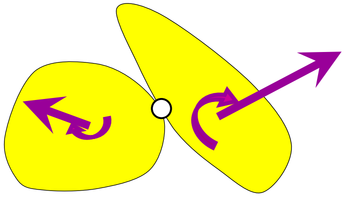
STATE VARIABLES OF DISCRETE SYSTEMS

→ Geometry
→ Stress tensors
→ Strain tensors

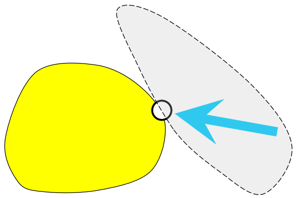
BASIC ASSUMPTIONS



Rigid elements: translate; rotate;

own deformations are small and

restricted to the neighborhood of contacts



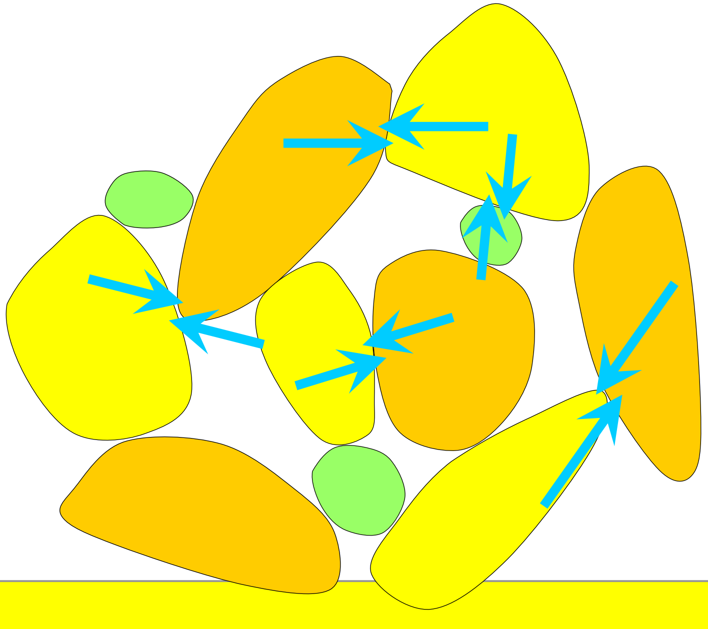
Contacts: small, point-like;

transmit concentrated forces

can be generalized!!!

[large contact surfaces;

contact moments etc.]



Averaged geometrical characteristics;

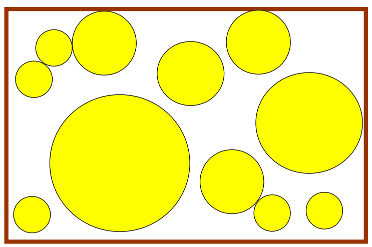
Stress tensors:

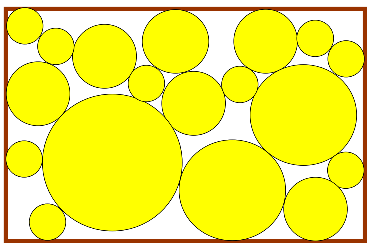
this presentation

GEOMETRICAL CHARACTERISTICS

1. Porosity
2. Coordination number
3. Fabric tensor

Porosity:

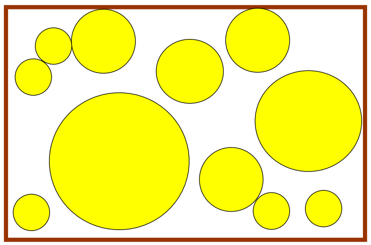


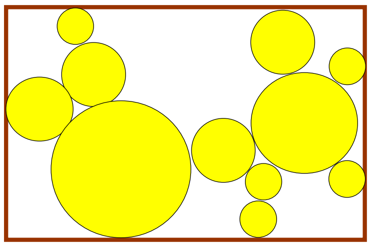


$$\rho = \frac{empty\ volume}{gross\ volume}$$

$$N_c = \frac{2M_{e\&e} + M_{e\&w}}{N_e}$$

Coordination number:

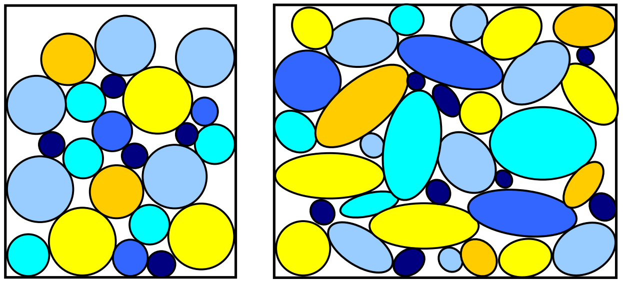


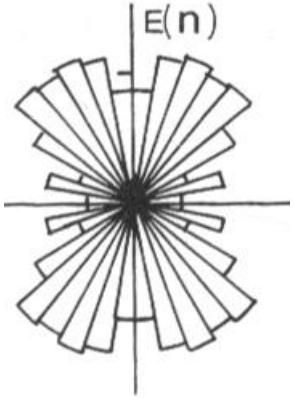


N_c : *average number of contact per element*

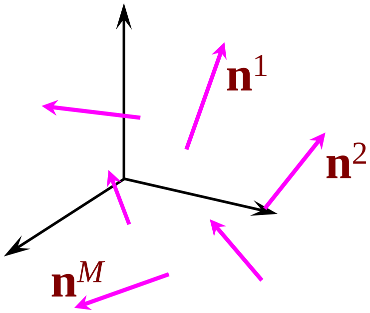
GEOMETRICAL CHARACTERISTICS

Characterize anisotropy:





Fabric tensors



Given: $\mathbf{n}^1$; $\mathbf{n}^2$; ... $\mathbf{n}^M$ unit vectors (2D, 3D, ...)

