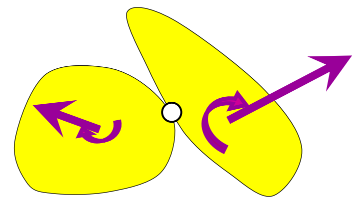
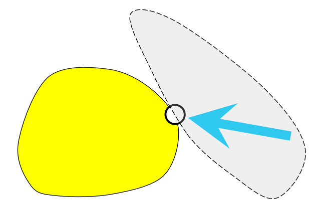
STATE VARIABLES OF DISCRETE SYSTEMS

→ Geometry
→ Stress tensors
→ $\boxed{\text{Strain tensors}}$

BASIC ASSUMPTIONS

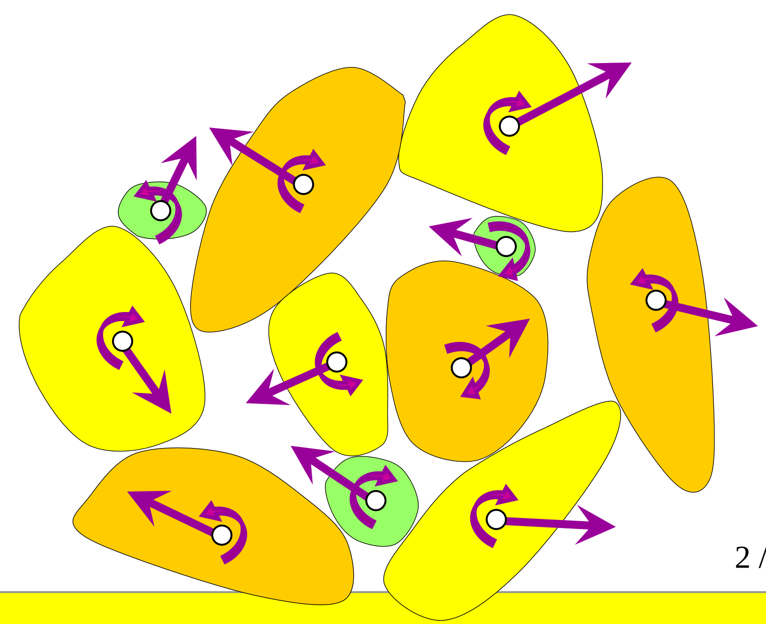


Rigid elements: translate; rotate;
own deformations are small and
restricted to the neighborhood of contacts



Contacts: small, point-like;
transmit concentrated forces

Overall deformations:
dominantly from the
displacements of the elements



Strain tensor:

this presentation

STRAIN TENSORS

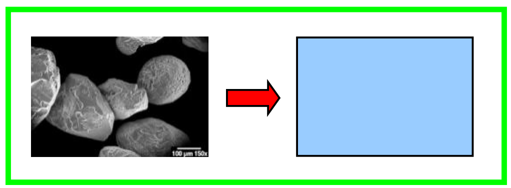
Content:

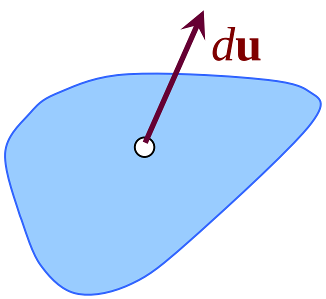
→ Strain tensor in continuum mechanics

→ Microstructural strain tensors based on
equivalent continuum

→ Microstructural strain tensors based on
least-square approximaions

Continuum mechanics:





du_i continuous translation field

$$de_{ij} = \frac{\partial u_j}{\partial x_i}$$

the gradient of the translation field

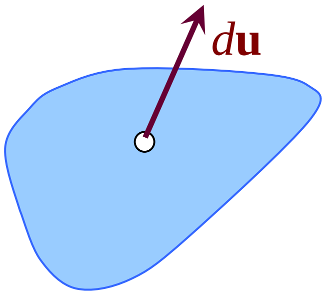
→ different strain tensors

linear strain tensor:

the symmetric part of de_{ij}

STRAIN TENSORS

Continuum mechanics :



du_i continuous translation field

$$de_{ij} = \frac{\partial u_j}{\partial x_i}$$ $\boxed{\text{the gradient of the translation field}}$

