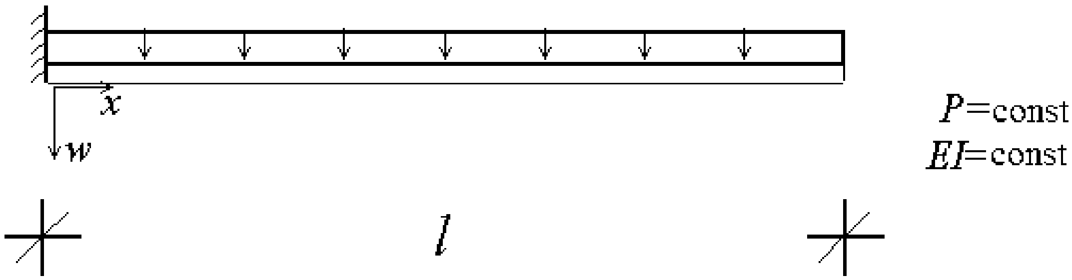
The Ritz-method

Sample 1.: Determine the displacement function and the deflection of the end of the console.



The shape functions are: $\varphi_1 = x^2$, $\varphi_2 = x^3$. Both satisfy the conditions $w = \frac{dw}{dx} = 0$ at x = *0*. The Ritz approximation function is:

$$w(x) = \sum_{i=1}^{2} c_i \varphi_i = c_1 x^2 + c_2 x^3 .$$

Let's substitute this approximation function into the expression of the potential energy:

$$\Pi(c_1, c_2) = \frac{EI}{2} \int_0^l \left(\frac{d^2 w}{dx^2} \right)^2 dx - \int_0^l w\, p\, dx = \frac{EI}{2} \int_0^l (2c_1 + 6c_2 x)^2 dx - \int_0^l p(c_1 x^2 + c_2 x^3) dx .$$

Having executed the integration, one gets:

$$\Pi(w) = \frac{EI}{2}(4lc_1^2 + 12l^3 c_2^2 + 12 c_1 c_2 l^2) - p(c_1 \frac{l^3}{3} + c_2 \frac{l^4}{4}) .$$

Applying the stationarity condition we have:

$$\frac{\partial \Pi}{\partial c_1} = 0 = 4lc_1 + 6l^2 c_2 - \frac{pl^3}{3EI}, \text{ illetve } \frac{\partial \Pi}{\partial c_2} = 0 = 12l^3 c_2 + 6l^2 c_1 - \frac{pl^4}{4EI} .$$

The solution of the system of equation gives:

$$c_1 = \frac{5l^2 p}{24EI} \text{ , } c_2 = -\frac{pl}{12EI} .$$

So, the approximate displacement function is:

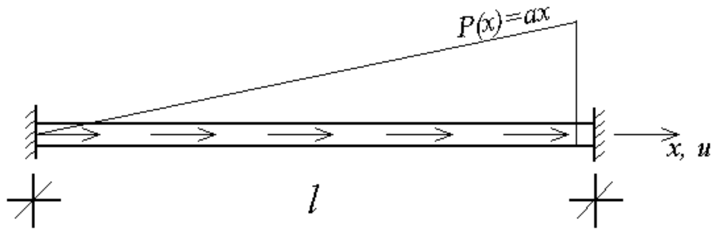
$w(x) = \frac{pl^4}{24EI}(\frac{5x^2}{l^2} - \frac{2x^3}{l^3})$. And the deflection of the end of the console is:

$$w(l) = \frac{pl^4}{8EI} .$$

For comparison, the:exact solution is

$$w(x) = \frac{pl^4}{24EI}(\frac{6x^2}{l^2} - \frac{4x^3}{l^3} + \frac{x^4}{l^4}) \text{ , } w(l) = \frac{pl^4}{8EI} .$$

Sample 2.: Determine the displacement function.



This is an axially loaded bar of which the potential energy is:

$$\Pi(u)=\frac{EA}{2}\int_l\left(\frac{du}{dx}\right)^2 dx-\int_l p\,u\,dx.$$

Let's use the shape functions : $\varphi_1 = x(l-x)$, $\varphi_2 = x^2(l-x)$.