Directional distribution?

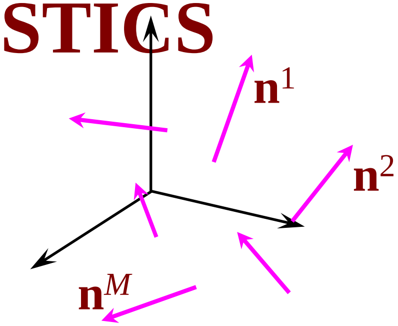
e.g. is there any preferred orientation?

e.g. how it changes during a loading process?

2nd order fabric tensor: (Satake, 1978)

$$\phi_{ij} = \frac{1}{M}\sum_{c=1}^{M} n_i^c n_j^c$$

GEOMETRICAL CHARACTERISTICS



Given: $\mathbf{n}^1$; $\mathbf{n}^2$; ... $\mathbf{n}^M$ unit vectors (2D, 3D, ...)

2nd order fabric tensor: (Satake, 1978)

$$\phi_{ij} = \frac{1}{M}\sum_{c=1}^{M} n_i^c n_j^c$$

→ characteristics: positive definit
symmetric (⇒ real eigenvalues, ⊥ eigenvectors)
trace = 1
n and –**n** : gives the same result

→ after determining its eigenvalues and eigenvectors:
differences between eigenvalues: shows the strength of anisotropy
eigenvector belonging to larges eigenvalue: preferred orientation of the vector set

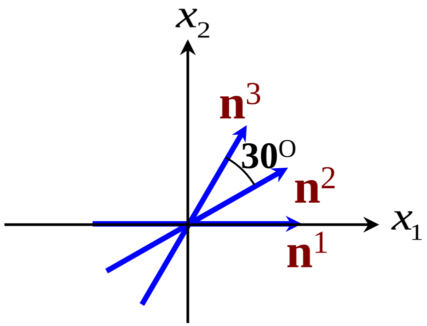
→ Visualization: „fabric ellipsoid” (in 2D: „fabric ellipse”)

GEOMETRICAL CHARACTERISTICS

2nd order fabric tensor: (Satake, 1978)

$$\phi_{ij} = \frac{1}{M}\sum_{c=1}^{M} n_i^c n_j^c$$

Example:



$$\mathbf{n}^1 = \begin{bmatrix} +1 \\ 0 \end{bmatrix}; \quad \mathbf{n}^2 = \begin{bmatrix} +0,866 \\ +0,5 \end{bmatrix}; \quad \mathbf{n}^3 = \begin{bmatrix} +0,5 \\ +0,866 \end{bmatrix};$$

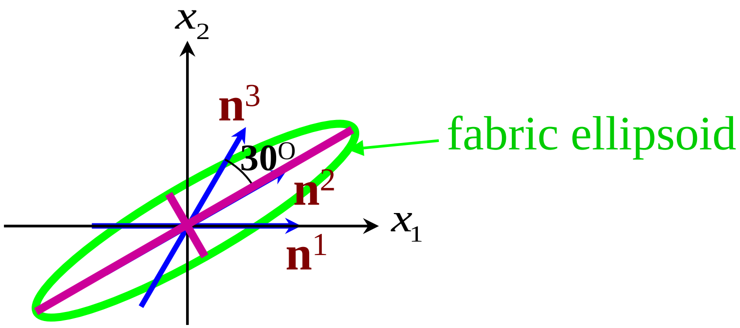
$$\mathbf{n}^1 \circ \mathbf{n}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{n}^2 \circ \mathbf{n}^2 = \begin{bmatrix} 0,75 & 0,433 \\ 0,433 & 0,25 \end{bmatrix}; \quad \mathbf{n}^3 \circ \mathbf{n}^3 = \begin{bmatrix} 0,25 & 0,433 \\ 0,433 & 0,75 \end{bmatrix};$$

$$\boldsymbol{\varphi} = \begin{bmatrix} 0,6667 & 0,2887 \\ 0,2887 & 0,3333 \end{bmatrix}$$

Eigenvalues, eigenvectors:

$$0,8333 \rightarrow \begin{bmatrix} 0,866 \\ 0,5 \end{bmatrix}$$

$$0,1667 \rightarrow \begin{bmatrix} -0,5 \\ 0,866 \end{bmatrix}$$

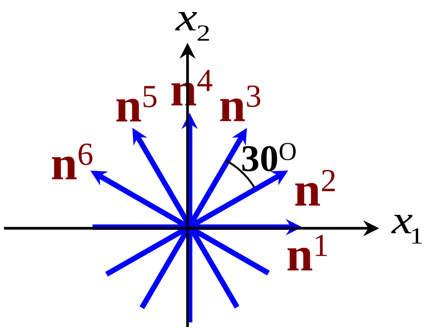


GEOMETRICAL CHARACTERISTICS

2nd order fabric tensor: (Satake, 1978)

$$\phi_{ij} = \frac{1}{M}\sum_{c=1}^{M} n_i^c n_j^c$$

Example:



$$\mathbf{n}^1 = \begin{bmatrix} +1 \\ 0 \end{bmatrix};\quad \mathbf{n}^2 = \begin{bmatrix} +0,866 \\ +0,5 \end{bmatrix};\quad \mathbf{n}^3 = \begin{bmatrix} +0,5 \\ +0,866 \end{bmatrix};$$

