Kinetic energy and work

1. Kinetic energy

The kinetic energy is associated with the state of motion of an object. The faster the object moves, the greater its kinetic energy. The kinetic energy of a stationary object is zero. The formula for calculating the kinetic energy is the following:

$$K = \frac{1}{2} \cdot m \cdot v^2$$

For example, an object with the mass of 3 kg moving at 2 m/s has a kinetic energy of 6 kg·m^2/s^2. The unit energy in the SI system is the joule (J). One J means exactly 1 kg·m^2/s^2.

Example 1:

In 1896 in Waco, Texas, William Crush parked two locomotives at opposite ends of a 6.4-km-long track, fired them up, tied their throttles open, and then allowed them to crash head-on at full speed in front of 30,000 spectators. Hundreds of people were hurt by flying debris; several were killed. Assuming each locomotive weighed $1.2 \cdot 10^6$ N and their acceleration was a constant 0.26 m/s^2 , what was the total kinetic energy of the two locomotives just before the collision?

First we need to calculate the velocity of the locomotives at the point of collision. In order to do this, we can use the two well-known equations of linear motion:

$$v = v_0 + a \cdot t$$

$$x = x_0 + v_0 \cdot t + \frac{1}{2} \cdot a \cdot t^2$$

where x and v are the current position and velocity of the object, x_0 and v_0 are the initial position and velocity of the object, a is the acceleration and t is the time passed. We can solve the first equation for t and substitute it into the second one, resulting in:

$$v^2 = v_0^2 + 2 \cdot a \cdot (x - x_0)$$

Substituting $v_0 = 0$ and $x - x_0 = 3200$ m (half of the track), this yields:

$$v^2 = 0 + 2 \cdot 0.26 \cdot 3200 = 40.8 \text{ m/s} = 147 \text{ km/h}$$

The mass of each locomotive can be found using their weight:

$$m = \frac{F_G}{g} = \frac{1.2 \cdot 10^6}{9.81} = 1.22 \cdot 10^5 \text{ kg}$$

The total kinetic energy of the two locomotives adds up and can be calculated using the formula above:

$$K = 2\left(\frac{1}{2} m v^2\right) = 1.22 \cdot 10^5 \cdot 40.8 = 2.0 \cdot 10^8 \text{ J}$$

The result shows that the collision had a massive amount of kinetic energy, akin to an exploding bomb.

2. Work and kinetic energy

When accelerating an object by applying a force to it, we increase its kinetic energy. If we try to stop an object by decelerating it, we decrease its kinetic energy. Whenever we change the kinetic energy of an object, we say that we transfer energy to the object or from the object. Acceleration means transferring energy to the object, while deceleration means transferring energy from the object. In all of these cases, it is said that the force applied to the object does work on it, or in other words, there is work being done on the object by the force. By convention, if the force transfers energy to the object, the work is positive, if it transfers energy from the object, the work is negative.

In the context of physics, whenever we say work, we do not mean the generic, common meaning of the word as in physical or mental labor, but rather the transfer of energy to an object or the transfer of energy from it.

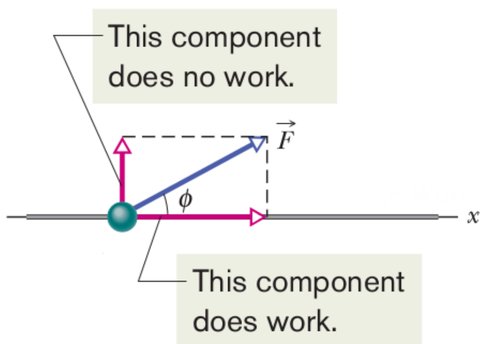


Figure 1. Work done on a particle by a force. (Image curtesy of [1].)

Consider a particle moving along a line (denoted by the x-axis) with an initial v_0 velocity. Apply a constant force to the particle (F) that creates a d displacement. . According to Newton's second law, the connection between the acceleration of the particle on the x-axis and the force:

$$F_x = m \cdot a_x$$

where F_x is the component of the force along the x-axis and a_x is the acceleration along the x-axis. After the force is applied to the object, it increases its velocity by some amount and the objects final velocity will be greater than the initial v_0 velocity. We can calculate the final velocity after d displacement using the following formula:

$$v^2 = v_0^2 + 2 \cdot a_x \cdot d$$

If we substitute F_x/m for a_x and rearrange the equation:

$$\frac{1}{2} \cdot m \cdot v^2 - \frac{1}{2} \cdot m \cdot v_0^2 = F_x \cdot d$$

