The Earth's gravity field. Interpretation of elevations and heights.

1. Gravitational and centrifugal forces

1.1. Newton's law of universal gravitation

Newton's law of universal gravitation states that if the particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of the gravitational force acting on them is:

$$F_g = G \cdot \frac{m_1 \cdot m_2}{r^2}$$

where G is the universal gravitational constant $G = 6.673 \cdot 10^{-11}$ Nm^2/kg^2. In vector form, we can write

$$\overrightarrow{F_g} = -G \cdot \frac{m_1 \cdot m_2}{r^2} \cdot \overrightarrow{r_{12}}$$

where $\overrightarrow{r_{12}}$ denotes the unit vector between particles 1 and 2 and the minus sign means that the particles are attracted to each other. The magnitude of the forces acting on both particles is the same, while their direction is opposite. The force at particle 1 "points" to particle 2 while the force at particle 2 "points" to particle 1.

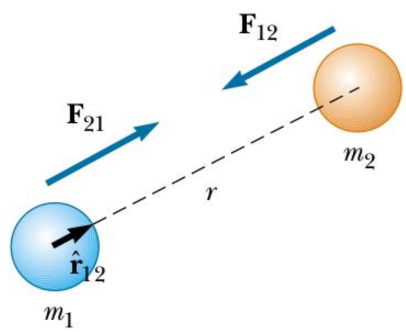

Figure 1. Gravitational force acting on two particles.

Analyzing the equation, we can see that the magnitude of the force is a function of the inverse of the distance between the particles, therefore the gravitational force decreases rapidly with increasing distance. Another important remark about the equation is that the gravitational force exerted by a finite-size, spherically symmetric mass on a particle which is outside this mass distribution is the same as if the entire mass was concentrated at the center.

1.2. Centrifugal force

During a uniform circular motion (just like the rotation of the Earth in our case) a particle moves with constant velocity v around an axis.

According to Newton's first law, the sum of the forces acting upon the particle is equal to the particle's mass times its acceleration. As the acceleration in case of the uniform circular motion is $a = v^2/r = \omega^2 r$, we can write that:

$$F_{cf} = m \cdot a = m \cdot \frac{v^2}{r} = m \cdot \omega^2 \cdot r$$

In case of the uniform circular motion, this force is called the centripetal (center-seeking) force and always points toward the axis of the circular motion. The reactive force which has the same magnitude but points away from the axis is called the centrifugal force.

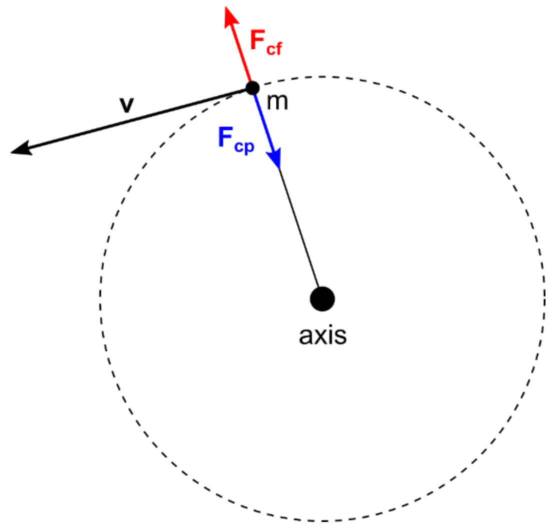

Figure 2. Uniform circular motion.

When it comes to being on the Earth's surface, our planet is rotating with a speed of $\sim 2\pi/24\text{h} \approx 0.262\ \text{rad/h}$ which is approximately 15° per hour. If we substitute the Earth with an ellipsoid that has a semi-major axis of 6378 km, we can calculate the centrifugal force acting on a particle with a mass of 1 kg at the equator:

$$F_{cf} = m \cdot \omega^2 \cdot r = 1\ \text{kg} \cdot \left(\frac{2\pi}{86400\ \text{s}}\right)^2 \cdot 6378000\ \text{m} = 0.03373\ \text{N}$$

Example 1: What is the magnitude of the forces acting on a person standing on the equator? The distance of the person from the Earth rotational axis is 6378 km and his mass is 75 kg. The mass of the Earth is $5.97 \cdot 10^{24}$ kg.