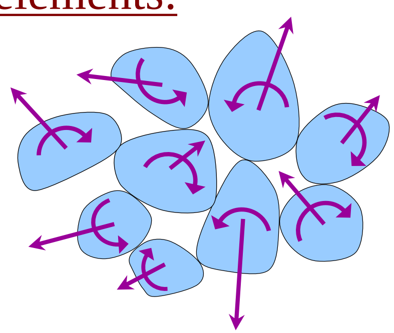
→ different strain tensors

linear strain tensor:

the symmetric part of de_{ij}

Basic assumption: du_i „nearly-everywhere” continuous and can be differentiated

Assemblies of discrete elements:

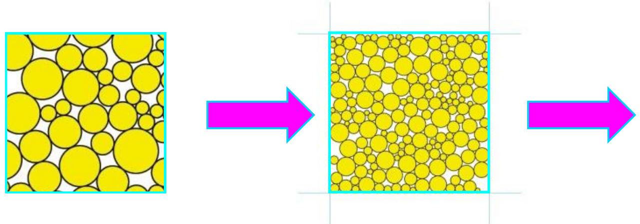


~~continuous translation field~~

THE CORE OF THE PROBLEM

Common misbelief:

assembly of more and more discrete elements → „tends to the continuum” ???



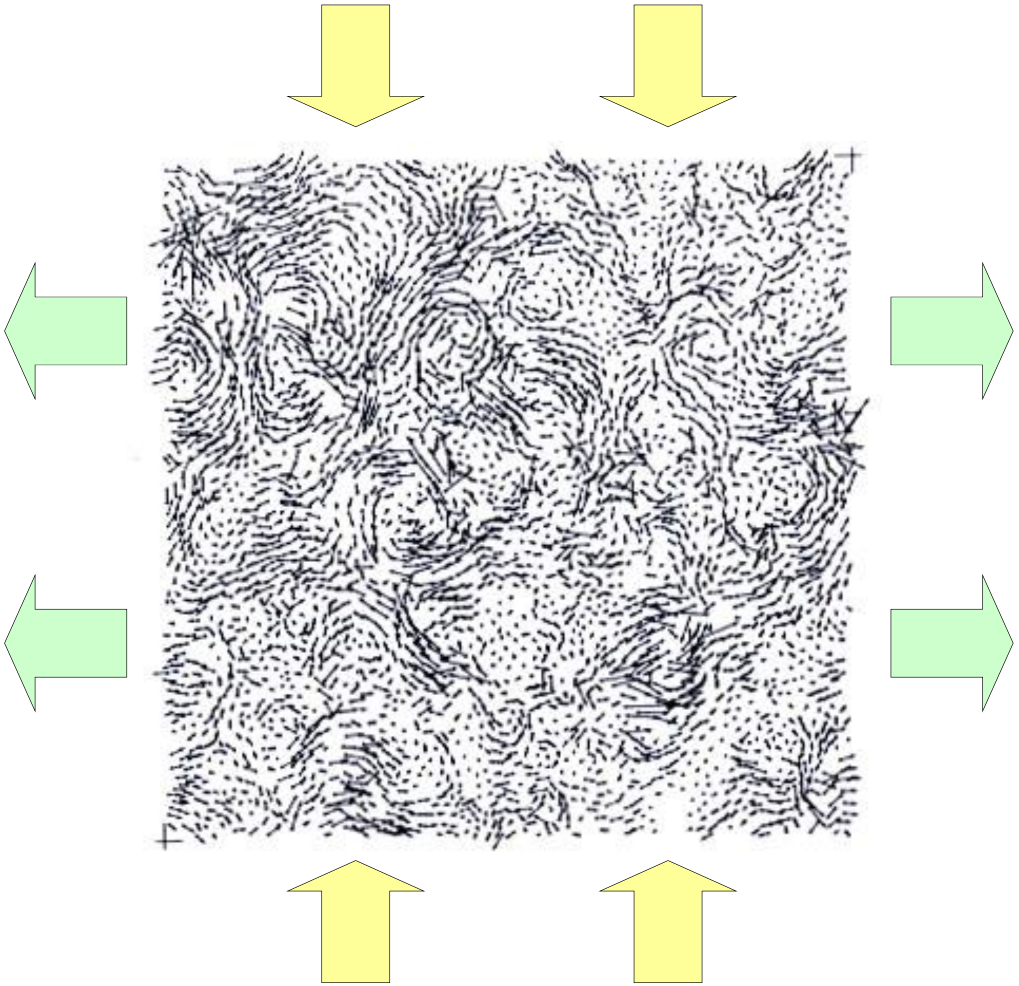
countably ∞

conti-
nuum -
∞

Number of elements:				
	finite	finite	countably infinite	continuum-infinite
Number of neighbours of the elements:				
	finite	finite	finite	continuum infinite
Degrees of freedom of the elements:				
	translation, rotation, …	translation, rotation, …	translation; rotation, …	[different versions]
Translation field:				
	strongly heterogeneous	strongly heterogeneous	strongly heterogeneous	continuous & differentiable

STRAIN TENSORS

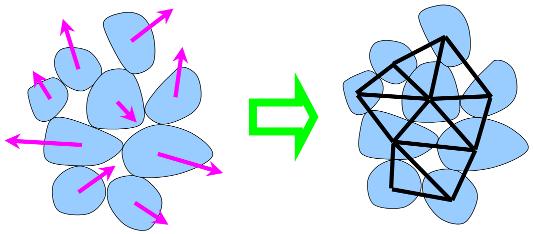
Translations of the elements in a biaxial compression test:



STRAIN TENSORS

Suggestions for microstructural strain tensors:

- based on an equivalent continuum: Bagi (1993) (2D, 3D)