The first term on the left side is the final kinetic energy of the object, while the second term is the initial kinetic energy. The difference between the two is the amount of work done by the force on the object:

$$W = F_x \cdot d$$

We can see, that only the component of the force that is parallel with the displacement does work on the object, so we can more generally write:

$$W = F \cdot d \cdot \cos(\phi)$$

where ϕ is the angle between the force and the displacement. This is equivalent to calculating the dot product of the vectors $\vec{F}$ and $\vec{d}$, if they are given in vector form:

$$W = \vec{F} \cdot \vec{d}$$

These equations are valid if the force or forces applied to the object are constant (neither their magnitude, nor their direction changes during the displacement) and the object is particle-like meaning that all its part accelerate in the same way and at the same time.

We can easily decide the sign for the value of work by taking a look at the value of the cosine inside the formula. If $\phi = 0$, $\cos(\phi) = 1$, which means that the our force doing work on the object is exactly parallel with the displacement. As we increase ϕ, the value of $\cos(\phi)$ decreases, but the sign is still + until we reach $\phi = 90°$ where $\cos(\phi) = 0$. As we go from 90° to 180°, where the force is pointing in the opposite direction of the displacement, the sign of $\cos(\phi)$ becomes negative. This means that the amount of work the force does on an object is positive, if the force has a component which points in the same direction as the displacement. It is negative, if the component that is parallel with the displacement points in the opposite direction as the displacement.

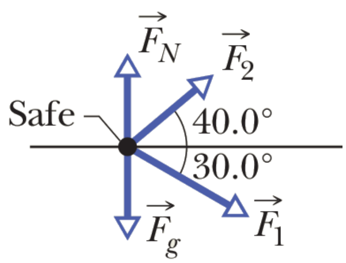
The SI unit for work is kg·m^2/s^2, or N·m.

Example 2:

Two workers are sliding an initially stationary 225 kg floor safe a displacement $\vec{d}$ of magnitude 8.50 m. The push $\vec{F_1}$ of worker 1 is 12.0 N at an angle of 30.0° downward from the horizontal; the pull $\vec{F_2}$ of worker 2 is 10.0 N at 40.0° above the horizontal. The magnitudes and directions of these forces do not change as the safe moves, and the floor and safe make frictionless contact.

(a) What is the net work done on the safe by forces $\vec{F_1}$ and $\vec{F_2}$ during the displacement $\vec{d}$?

(b) The safe is initially stationary. What is its speed v_f at the end of the 8.50 m displacement?



The net work W done on the safe by the two forces is the sum of the works they do individually. Only force components that are parallel to the displacement actually do work on the safe. This means, that we do not have to consider gravity and the restoring in the example. We can treat the safe as a particle and the forces are constant in both magnitude and directions, we can use the following formula to calculate the work of the individual forces:

$$W_1 = F_1 \cdot d \cdot \cos(\phi_1) = 12 \cdot 8.5 \cdot \cos(30°) = 88.33 \text{ J}$$

$$W_2 = F_2 \cdot d \cdot \cos(\phi_2) = 10 \cdot 8.5 \cdot \cos(40°) = 65.11 \text{ J}$$

$$\sum W = W_1 + W_2 = 153.44 \text{ J}$$

The speed of the safe changes because its kinetic energy changed when energy was transferred to it by $\vec{F_1}$ and $\vec{F_2}$. The change in the kinetic energy equals the amount of work done:

$$\sum W = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

As the initial speed $v_i = 0$, we can solve this for v_f:

$$v_f = \sqrt{\frac{2W}{m}} = \sqrt{2 \cdot \frac{153.44}{225}} = 1.17 \text{ m/s}$$

3. Work done by the gravitational force

Let us consider a particle moving upward in the vertical direction with speed v_0 and thus $K = 1/2\, mv_0^2$. As the particle is slowed down by the gravitational force, its kinetic energy decreases as the gravitational force does work on the particle. We can express the amount of work done by the gravitational force during a displacement d:

$$W = m \cdot g \cdot d \cdot cos(\phi)$$

If the object is rising, the gravitational force is directed opposite the displacement ($\phi = 180°, \cos(\phi) = -1$), therefore:

$$W = m \cdot g \cdot d \cdot \cos(180°) = -m \cdot g \cdot d$$

After the object has reached its final height, it starts falling back down, ϕ becomes 0° and thus:

$$W = m \cdot g \cdot d \cdot \cos(0°) = +m \cdot g \cdot d$$

3.1. Work done while lifting and lowering an object

Let's suppose we lift a particle-like object by applying an $\vec{F}$ force to it. During the upward motion, our applied force does positive work, while the gravitational force does negative work on the object. We can imagine this as our force transferring energy to the object, while the gravitational force is transferring energy from the object. The difference in the object's kinetic energy will be the difference between the kinetic energies of the final and the initial state, which is equal to the net work done by the forces applied:

$$\Delta K = K_f - K_i = W_a + W_g$$

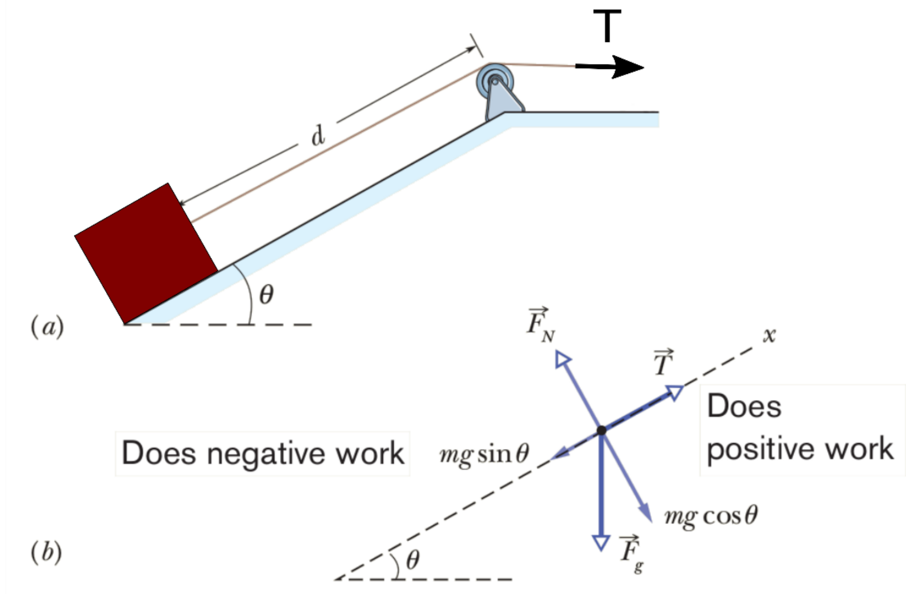
where W_a is the work of the applied force and W_g is the work of the gravitational force. If an object is stationary before and after the lift, it means that the difference in its kinetic energy is 0 (it started at 0 and it ended at 0). This means of course, that:

$$\Delta K = 0 \rightarrow W_a + W_g = 0 \rightarrow W_a = -W_g$$

From this, we can see, that the amount of work done by our applied force and the work done by the gravitational force are equal in magnitude but have different signs. The applied force transfers the same amount of energy to the target as the gravitational force transfers from it.

Example 3:

A rope pulls a 200 kg box up a slope which has $\Theta = 30°$ angle of inclination. The length of the slope is $d = 20$ m. The slope is frictionless. How much work is done by each force acting on the box?



Work by the normal force:

$$W_N = F_N \cdot d \cdot \cos(90°) = 0$$

Work by the gravitational force:

$$F_{gx} = mgsin(\Theta) = 200 \cdot 9.8 \cdot \sin(30°) = 981 \text{ N}$$

$$W_{gx} = F_{gx} \cdot d \cdot \cos(180°) = 981 \cdot 20 \cdot \cos(180°) = -19620 \text{ J} = -19.62 \text{ kJ}$$

As we have a negative sign for the work, it means that the gravitational force is removing energy from the box.

Work by the rope's force: as the box started stationary and ended in a stationary state as well, the difference in its kinetic energy was 0. We can use this fact to calculate the work of the rope's force:

$$\Delta K = \sum W = 0 \rightarrow W_N + W_g + W_T = 0 \rightarrow W_T = -W_N - W_g = 0 - (-19.62 \text{ kJ}) = +19.62 \text{ kJ}$$

We can also calculate this work by using Newton's second law. Assuming that the acceleration is zero along the slope, we can have an expression for the forces along the slope:

$$F_{\text{net},x} = m \cdot a_x$$

$$T - m \cdot g \cdot \sin(30°) = m \cdot 0$$

$$T = m \cdot g \cdot \sin(30°) = 981 \text{ N}$$

$$W_T = T \cdot d \cdot \cos(0°) = 981 \cdot 20 \cdot 1 = 19.62 \text{ kJ}$$