If the person is standing still, there are three forces acting on him in the non-inertial reference frame rotating with the Earth and the sum of these forces is zero:

$$\sum \vec{F} = \overrightarrow{F_g} + \vec{N} + \overrightarrow{F_{cf}} = 0$$

The magnitude of the forces:

$$F_g - N - F_{cf} = 0$$

$$F_g = G \cdot \frac{m_1 M_E}{r^2} = 6.67 \cdot 10^{-11} \text{Nm}^2/\text{kg}^2 \cdot \frac{75\ \text{kg} \cdot 5.97 \cdot 10^{24}\ \text{kg}}{(6378\ 000\ \text{m})^2} = 734.2\ \text{N}$$

$$F_{cf} = m \cdot \omega^2 \cdot r = 75\ \text{kg} \cdot \left(\frac{2\pi}{86400\ \text{s}}\right)^2 \cdot 6378\ 000\ \text{m} = 2.5\ \text{N}$$

$$N = F_g - F_{cf} = 734.2\ \text{N} - 2.5\ \text{N} = 731.7\ \text{N}$$

As we can see, the restoring force is not exactly equal to the gravitational force, as it also has to act against the centrifugal force.

2. Gravitational and centrifugal potential

2.1. Gravitational potential

The gravitational field is a central force field, meaning that the force acting on a particle inside the gravitational field is always along the radial line to a fixed center and the magnitude of the force only depends on the radial distance from the center.

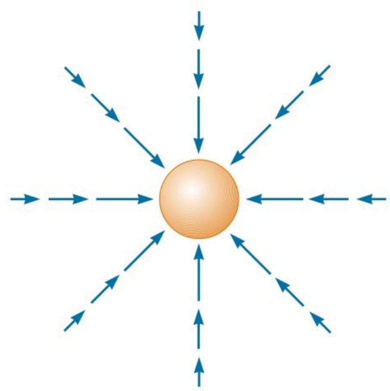

Figure 3. Central force field.

The gravitational force is also conservative, which means that the work done by the force on a particle moving inside the field only depends on the starting and the endpoint of the particle's movement. This definition also comes from the fact that the gravitational field is a central field as all central fields are conservative.

The gravitational potential energy of an object inside the gravitational field of the Earth tells us how much energy (or work) is needed to move the object from a defined reference point to its current position in the field. In case of the gravitational field of the real Earth, instead of a reference point, we define a reference surface and measure the potential values compared to that surface.

By definition, the work done by any force that is perpendicular to the displacement is zero. If we take a look at Figure 4 and divide the path between points P and Q into perpendicular and radial segments, we can see that as the particle moves between P and Q, work is only being done as we move along the radial segments of the path. This also means that the work done is the same on any path between P and Q, in other words, in a conservative field, the work done only matters on the starting and the endpoint of the motion.

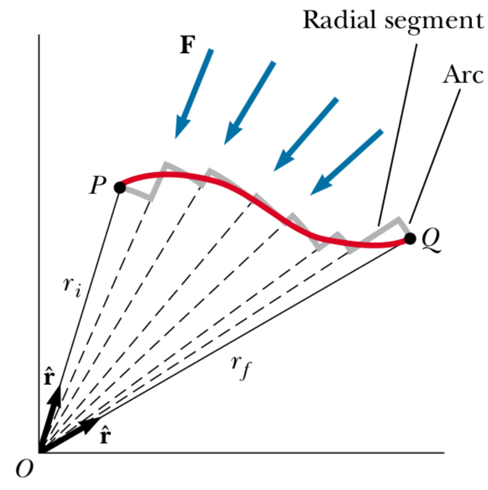

Figure 4. Conservative force.