Rothenburg & Kruyt (1996) (2D)
Kuhn (1999) (2D)
Dedecker et al (2000) (2D)
Kruyt (2003) (2D)

- based on least-square approximatrions: „best-fit”

Cundall (????) (2D, 3D)
Liao et al (1997) (2D, 3D)
Cambou I., Cambou II. (2000) (2D)

- Other ideas: Satake (2004) (2D, 3D)

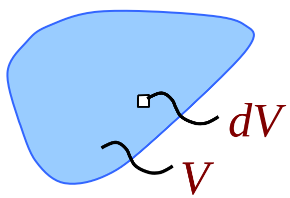
Common for all: define a translation gradient, and then take its symmetric part

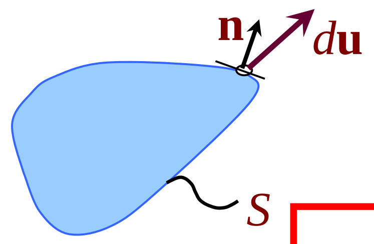
- small displacements are assumed
- initial configuration is used

STRAIN TENSORS

<u>Microstructural strain tensor based on equivalent continuum:</u>

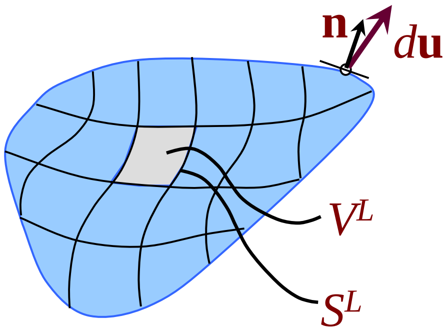
Reminder from continuum mechanics:





$$\boxed{d\bar{e}_{ij} = \frac{1}{V}\oint_{(V)} de_{ij}\, dV} \Longrightarrow \boxed{d\bar{e}_{ij} = \frac{1}{V}\oint_{(S)} du_{,j} n_i\, dS}$$

Continuum divided into subdomains:

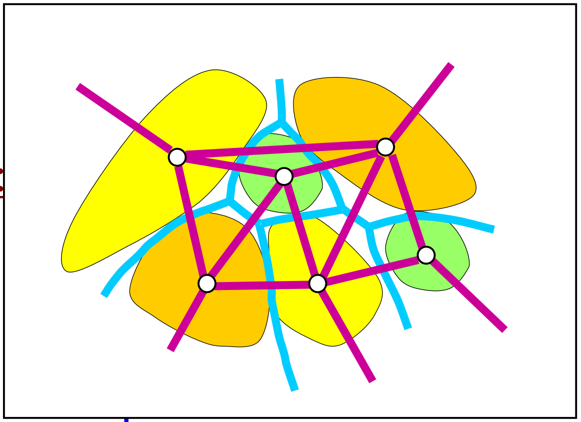


<u>L-th cell:</u> $$d\bar{e}_{ij}^{L} = \frac{1}{V^L}\oint_{(S^L)} du_{,j} n_i\, dS$$

$$d\bar{e}_{ij} = \frac{1}{V}\sum_{(L)}\left(de_{ij}^{L} V^{L}\right) = \frac{1}{V}\sum_{(L)}\left(\oint_{(S^L)} du_{,j} n_i\, dS\right) = \frac{1}{V}\oint_{(S)} du_{,j} n_i\, dS$$

