Budapest University of Technology and Economics
Short Course on Topology Optimization of Structures

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# Short Course on Topology Optimization of Structures

*Matteo Bruggi*, Department of Civil and Environmental Engineering DICA, Politecnico di Milano, Milano, Italy

# Schedule

DAY 1

**Topology optimization of structures: basics**

DAY 2

**Stress-constrained topology optimization**

DAY 3

**Mixed finite elements for the optimal design of structures**

DAY 4

**Analysis and design of no-tension structures, by formulating optimization problems**

8 April 2020
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# Topology optimization of structures: basics

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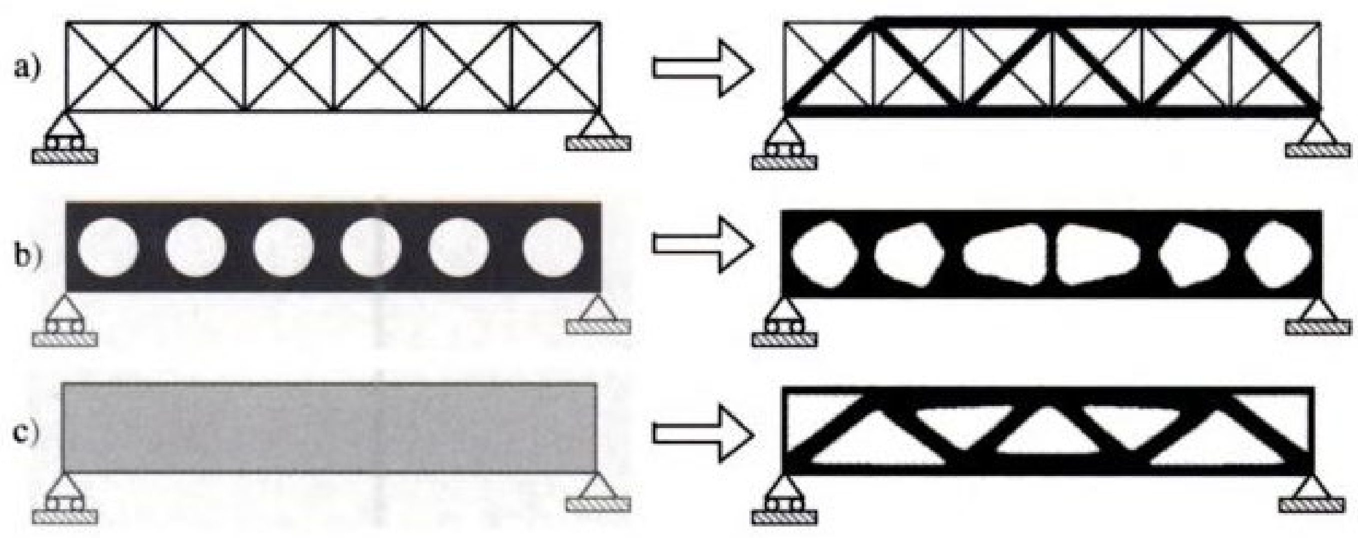
# Structural and topology optimization



(Topology Optimization:
Methods and Applications,
Bendsøe and Sigmund, 2003)

a) sizing optimization:

the areas of the elements of a fixed truss “ground structure” are unknown

b) shape optimization:

the parameters describing the geometry of the boundaries are unknown

*c) topology optimization:*

*the distribution of material is unknown*

# Structural topology optimization: a design tool

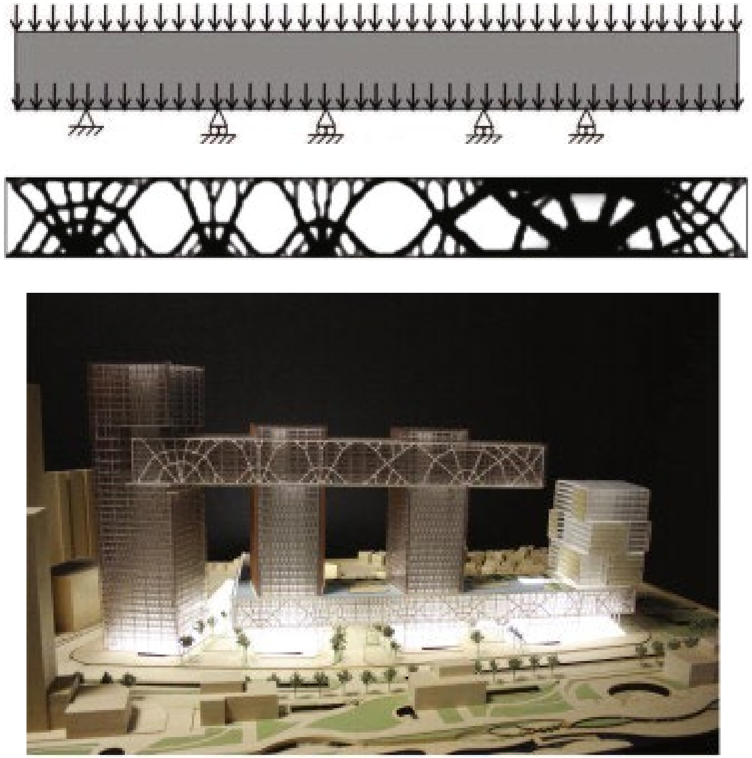
“The art of structure is where to put the holes”