The change in gravitational potential energy associated with a given displacement is defined as the negative work done by the gravitational force during that displacement. As the gravitational field is a conservative force field, the path of the displacement doesn't matter, we can simply compute the difference between the potential value at the final position (U_f) and the potential value between the initial position (U_i):

$$\Delta U = U_f - U_i$$

where U_f is the potential at the endpoint of the displacement and U_i is the initial potential at the starting point of the displacement. The formula for calculating the potential inside the Earth's gravitational field:

$$U_f - U_i = -\left(\frac{GM_E m}{r_f} - \frac{GM_E m}{r_i}\right) = -GM_E m\left(\frac{1}{r_f} - \frac{1}{f_i}\right)$$

where r_f and r_i are the vectors pointing to the starting and the endpoint of the displacement from the center of the field. The initial reference point where we measure the potential from can be completely arbitrary and it is customary to choose a point where the potential is zero (and $r_i = \infty$). With this customary substitution we get the final form of the formula for calculating the potential in the Earth's gravitational field for a mass m which is at distance r from the center:

$$U_g = -\frac{GM_E m}{r}$$

2.2. Centrifugal potential

In case of the centrifugal potential, we are interested in the work done by the centrifugal force on a rotating particle. If we look at Figure 5, we that the centrifugal force is always parallel with the radial line connecting the moving particle and the axis around which it rotates.

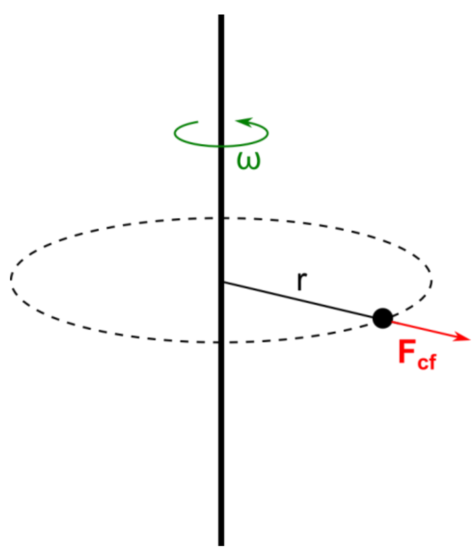

Figure 5. Centrifugal potential.

The formula for calculating the change in centrifugal potential energy:

$$U_{cf} = \frac{1}{2} \cdot \omega^2 \cdot r_f^2 - \frac{1}{2} \cdot \omega^2 \cdot r_i^2 = \frac{1}{2} \cdot \omega^2 \cdot \left(r_f^2 - r_i^2\right)$$

where ω is the angular velocity measured in rad/s, r_f is the distance of the object's final position from the axis and r_i is the distance of the object's initial position from the axis. If at our chosen initial reference position $r_i = 0$, the formula changes to

$$U_{cf} = \frac{1}{2} \cdot \omega^2 \cdot r_f^2 = \frac{1}{2} \cdot \omega^2 \cdot r^2$$

where r is the object's distance from the axis.

If we consider the centrifugal potential on the surface of the Earth, we can see that the distance from the rotational axis gets larger as we move from the poles to the equator and it reaches its maximum at the equator. This means, that the centrifugal potential is zero at the poles (where the distance from the rotational axis is zero) and is maximum at the equator (where the distance from the rotational axis is maximum as well).

3. The Earth's gravity field

In physical geodesy, we differentiate between gravity and gravitation. When we talk about gravitation, we mean the gravitational force or gravitational potential as depicted by Newton's law of universal gravitation. In case of gravity, we talk about the combined effects of the gravitational, centrifugal and tidal forces.

