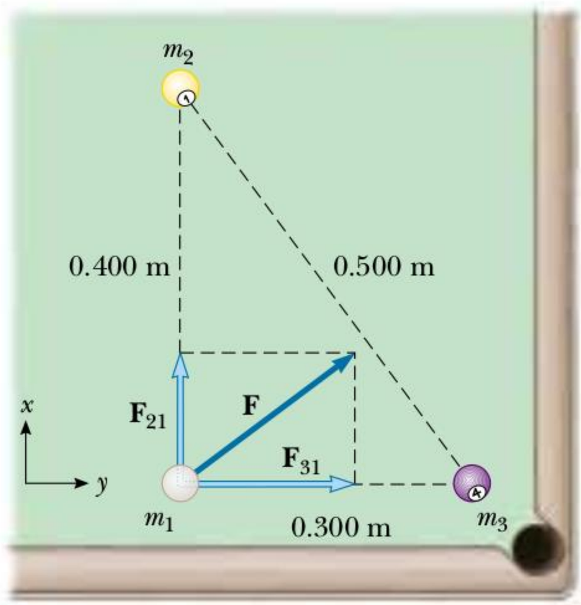
Exercises: gravitation, kinematic energy and work

1. We have 3 billiard balls on a table in the configuration shown on the figure below. Each ball has a mass of 0.3 kg. What is the magnitude of the net gravitational force and its angle from the y-axis, that is acting on the cue ball (white) from the two other balls?



Newton's law of universal gravitation can give us the gravitational forces between the cue ball and each of the other balls:

$$F_{21} = G \cdot \frac{m_2 \cdot m_1}{r_{21}^2} = 6.67 \cdot 10^{-11} \cdot \frac{0.3 \cdot 0.3}{0.4^2} \approx 3.75 \cdot 10^{-11}\ \text{N}$$

$$F_{31} = G \cdot \frac{m_3 \cdot m_1}{r_{31}^2} = 6.67 \cdot 10^{-11} \cdot \frac{0.3 \cdot 0.3}{0.3^2} \approx 6.67 \cdot 10^{-11}\ \text{N}$$

Using the figure below, we can calculate the magnitude of the net force as the length of the sum of the two vectors $\overrightarrow{F_{21}}$ and $\overrightarrow{F_{31}}$:

$$F = \sqrt{F_{21}^2 + F_{31}^2} = \sqrt{3.75^2 + 6.67^2} \cdot 10^{-11} \approx 7.65 \cdot 10^{-11}\ \text{N}$$

The tangent of the angle between the net force and the y-axis is given by

$$\tan(\phi) = \frac{F_{21}}{F_{31}} \rightarrow \phi = \arctan\left(\frac{F_{21}}{F_{31}}\right) = \arctan\left(\frac{3.75}{6.67} \cdot 10^{-11}\right) = 29.35°$$

2. The international space station is designed to operate at an altitude of 350 km. When completed, it will have a weight of $4.22 \cdot 10^6$ N, measured at the Earth's surface. What is its weight when it is orbit? (The radius of the Earth is 6378 km. Its mass is $6 \cdot 10^{24}$ kg.)

As the station is above the surface of the Earth, we expect its weight in orbit to be less than on the surface of the Earth. Its mass on the Earth is, calculated with 9.81 m/s^2 as the gravitational acceleration:

$$F_E = m_E g_E \rightarrow m_E = \frac{F_E}{g_E}$$

To calculate its weight in orbit, we need to know the gravitational acceleration in orbit. To calculate this, we use the mass and the radius of the Earth and the height of the orbit above the surface:

$$g_o = G \cdot \frac{M}{(R_E + h)^2} = 6.67 \cdot 10^{-11} \cdot \frac{6 \cdot 10^{24}}{(6378 \cdot 1000 + 350 \cdot 1000)^2} = 8.84 \text{ m/s}^2$$

In order to get the weight in orbit, we can take

$$F_o = m_E \cdot g_o = \frac{F_E}{g_E} \cdot g_o = F_E \cdot \frac{g_o}{g_E} = 4.22 \cdot 10^6 \cdot \frac{8.84}{9.81} = 3.80 \cdot 10^6 \text{ N}$$

3. A m_1 = 200-kg mass and a m_2 = 500-kg mass are separated by 0.400 m.

a) Find the net gravitational force exerted by these masses on a m_3 = 50.0-kg mass placed midway between them.

The forces acting between the masses:

$$F_{13} = G \cdot \frac{m_1 \cdot m_3}{r_{13}^2} = 6.67 \cdot 10^{-11} \cdot \frac{200 \cdot 50}{0.2^2} = 1.67 \cdot 10^{-5} \text{ N}$$

$$F_{23} = G \cdot \frac{m_2 \cdot m_3}{r_{23}^2} = 6.67 \cdot 10^{-11} \cdot \frac{500 \cdot 50}{0.2^2} = 4.17 \cdot 10^{-5} \text{ N}$$

If our positive direction points towards the heavier mass, then the net force is:

$$\sum F = F_{23} - F_{13} = (4.17 - 1.67) \cdot 10^{-5} = 2.50 \cdot 10^{-5} \text{ N}$$

b) At what position (other than infinitely remote ones) can the 50.0-kg mass be placed so as to experience a net force of zero?