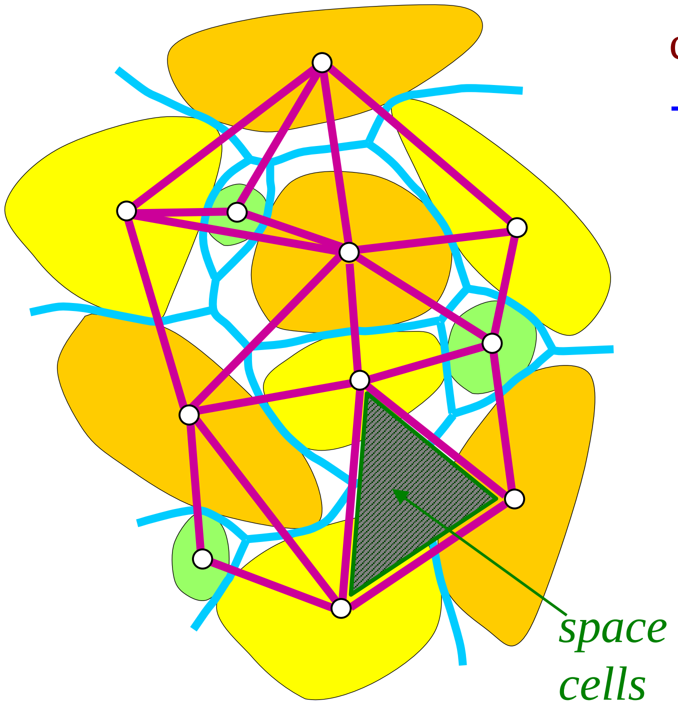
STRAIN TENSORS

<u>Microstructural strain tensor based on equivalent continuum:</u>



The equivalent continuum: „Space cell system”

in 2D:



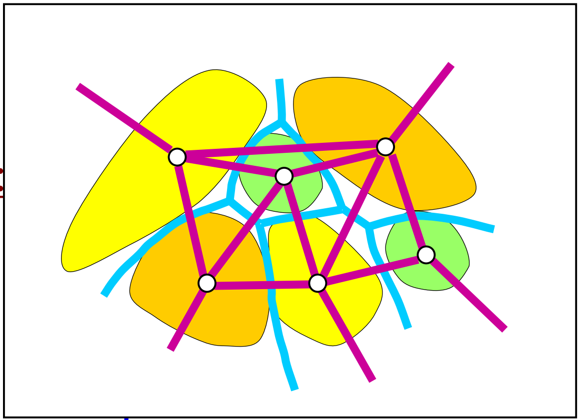
material cell system	space cell system	the discrete elements
cell	node	centroid of the element
face (*a line!*)	edge	neghbouring pair of elements
node	cell	(≈) void

→ simplex cells

→ space covered „unilayerly”

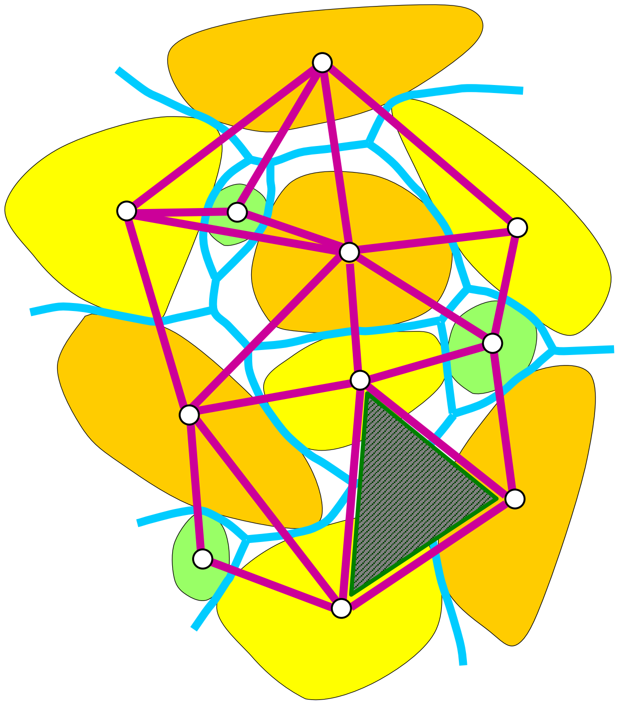
STRAIN TENSORS

<u>Microstructural strain tensor based on equivalent continuum:</u>



The equivalent continuum: „Space cell system”