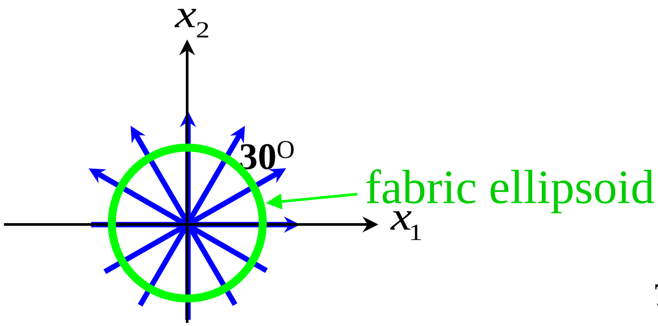
$$\mathbf{n}^4 = \begin{bmatrix} 0 \\ +1 \end{bmatrix};\quad \mathbf{n}^5 = \begin{bmatrix} -0,5 \\ +0,866 \end{bmatrix};\quad \mathbf{n}^6 = \begin{bmatrix} -0,866 \\ +0,5 \end{bmatrix};$$

$$\mathbf{n}^1 \circ \mathbf{n}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\quad \mathbf{n}^2 \circ \mathbf{n}^2 = \begin{bmatrix} 0,75 & 0,433 \\ 0,433 & 0,25 \end{bmatrix};\quad \mathbf{n}^3 \circ \mathbf{n}^3 = \begin{bmatrix} 0,25 & 0,433 \\ 0,433 & 0,75 \end{bmatrix};$$

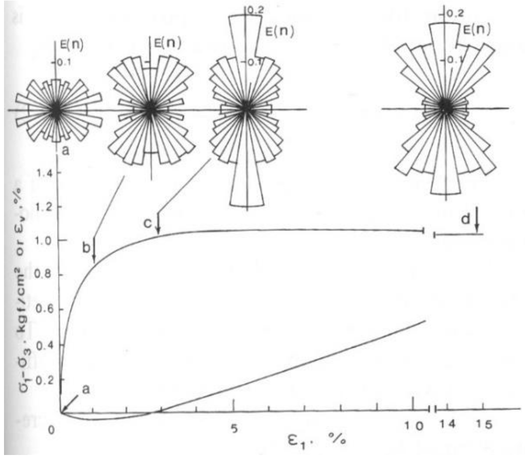
$$\mathbf{n}^4 \circ \mathbf{n}^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\quad \mathbf{n}^5 \circ \mathbf{n}^5 = \begin{bmatrix} 0,25 & -0,433 \\ -0,433 & 0,75 \end{bmatrix};\quad \mathbf{n}^6 \circ \mathbf{n}^6 = \begin{bmatrix} 0,75 & -0,433 \\ -0,433 & 0,25 \end{bmatrix};$$

$$\boldsymbol{\varphi} = \begin{bmatrix} 0,5 & 0 \\ 0 & 0,5 \end{bmatrix}$$

Eigenvalues: 0,5 ; 0,5 →



GEOMETRICAL CHARACTERISTICS



Application of fabric tensors

n can be: e.g. contact normals;
directions of voids;
longest axis of grains etc.

- diagnostics, follow the change of anisotropy during loading procedures
- theoretical modelling: state variables involved in constitutive relations

(e.g. Nemat-Nasser et al, 2002)

Higher order fabric tensors:

e.g. 4th order fabric tensor: $\phi_{ijhl} = \frac{1}{M} \sum_{c=1}^{M} n_i^c n_j^c n_h^c n_l^c$

Weighted fabric tensors:

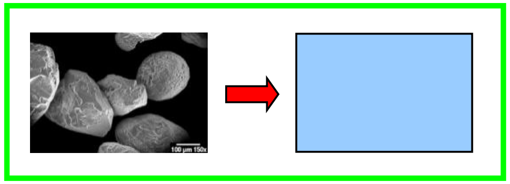
e.g. weighted with contact normal force magnitude: $\phi_{ij}^N = \frac{1}{M} \sum_{c=1}^{M} N^c n_i^c n_j^c$

e.g. weighted with the length of branch vector: $\phi_{ij}^l = \frac{1}{M} \sum_{c=1}^{M} l^c n_i^c n_j^c$

STRESS TENSOR

→ Repetition: Stress tensor in continuum mechanics

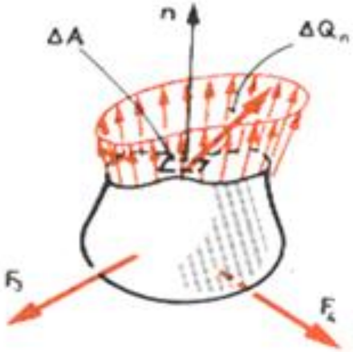
→ Microstructural stress tensors based on equivalent continua



<u>Stress tensor in continuum mechanics:</u>

stress vector belonging to a point P and plane $\mathbf{n}$:

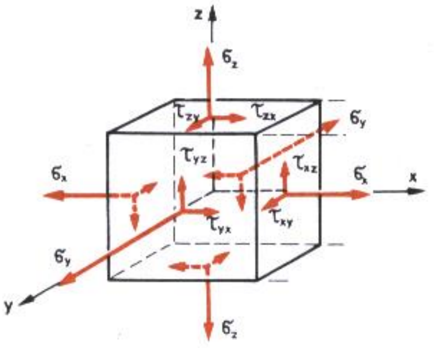
$$\mathbf{p}_n = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{Q}_n}{\Delta A}$$



$$p_i = \lim_{\Delta A \to 0} \frac{\Delta Q_i}{\Delta A}$$

stress tensor belonging to a point P :