The equation that has to be satisfied:

$$\sum F = F_{23} - F_{13} = 0$$

$$G \cdot \frac{m_2 \cdot m_3}{r_{23}^2} - G \cdot \frac{m_1 \cdot m_3}{r_{13}^2} = 0$$

$$\frac{r_{13}}{r_{23}} = \sqrt{\frac{m_2}{m_1}} = 1.581 \Rightarrow r_{13} = 1.581 \cdot r_{23}$$

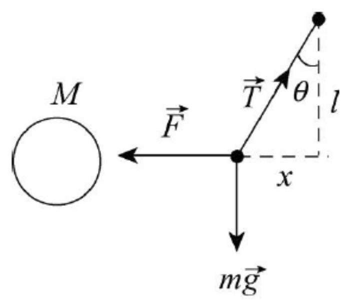
This means, that if the ratio of the distance between m_3 and m_1 and the distance between m_3 and m_2 is 1.581, the net force on m_3 is zero.

$$r_{13} + r_{23} = 0.400 \Rightarrow 1.581 \cdot r_{23} + r_{23} = 0.400$$

$$r_{23} = 0.155 \text{ m} \qquad r_{13} = 0.245 \text{ m}$$

4. A large body of mass can slightly affect the direction of the "down" as determined by a plumb line. Assume that we can model a large mass as a sphere of radius R = 2 km and density of 2600 kg/m^3. Assume also that we hang a l = 0.5 m plumb line at a distance of $3R$ from the sphere's center and such that the sphere pulls horizontally on the lower end. How far would the lower end move toward the sphere?

   The figure below shows the forces acting on the plumb line:



   We can calculate the mass of the sphere using its radius and density:

$$M = \rho V = \rho \cdot \left(\frac{4}{3}\pi R^3\right) = 2600 \cdot \left(\frac{4}{3}\pi \cdot 2000^3\right) = 8.71 \cdot 10^{13} \text{ kg}$$

   The force between the spherical mass and the plumb line

$$F = G \cdot \frac{M \cdot m}{r^2},$$

   where m is the mass of the plumb line, r is the distance between the plumb line and the sphere. We suppose, that at equilibrium, the line makes an angle Θ with the vertical and the net force acting on the line is zero.

   The sum of the forces acting in the horizontal and vertical directions:

$$\sum F_x = T \cdot \sin(\Theta) - F = 0$$

$$\sum F_y = T \cdot \cos(\Theta) - mg = 0$$

   Dividing the first equation with the second one gives:

$$\frac{\sin(\Theta)}{\cos(\Theta)} = \frac{F}{mg} \to \tan(\Theta) = G\frac{\frac{M \cdot m}{r^2}}{mg} \to \tan(\Theta) = G \cdot \frac{M}{r^2 g}$$

   The distance that the lower end of the plumb line moves is

$$x = l \cdot \tan(\Theta) = l \cdot G \cdot \frac{M}{r^2 g} = 0.5 \cdot 6.67 \cdot 10^{-11} \cdot \frac{8.71 \cdot 10^{13}}{6000^2 \cdot 9.81} = 8.2 \cdot 10^{-6} \text{ m}$$

5. A person pushes a crate with a total mass of 14 kg across a concrete floor with a constant horizontal force F of 40 N in a straight line. The displacement is 0.5 m and the speed of the crate decreases from $v_0 = 0.6\ \mathrm{m/s}$ to $v = 0.2\ \mathrm{m/s}$.

   (a) How much work is done by F and on what system does it do work?

   As the force F is constant and we know the displacement as well, we can use the simple formula for calculating the work done by it:

$$W = Fd\cos(\phi) = 40 \cdot 0.5 \cdot \cos 0° = 20\ \mathrm{J}\,.$$

   To determine the system on which the work is done, we have to consider the energy changes. As the speed of the crate changes, there is certainly a change in the crate's kinetic energy. Also, as the direction of F and the displacement coincide, if there were no friction, F should be accelerating the crate. However, the crate is slowing, so this means that there is friction and therefore a change in thermal energy between the crate and the floor. From these, we can deduce that F does work on the crate–floor system as all the energy changes occur in that system.

   (b) What is the increase ΔE_{th} in the thermal energy of the crate–floor system?

   Previously we determined that in case of friction, an external force on the system causes changes in the mechanical energy and the thermal energy:

$$W = \Delta E_{\mathrm{mec}} + \Delta E_{\mathrm{th}}\,.$$

   We already calculated the value of the work done by the external force (20 J). As the height of the crate above a reference level doesn't change, there is no change in the gravitational potential, so ΔE_{mec} only covers the change in kinetic energy:

$$\Delta E_{\mathrm{mec}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{1}{2} \cdot 14 \cdot (0.2^2 - 0.6^2) = -2.24\ \mathrm{J}\,.$$

   Now, we can use the equation above to find the change in thermal energy:

$$\Delta E_{\mathrm{th}} = W - \Delta E_{\mathrm{mec}} = 20 - (-2.24) = 22.24\ \mathrm{J}\,.$$