in 3D:



material cell system	space cell system	the discrete elements
cell	node	centroid of the element
face (*surface*)	edge (*line*)	neghbouring pair of elements
edge	face (*triangle*)	???
node	cell	(≈) void

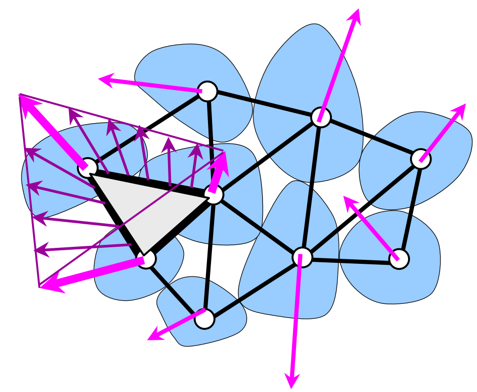
STRAIN TENSORS

Microstructural strain tensor based on equivalent continuum:

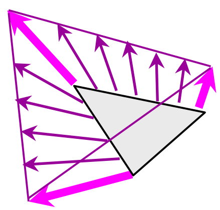
→ how to define a continuous translation field?

in the nodes: average rigid-body translations of the elements

inside the cells: linear interpolation



L-th cell:



$$d\bar{e}_{ij}^{L} = \frac{1}{V^{L}} \oint_{(S^{L})} du_{j} n_{i} dS$$

⇒ average for the whole space cell system:

$$d\bar{e}_{ij} = \frac{1}{V} \sum_{(L)} V^{L} d\bar{e}_{ij}^{L}$$

⇒ discretization:

$$\boxed{d\bar{e}_{ij} = \frac{1}{V} \sum_{(c)} d\Delta u_{j} d_{i}^{c}}$$

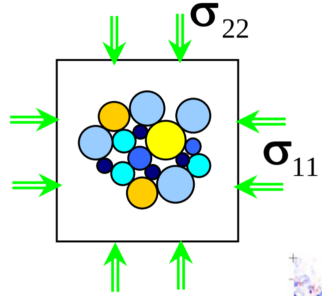
relative translation of neighbouring elements

summing up for all neighbouring pairs

„complementary area vector”

CRITICAL REMARK:

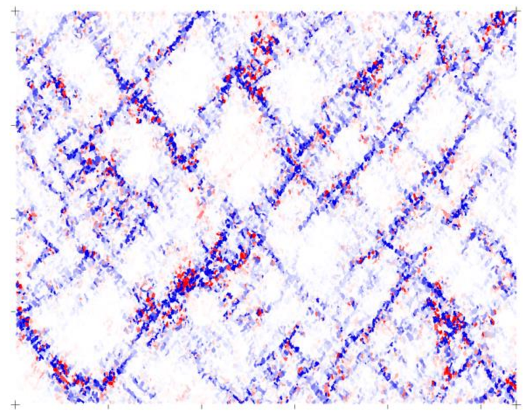
Biaxial shear tests:



Kuhn & Bagi, 2006

Deformation patterns:

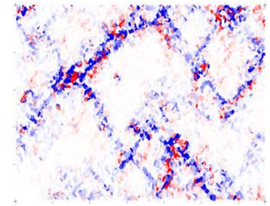
≈ 66 600 grains



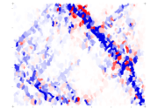
blue: volume increase

red: volume decrease

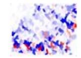
≈ 16 600 grains



≈ 4150 grains



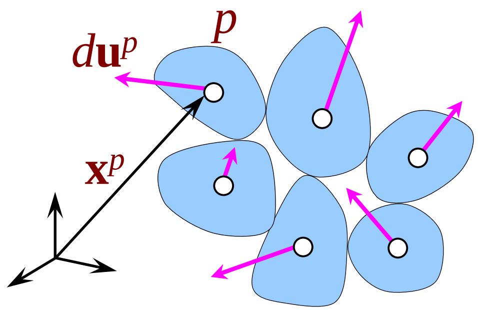
≈ 1040 grains



„AVERAGE STRAIN" DOES NOT DESCRIBE WHAT HAPPENS ON THE ELEMENT LEVEL!