Their derivatives are: $\frac{d\varphi_1}{dx}=l-2x$, $\frac{d\varphi_2}{dx}=2lx-3x^2$.

Without computing the potential energy, one can express directly the system of equation for the unknown constants.:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$a_{11}=\langle R\varphi_1 , R\varphi_1\rangle = EA\int_0^l (l^2-4x+4x^2)dx = EA\left[l^2x-2x^2+\frac{4}{3}x^3\right]_0^l=\frac{EAl^3}{3},$$

$$a_{12}=EA\int_0^l (2l^2x-7lx^2+6x^3)dx = EA\left[l^2x^2-\frac{7}{3}lx^3+\frac{3}{2}x^4\right]_0^l=\frac{EAl^4}{6},$$

$$a_{22}=EA\int_0^l (4l^2x^2-12lx^3+9x^4)dx = EA\left[\frac{4}{3}l^2x^3-3lx^4+\frac{9}{5}x^5\right]_0^l=\frac{2EAl^5}{15},$$

$$b_1=\langle ax, x(l-x)\rangle = a\int_0^l (lx^2-x^3)dx = a\left[\frac{lx^3}{3}-\frac{x^4}{4}\right]_0^l=\frac{al^4}{12},$$

$$b_2=a\int_0^l (lx^3-x^4)dx = a\left[\frac{lx^4}{4}-\frac{x^5}{5}\right]_0^l=\frac{al^5}{20}.$$

So, the systeme of equations is:

$$EA\begin{bmatrix} l^3/3 & l^4/6 \\ l^4/6 & 2l^5/15 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=a\begin{bmatrix} l^4/12 \\ l^5/20 \end{bmatrix}.$$ The solution is: $$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} \frac{al}{6EA} \\ \frac{a}{6EA} \end{bmatrix}$$

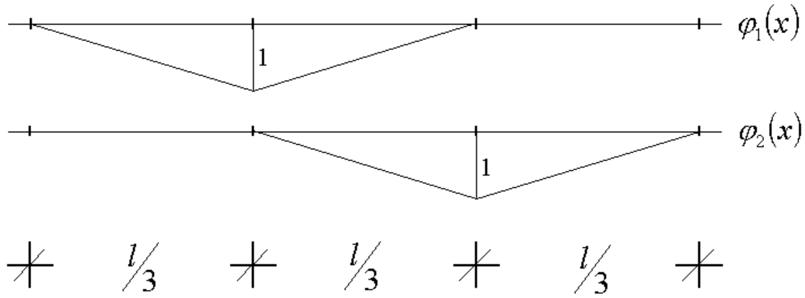
The approximative displacement function is:

$$u(x)=\frac{al}{6EA}x(l-x)+\frac{a}{6EA}x^2(l-x)=\frac{ax(l^2-x^2)}{6EA}.$$

The normal force function can be computed by its law:

$$N(x)=EA\frac{du}{dx}=\frac{a}{6}(l^2-3x^2).$$

Sample 3.: Determine the displacement function by using the following shape functions.



The shape functions are:

$$\varphi_1=\frac{3x}{l},if\quad x\le\frac{l}{3},\varphi_1=2-\frac{3x}{l},if\ \frac{l}{3}\le x\le\frac{2l}{3},\varphi_1=0,\ if\ \frac{2l}{3}\le x,$$
$$\varphi_2=0,if\ x\le\frac{l}{3},\varphi_2=-1+\frac{3x}{l},if\ \frac{l}{3}\le x\le\frac{2l}{3},\varphi_2=3-\frac{3x}{l},if\ \frac{2l}{3}\le x.$$

a./ Solution by the potential energy.

$$\Pi(u)=\frac{EA}{2}\left\{\int_0^{l/3}(c_1\frac{3}{l}+c_2 0)^2dx+\int_{l/3}^{2l/3}(c_1(-\frac{3}{l})+c_2\frac{3}{l})^2dx+\int_{2l/3}^{l}(c_2(-\frac{3}{l}))^2dx\right\}-$$
$$-a\left\{\int_0^{l/3}xc_1\frac{3x}{l}dx+\int_{l/3}^{2l/3}x\left[c_1(2-\frac{3x}{l})+c_2(-1+\frac{3x}{l})\right]dx+\int_{2l/3}^{l}xc_2(3-\frac{3x}{l})dx\right\}.$$