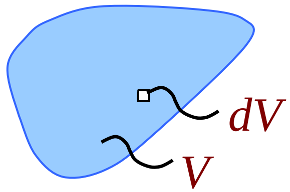
gives for any $\mathbf{n}$: $\boldsymbol{\sigma}\mathbf{n} = \mathbf{p}_n$

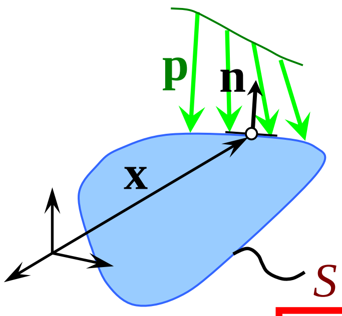


$$\sigma_{ij} n_j = p_i$$

STRESS TENSOR

Average stress tensor in a finite size domain:

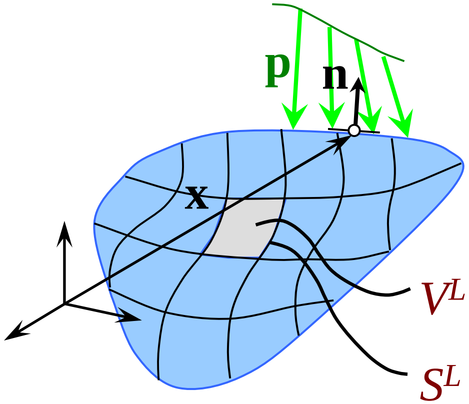




$$\bar{\sigma}_{ij} = \frac{1}{V} \oint_{(V)} \sigma_{ij} dV \quad \Rightarrow \quad \bar{\sigma}_{ij} = \frac{1}{V} \oint_{(S)} x_i (p_j dS)$$

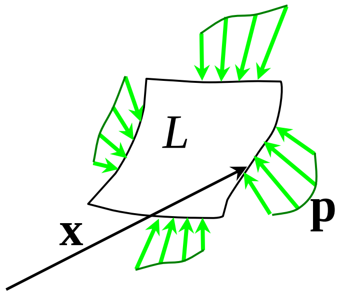
(Gauss-Osztrogradskij theorem)

Finite domain subdivided into finite cells:



L-th cell:

$$\bar{\sigma}_{ij}^{L} = \frac{1}{V^L} \oint_{(S^L)} x_i p_j dS$$

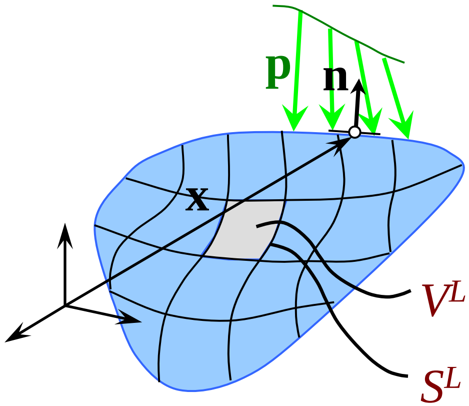


For the whole domain:

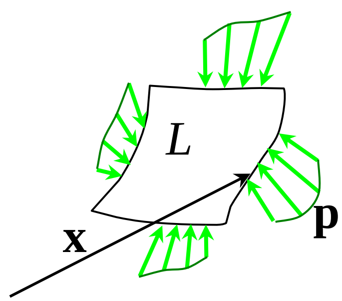
$$\bar{\sigma}_{ij} = \frac{1}{V} \sum_{(L)} \left(\sigma_{ij}^{L} V^L \right) = \frac{1}{V} \sum_{(L)} \left(\oint_{(S^L)} x_i p_j dS \right) = \frac{1}{V} \oint_{(S)} x_i p_j dS$$

STRESS TENSOR

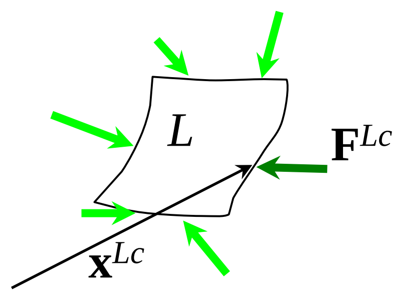
Continuous domain subdivided into finite cells, with concentrated forces transmitted between the cells:



L-th cell:



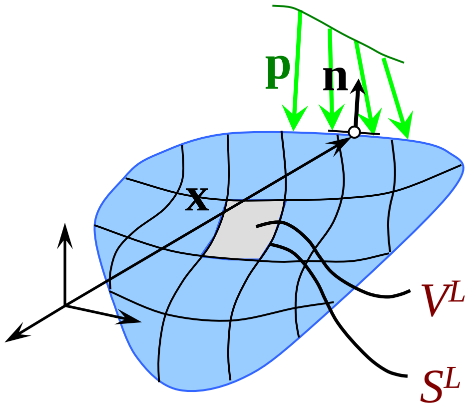
$$\bar{\sigma}_{ij}^{L} = \frac{1}{V^{L}} \oint_{(S^{L})} x_i (p_j dS)$$



$$\bar{\sigma}_{ij}^{L} = \frac{1}{V^{L}} \sum_{(Lc \in S^{L})} x_i^{Lc} F_j^{Lc}$$