On the surface of the Earth, these three forces combined create the Earth's gravity field. In our further discussion, we omit the tidal effects (the gravitational pull of the Moon, other planets and the oceanic tides), but we note that when needed, these effect have to be compensated for as well. We can write the force of gravity as the sum of the gravitational and centrifugal forces:

$$F_G = F_g + F_{cf}$$

As mentioned before, the centrifugal force and potential reach their maximum at the equator. On the surface of the Earth, the force of gravity will always be the sum of the two forces which means that it only points to the center of mass of the Earth at the poles and at the equator (see Figure 6).

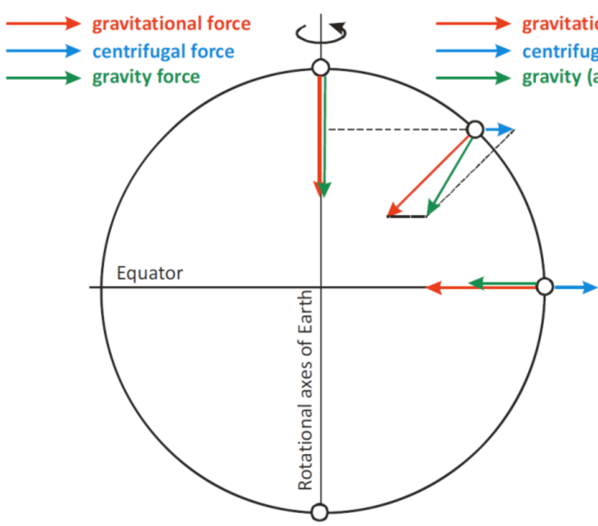

Figure 6. The force of gravity as the sum of the gravitational and the centrifugal forces.

The potential of the force of gravity can be calculated as the sum of the potentials of the two force fields comprising the field of gravity.

$$U_G = U_g + U_{cf} = \frac{1}{2}\omega^2 r^2 - \frac{G M_E m}{r}$$

Whenever we move an object in the Earth's gravity field and change its potential, we do work against the forces that create the gravity field. The amount of work done is equal to the displacement multiplied by the component of the force that is parallel with the displacement:

$$W = F \cdot s$$

As the centrifugal force act in the opposite direction of the gravitational force it means that the magnitude of the force of gravity (the sum of the two) will be smaller at the equator than at the poles. If we want to do the same amount of work on an object (change its energy by the same amount) at the equator as on an object at the poles, we have to take the object at the equator higher (increase the value of s in the equation above) as the force gravity acting on it is smaller than on the object at the poles.

We can imagine the potential of the Earth's gravity field as layers of equipotential surfaces. On each equipotential surface the value of the potential is the same and if we want to increase the potential, we have to move to a higher surface. As mentioned above, when we are at the equator, we have to travel a longer distance to reach the next equipotential surface as opposed to when we are at the pole. This means that the distance between the equipotential surfaces gets smaller as we approach the poles (see Figure 7).

The equipotential surfaces are what we call level or horizontal surfaces in everyday use as water always flows from the place with the higher potential to the place with the lower potential value. We define the vertical direction as the direction which is perpendicular to the level surface. The problem with the equipotential surfaces approaching each other at the poles, is that, as the vertical line (the plumb line) is always perpendicular to the horizontal, the actual vertical is not a straight line but rather a curve (see Figure 7).