After integration one has:

$$\Pi=\frac{EA}{2}\left(\frac{3}{l}\right)^2\left\{c_1^2\frac{l}{3}+(c_1^2-2c_1c_2+c_2^2)\frac{l}{3}+c_2^2\frac{l}{3}\right\}-a(c_1\frac{(l/3)^3}{l}+c_1\left((\frac{2l}{3})^2-(\frac{l}{3})^2\right)-$$
$$-\frac{c_1}{l}\left[(\frac{2l}{3})^3-(\frac{l}{3})^3\right]-\frac{c_2}{2}\left[(\frac{2l}{3})^2-(\frac{l}{3})^2\right]+\frac{c_2}{l}\left[(\frac{2l}{3})^3-(\frac{l}{3})^3\right]+c_2\frac{3}{2}\left[l^2-(\frac{2l}{3})^2\right]-$$
$$-\frac{c_2}{l}\left[l^3-(\frac{2l}{3})^3\right])=\frac{3EA}{l}(c_1^2-c_1c_2+c_2^2)-\frac{al^2}{27}(c_1(1+9-7)+c_2(-\frac{9}{2}+7+\frac{45}{2}-19))=$$
$$=\frac{3EA}{l}(c_1^2-c_1c_2+c_2^2)-a\frac{l^2}{27}(3c_1+6c_2).$$

From the stationarity condition, one has:

$$\frac{\partial\Pi}{\partial c_1}=\frac{3EA}{l}(2c_1-c_2)-a\frac{l^2}{9}=0\ ,\ \frac{\partial\Pi}{\partial c_2}=\frac{3EA}{l}(-c_1+2c_2)-a\frac{2l^2}{9}=0\ .$$

The result is: $c_1=\frac{4al^3}{81EA}$, $c_2=\frac{5al^3}{81EA}$. The approximative displacement function is:

$u(x) = \dfrac{4al^2 x}{27EA}$ *if* $x \le \dfrac{l}{3}$, $u(x) = \dfrac{al^2(l+x)}{27EA}$ *if* $\dfrac{l}{3} \le x \le \dfrac{2l}{3}$, *and*

$u(x) = \dfrac{5l^2 a(l-x)}{27EA}$ *if* $\dfrac{2l}{3} \le x$.

b./ Solution by writing the system of equations directly:

$$a_{11} = EA(\int_0^{l/3} (\frac{3}{l})^2 dx + \int_{l/}^{2l/3} (-\frac{3}{l})^2 dx + \int_{2l/3}^{l} 0^2 dx) = EA(\frac{3}{l} + \frac{3}{l}) = \frac{6EA}{l},$$

$$a_{22} = a_{11}\ ,\ a_{12} = a_{21} = EA \int_{l/3}^{2l/3} \frac{3}{l}(-\frac{3}{l})\, dx = -\frac{3EA}{l},$$

$$b_1 = \frac{3a}{l} \int_0^{l/3} x^2 dx + a \int_{l/3}^{2l/3} (2x - \frac{3x^2}{l}) dx = \frac{a}{l}(\frac{l}{3})^3 + a\left[x^2 - \frac{x^3}{l}\right]_{l/3}^{2l/3} = \frac{al^2}{9},$$

$$b_2 = a \int_{l/3}^{2l/3} (-x + \frac{3x^2}{l}) dx + a \int_{2l/3}^{l} (3x - \frac{3x^2}{l}) dx = a\left[-\frac{3}{2}(\frac{l}{3})^2 + \frac{7}{l}(\frac{l}{3})^3\right] + a\left[\frac{3}{2}5(\frac{l}{3})^2 - \frac{19}{l}(\frac{l}{3})^3\right]$$

$= \dfrac{2al^2}{9}$. So the system of equation is:

$$\frac{3EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{al^2}{9}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

. From this, the solution is: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \dfrac{al^3}{81EA}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

It can be seen that this way is faster.

The normal force function by using the differential relation is

$N(x) = EA\dfrac{du}{dx} = \dfrac{al^2}{27}\begin{Bmatrix} 4 \\ 1 \\ -5 \end{Bmatrix}$, for the three $l/3$ long interval. The second part of the figure shows the exact solution.