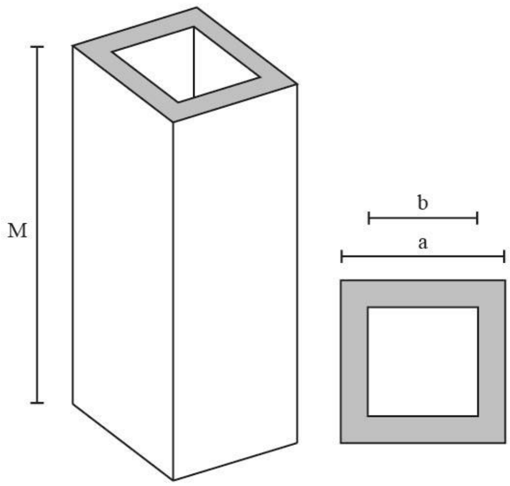
$$m_V = \pm\sqrt{v_a^2 \cdot m_a^2} = \pm v_a \cdot m_a = \pm 2.535 \text{ m}^3$$

The computed mean error corresponds to 1σ confidence level. The mean error for the 3σ confidence level is therefore $3 \cdot 2.535 = 7.605$ m^3.

The least amount of water the tank can hold on the given confidence level:

$$V_{\text{min}} = 274.625 - 7.605 = 267.02 \approx 267 \text{ m}^3$$

4. Our task is to determine the volume of a hollow concrete pillar with a maximum mean error of ± 1.5 m^3. In order to do this, we measured the sides of the base of the pillar and its height. What is the volume of the pillar? What can the maximum mean error of the height measurement be if we do not want to exceed the given threshold for the volume?



a = 5.00 m ± 3 mm

b = 2.50 m ± 5 mm

M = 13.4 m

The volume of the pillar:

$$V = a^2 \cdot M - b^2 \cdot M = 5^2 \cdot 13.4 - 2.5^2 \cdot 13.4 = 251.25 \text{ m}^3$$

The partial derivatives of the volume function:

$$\frac{\partial V}{\partial a} = 2aM = 134 \text{ m}^2$$

$$\frac{\partial V}{\partial b} = -2bM = -67 \text{ m}^2$$

$$\frac{\partial V}{\partial M} = a^2 - b^2 = 18.75 \text{ m}^2$$

The maximum acceptable mean error of the height measurement:

$$m_M = \pm \sqrt{\frac{m_V^2 - \left(\left(\frac{\partial V}{\partial a}\right)^2 \cdot m_a^2 + \left(\frac{\partial V}{\partial b}\right)^2 \cdot m_b^2\right)}{\left(\frac{\partial V}{\partial M}\right)^2}} = \pm 0.075 \text{ m}$$