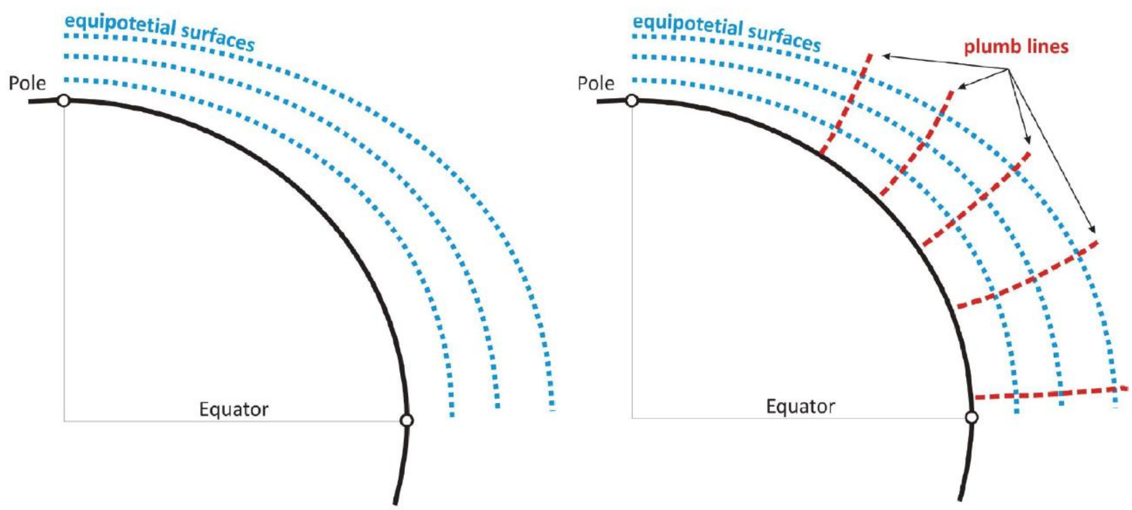

Figure 7. Equipotential surfaces of the Earth's gravity field and the plumb line.

4. Interpreting heights

As we mentioned before, in everyday use, we say that point A is located higher than point B if water flows from A to B. We have seen above that this is very hard to define geometrically as the equipotential surfaces are not parallel with each other. The correct definition of a point's height can only be given using its potential with respect to some sort of reference potential. This reference potential is the basis of the so-called vertical datum from which all the heights in a given country or region are measured.

The most notable vertical datum is the geoid which is also the mathematical shape of the Earth. The geoid is a special equipotential surface of the Earth's gravity field; it is defined as the surface that the oceans would take if only the effects of the Earth's gravity were present (no tides or winds).

Increasing a point's height means that we do work inside the gravity field. As the force of gravity always acts parallel with the plumb line (the "real", curved vertical) and points roughly toward the center of the gravity field (the mass center of the Earth), whenever we increase a point's height, we do work against this force. This work can be written as:

$$\Delta W = W_f - W_i = U_f - U_i = g \cdot \Delta h$$

where W_f is the amount of work done, the value of the potential, when the point we moved reached its final position, W_i is the point's initial potential, g is the gravitational acceleration and Δh is the distance between the initial and the final position of the point measured along the plumb line.

If we assume that the initial position of the point is on the reference surface, the geoid, then the potential at the final position will be equal to the force of gravity measured along the plumb line multiplied by the height measured along the plumb line. If the potential of one point is higher than another, the equipotential surface it is located on is farther from the geoid, which means that water flows from that point to the other point. In other words, the point with the higher potential is located higher in an everyday sense, but not necessarily in a geometrical sense.

Even though the ultimate definition of a point's height can only be given as the point's potential with respect to a vertical datum (the geoid for example), using the potential in everyday life or engineering work would be tedious and unintuitive. Therefore, we convert the potential values to a geometric quantity. This is done by dividing the potential of a point by some defined value. As seen above, this value has to be some force. For example if we could measure the value of the gravitational acceleration along the plumb line, we could write:

$$\Delta W = g \cdot \Delta h \rightarrow \Delta h = \frac{\Delta W}{g}$$

and get the geometrical height of the point measured along the plumb line. As the exact value of the gravitational acceleration cannot be measured along the plumb line, we have to make some kind of approximation. One approximation could be to use the average value of the gravitational acceleration along the plumb line. This definition would give the so-called orthometric height of the point. There are also other height definitions such as the dynamic or normal height. There can be considerable differences between the different height interpretations of a given point, so it is imperative to know the particular height system in our project or field of work.