STRAIN TENSORS

<u>Microstructural strain tensor based on least-square approximation:</u>



e.g. Cundall (??): $\tilde{x}_i^p = x_i^p - \frac{1}{N}\sum_{q=1}^{N} x_i^q \; ; \; d\tilde{u}_i^p = du_i^p - \frac{1}{N}\sum_{q=1}^{N} du_i^q$

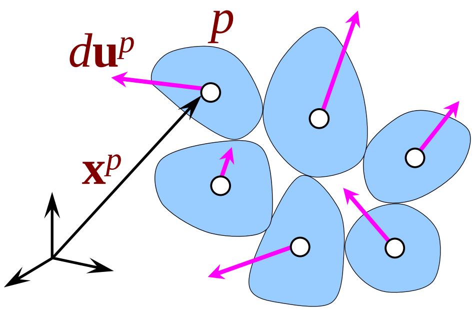
- if elements would move exactly according to a de_{ij} : their translation: $d\tilde{u}_i^p = de_{ji}\tilde{x}_j^p$
- but usually this is not the case, hence: $d\tilde{u}_i^p - de_{ji}\tilde{x}_j^p \neq 0$.
- Determine that de_{ij} tensor for which: $\Sigma = \sum_{p=1}^{N^p} (d\tilde{u}_i^p - \alpha_{ji}\tilde{x}_j^p)(d\tilde{u}_i^p - \alpha_{ji}\tilde{x}_j^p) \rightarrow \min!$
- minimization problem; its solution:

$$\begin{bmatrix} \sum_{p=1}^{N^p} \tilde{x}_1^p \tilde{x}_1^p & \sum_{p=1}^{N^p} \tilde{x}_2^p \tilde{x}_1^p & \sum_{p=1}^{N^p} \tilde{x}_3^p \tilde{x}_1^p \\ \sum_{p=1}^{N^p} \tilde{x}_1^p \tilde{x}_2^p & \sum_{p=1}^{N^p} \tilde{x}_2^p \tilde{x}_2^p & \sum_{p=1}^{N^p} \tilde{x}_3^p \tilde{x}_2^p \\ \sum_{p=1}^{N^p} \tilde{x}_1^p \tilde{x}_3^p & \sum_{p=1}^{N^p} \tilde{x}_2^p \tilde{x}_3^p & \sum_{p=1}^{N^p} \tilde{x}_3^p \tilde{x}_3^p \end{bmatrix} \begin{bmatrix} de_{1i} \\ de_{2i} \\ de_{3i} \end{bmatrix} = \begin{bmatrix} \sum_{p=1}^{N^p} d\tilde{u}_i^p \tilde{x}_1^p \\ \sum_{p=1}^{N^p} d\tilde{u}_i^p \tilde{x}_2^p \\ \sum_{p=1}^{N^p} d\tilde{u}_i^p \tilde{x}_3^p \end{bmatrix}$$

$$\boxed{d\bar{e}_{ij} = z_{ik}\sum_{p=1}^{N^p} d\tilde{u}_j^p \tilde{x}_k^p}$$

$[\ \ ]^{-1}$

STRAIN TENSORS



Microstructural strain tensor based on least-square approximation:

e.g. Cundall (??):

$$\boxed{d\bar{e}_{ij} = z_{ik} \sum_{p=1}^{N^p} d\tilde{u}_j^p \tilde{x}_k^p}$$

Application: in DEM codes (PFC-2D, -3D, TRUBAL, ...)

Other „best-fit" strain tensors:

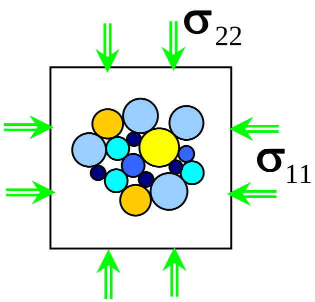
built in other translation characteristics, e.g.