STRESS TENSOR

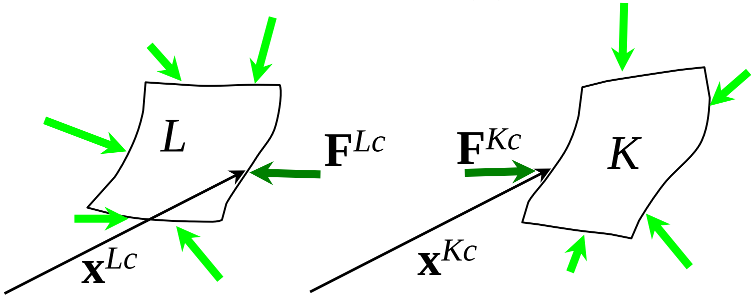
Continuous domain subdivided into finite cells, with concentrated forces transmitted between the cells:



L-th cell: $$\bar{\sigma}_{ij}^{L} = \frac{1}{V^{L}} \sum_{(Lc \in S^{L})} x_i^{Lc} F_j^{Lc}$$

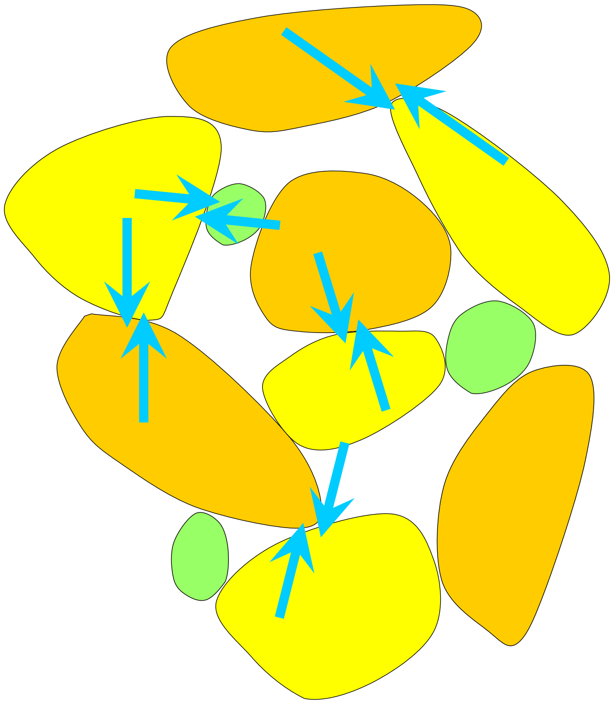
For the whole domain:

$$\bar{\sigma}_{ij} = \frac{1}{V} \sum_{(L)} \left(\sigma_{ij}^{L} V^{L} \right) = \frac{1}{V} \sum_{(L)} \left(\sum_{(Lc \in S^{L})} x_i^{Lc} F_j^{Lc} \right)$$



STRESS TENSOR

Discrete system:



Weber (1966)

Rothenburg et al (1981)

Christoffersen et al (1981)

Bagi (1996, 1999)

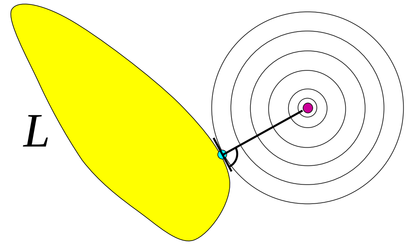
...

Aim: to derive a stress tensor from the contact forces between the elements

STRESS TENSOR

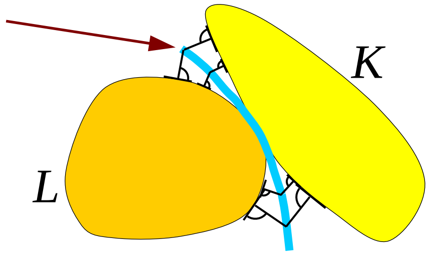
Discrete system:

The equivalent continuum: „material cell system”

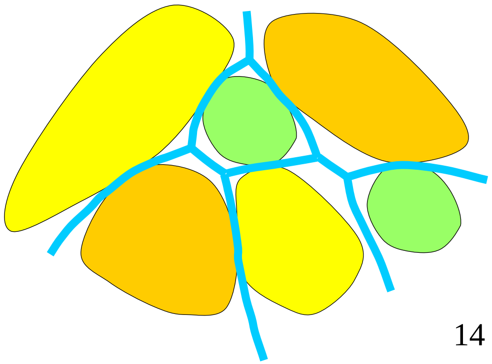


1. Definition of the distance of a point from element L :

2. Those points that have equal distance from elements L and K :



3. Assign every point of the space to that element being at shortest distance from:

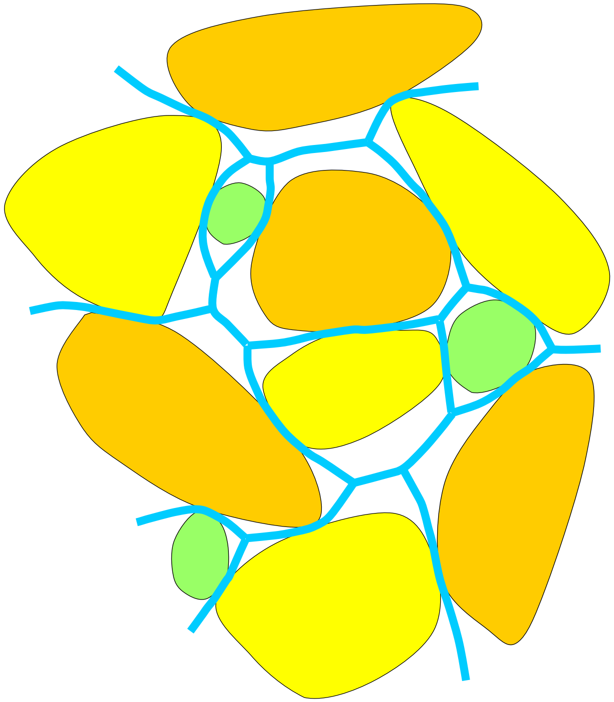


⇒ „material cell system”

STRESS TENSOR

Discrete system:

The equivalent continuum: „material cell system”



Characteristics:

→ every element is in exactly one cell,

every cell contains exactly one element

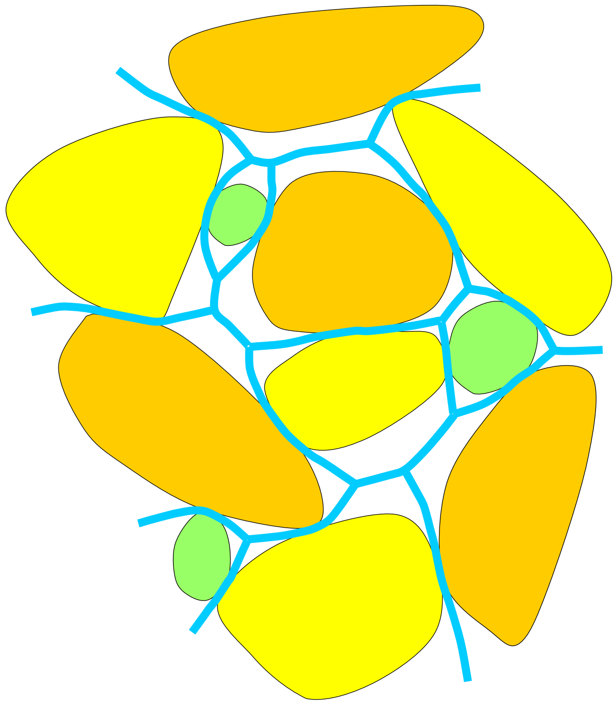
→ common face between neighbouring cells:

contains the contact point of the two elements

STRESS TENSOR

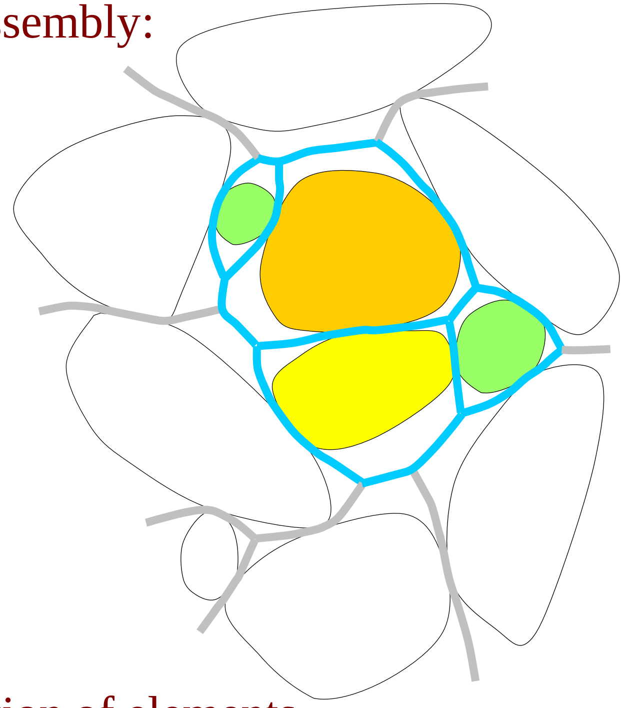
<u>Discrete system:</u>

The equivalent continuum: „material cell system”



4. Finite sub-assembly:

e.g.:



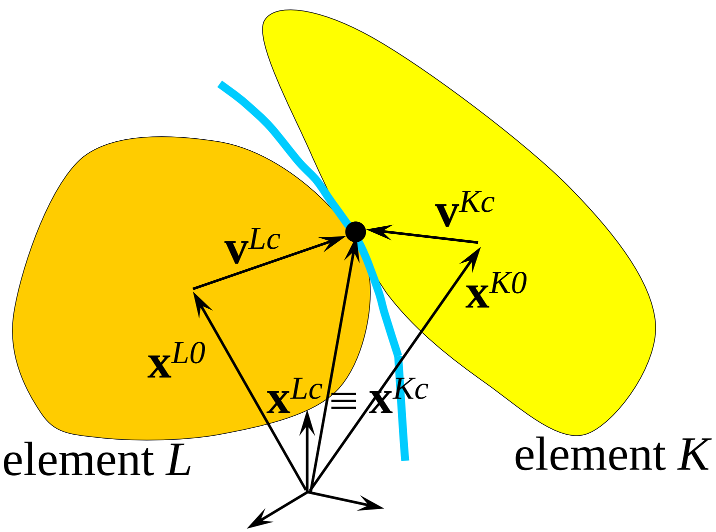
any collection of elements

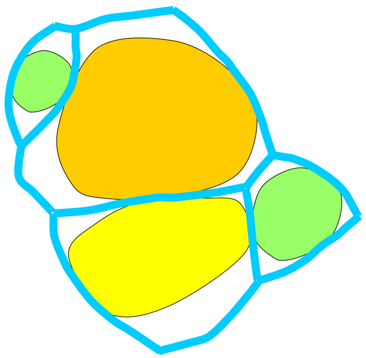
with finite material cell

STRESS TENSOR

Discrete system:

The equivalent continuum: „material cell system”





$$\sum_{(Lc \in S^L)} x_i^{L0} F_j^{Lc} = 0\,; \quad \text{remains:} \; \sum_{(Lc \in S^L)} v_i^{Lc} F_j^{Lc}$$

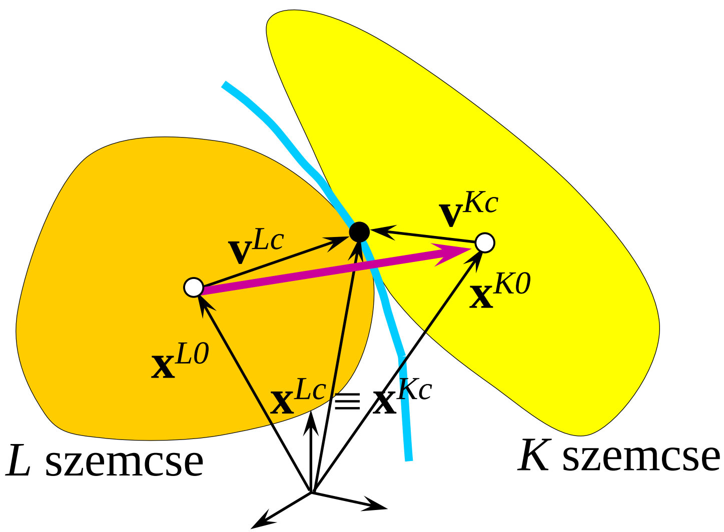
$$\bar{\sigma}_{ij} = \frac{1}{V} \sum_{(L)} \left(\sigma_{ij}^L V^L \right) = \frac{1}{V} \sum_{(L)} \left(\sum_{(Lc \in S^L)} x_i^{Lc} F_j^{Lc} \right) = \frac{1}{V} \sum_{(L)} \left(\sum_{(Lc \in S^L)} (x_i^{L0} + v_i^{Lc}) F_j^{Lc} \right) =$$

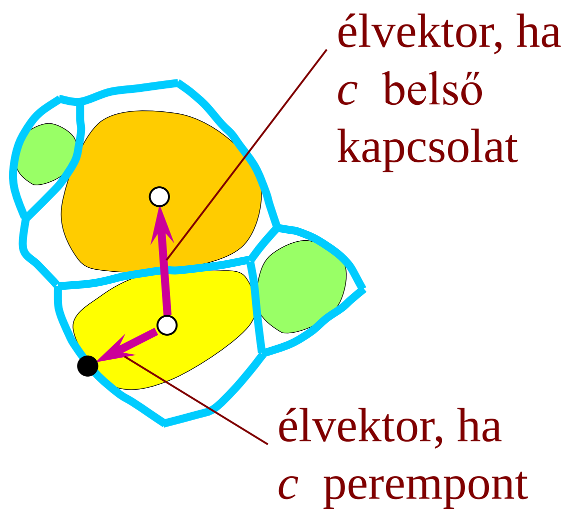
$$= \frac{1}{V} \sum_{(L)} \left(\sum_{(Lc \in S^L)} v_i^{Lc} F_j^{Lc} \right) = \frac{1}{V} \left(\sum_{(c\ \text{inside})} (v_i^{Lc} - v_i^{Kc}) F_j^{Lc} + \sum_{(c\ \text{boundary})} v_i^{Lc} F_j^{Lc} \right)$$

STRESS TENSOR

Discrete system:

The equivalent continuum: „material cell system”





$$\bar{\sigma}_{ij} = \frac{1}{V}\sum_{(L)}\left(\sum_{(Lc\in S^L)} v_i^{Lc} F_j^{Lc}\right) = \frac{1}{V}\left(\sum_{(c\ \text{inside})}\left(v_i^{Lc}-v_i^{Kc}\right)F_j^{Lc} + \sum_{(c\ \text{boundary})} v_i^{Lc} F_j^{Lc}\right) =$$

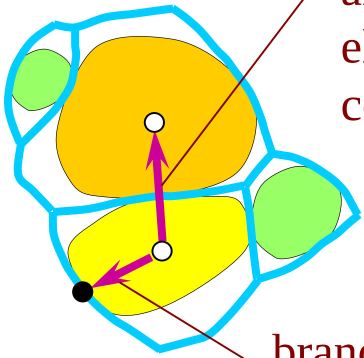
$$\boxed{= \frac{1}{V}\sum_{(c)} l_i^c F_j^c}$$

STRESS TENSOR

Discrete system:

The equivalent continuum: „material cell system”

branch vector, when is c an element-element contact



branch vector, when c is on the boundary

$$\bar{\sigma}_{ij} = \frac{1}{V}\sum_{(L)}\left(\sum_{(Lc\in S^L)} v_i^{Lc} F_j^{Lc}\right) = \frac{1}{V}\left(\sum_{(c\ inside)}(v_i^{Lc} - v_i^{Kc})F_j^{Lc} + \sum_{(c\ boundary)} v_i^{Lc} F_j^{Lc}\right) =$$

$$= \frac{1}{V}\sum_{(c)} l_i^c F_j^c$$

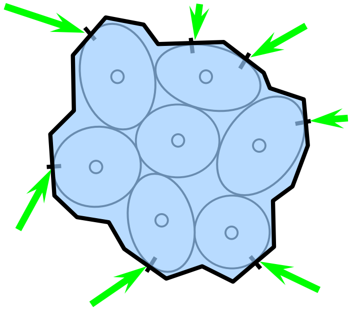
⟹ **SYMMETRIC!**

STRESS TENSOR

The most widespread version:

$$\bar{\sigma}_{ij} = \frac{1}{V} \sum_{(c)} l_i^c F_j^c$$

for every inside and boundary contact



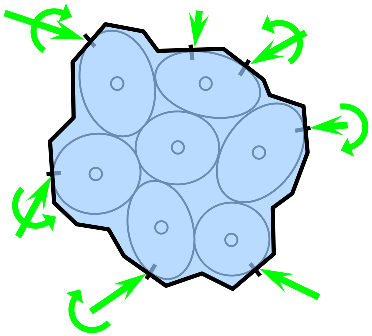
can be proven:

also valid in the presence of volume forces

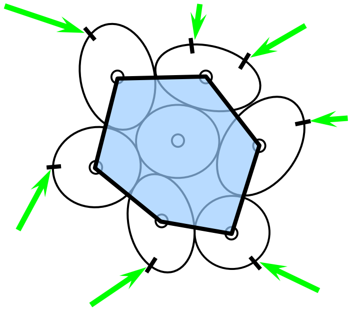
STRESS TENSOR

<u>Non-symmetric microstructural stress definitions:</u>

→ in the presence of contact moments:



→ in case of the boundary of the domain cutting through the elements:



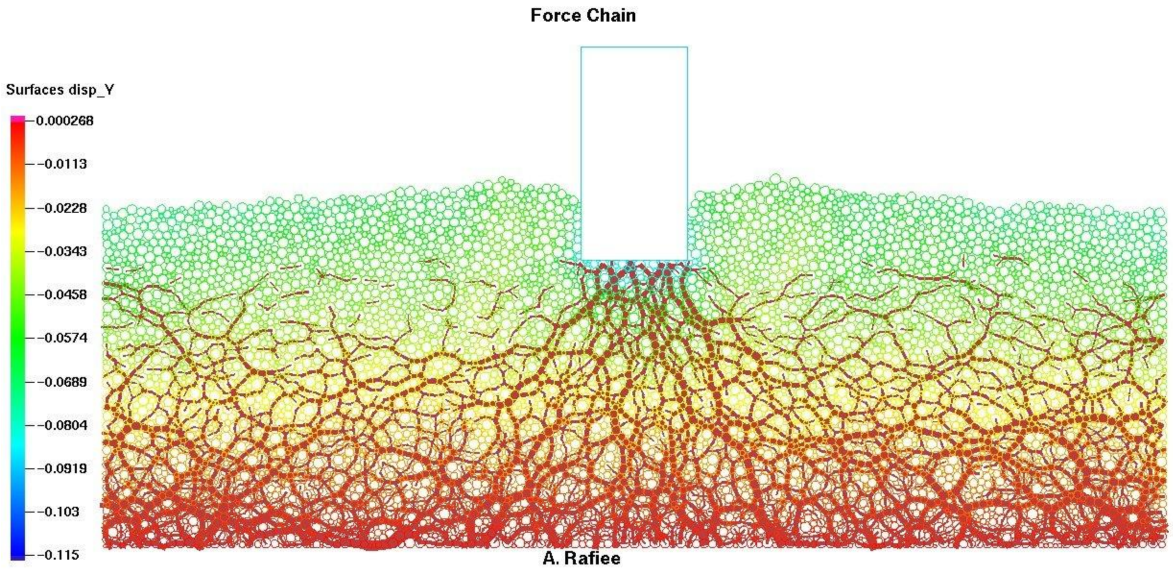
Diebels et al (2001)

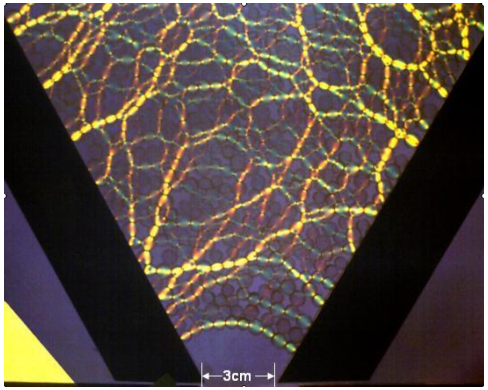
Bardet & Vardoulakis (2001)

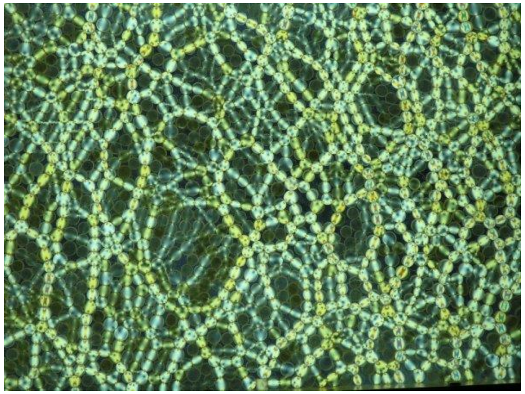
Fortin et al (2003)

STRESS TENSOR

Load carrying of granular assemblies: FORCE CHAINS







QUESTIONS

1. Explain what is „porosity” and „coordination number”.
2. Introduce the 2nd order fabric tensor defined by Satake.
   What other types of fabric tensors do you know?
3. How can we express the average stress tensor of a finite-sized continuous domain in terms of the forces acting on its boundary?
4. What is the distance of a point from an element?
5. Define the material cell system.
6. Define the microstructural stress tensor of a finite sub-assembly.