$\rightarrow$ relative translation of contacting material points (Liao et al)

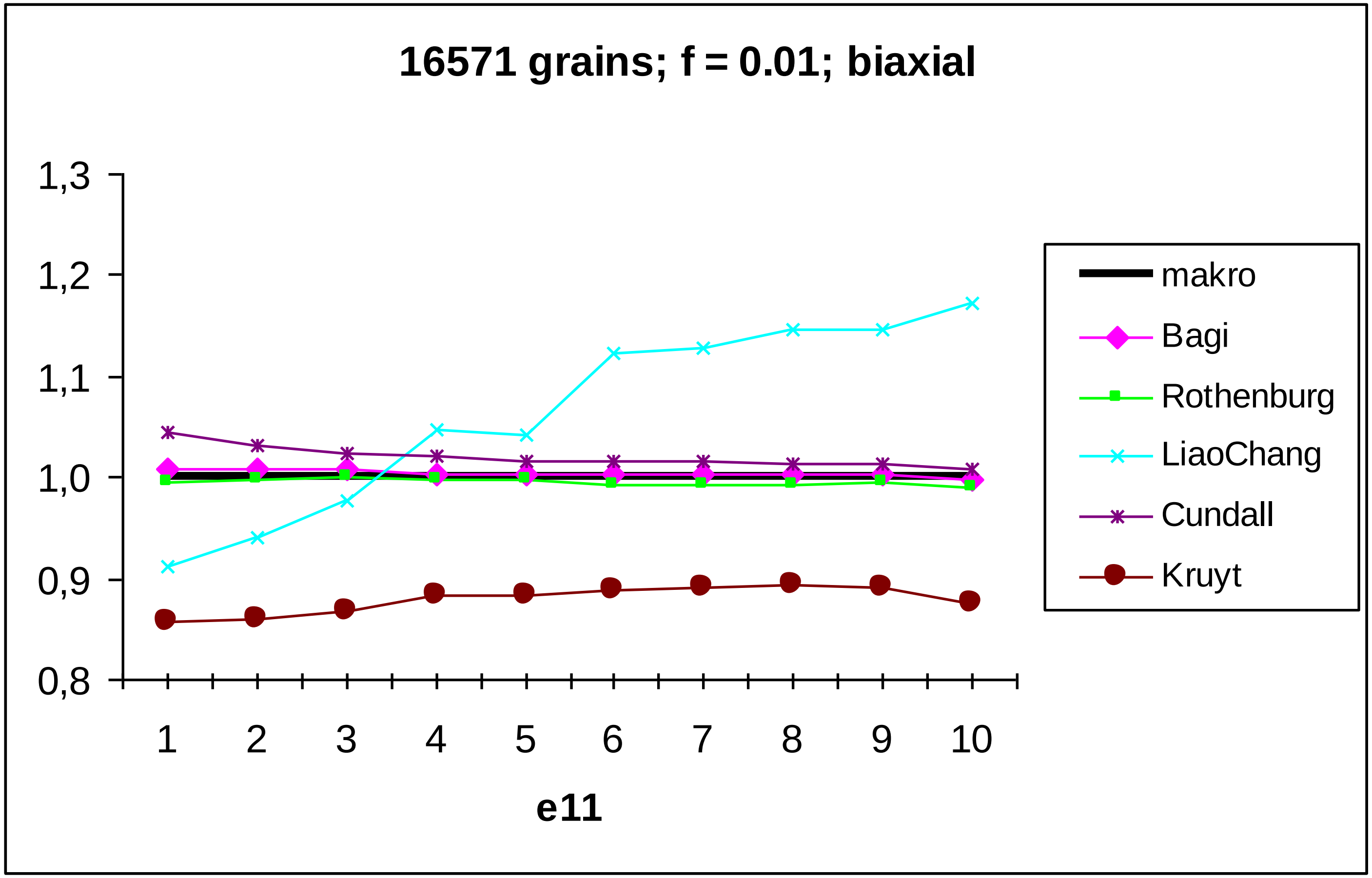
$\rightarrow$ relative translation of the centroids of neighbouring elements (Cambou et al)

[coincides well with the equivalent-continuum strain]

STRAIN TENSORS



Comparison of microstructural strain tensors



QUESTIONS

1. In case of a 2D assembly of discrete elements, define the space cell system. How is it generalized for 3D?
2. How to calculate the microstrucrural strain tensor based on an equivalent continuum?
3. Introduce Cundall's best-fit strain.