*Robert Le Ricolais (1894–1977)*
*French/American Engineer, “Father of Spatial Structures”*

# Structural topology optimization: a design tool

Gigantic tree-like columns support the overhanging roof canopy of the Qatar National Convention Centre by Japanese Architect Arata Isozaki

# Structural topology optimization: a design tool



A one-of-a-kind project: a conceptual design for the Zendai competition (China) created with topology optimization by Prof. Glaucio Paulino's research group along with Skidmore, Owings & Merrill LLP

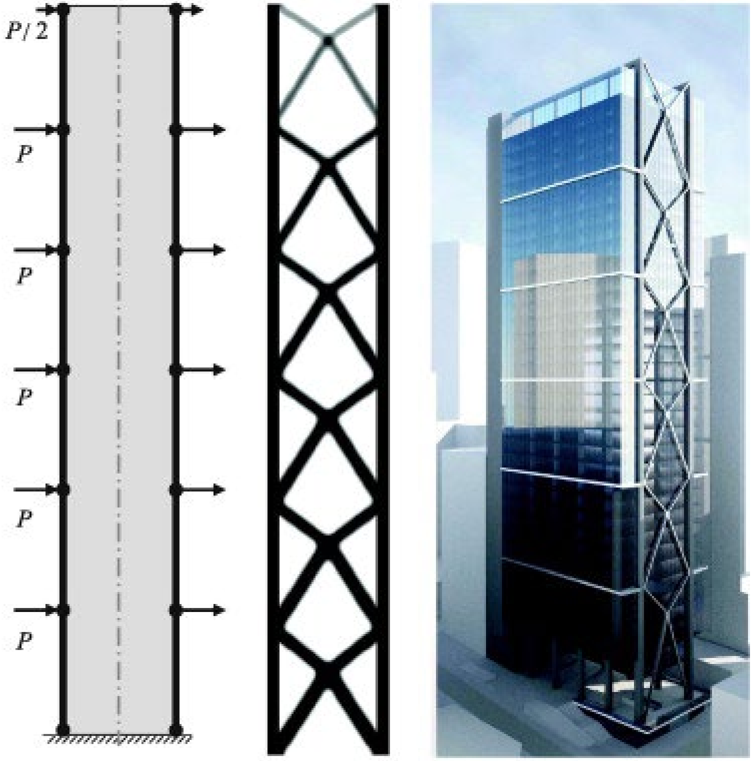


Illustration for the concept design of a 288 m tall high-rise in Australia, showing the engineering and architecture expressed together at Skidmore, Owings & Merrill LLP (Beghini, Katz, Baker and Paulino, 2014)

# Outline

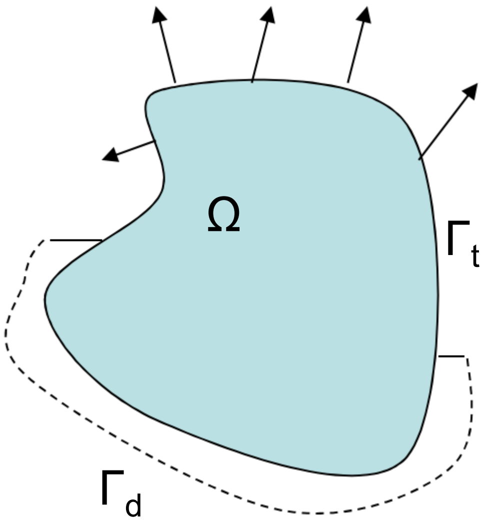
- Governing equations:
    - “penalized” elasticity problem and structural compliance

- Problem formulation:
    - conventional volume-constrained minimum compliance formulation

- Solution of the minimization problem:
    - gradient-based algorithms
    - sensitivity computation
    - numerical issues

- Applications:
    - design of stiff structures (with 88-line Matlab code)
    - design of compliant mechanisms
    - design of periodic microstructures

# Structural topology optimization: state equations (linear elastic problem)



A homogeneous domain Ω ∈ $R^2$ with a regular boundary Γ is considered, assuming that $\Gamma = \Gamma_d \cup \Gamma_t$.

Prescribed displacements with components $u_{0j}$ and tractions with components $t_{0j}$ are assigned on $\Gamma_d$ and $\Gamma_t$
$g_j$ are the components of the vector of body loads in Ω (generally neglected)

$C_{ijhk}$ are the component of the 4th order elasticity tensor

Let be $\underline{u}$ the unknown displacement field, $\underline{\underline{\sigma}}$ the unknown stress field and $\underline{\underline{\varepsilon}}$ the unknown strain field. One has:

equilibrium $\sigma_{ij,i} + \cancel{g}_j = 0$ in Ω, along with $\sigma_{ij} n_i \,|_{\Gamma_t} = t_{0j}$

compatibility $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ in Ω, along with $u_j \,|_{\Gamma_d} = u_{0j}$

constitutive law $\sigma_{ij} = C_{ijhk} \varepsilon_{hk}$

# Structural topology optimization: mathematical formulation

- Given a domain with assigned loads and boundary conditions, find the distribution of linear elastic isotropic material that minimizes an assigned scalar function for a fixed set of constraints

$$\psi(\chi) = \begin{cases} 1 \quad , \text{if} \quad \chi \in \Omega_{mat} & \text{“Full material”} \\ 0 \quad , \text{if} \quad \chi \in \Omega / \Omega_{mat}, & \text{“Void”} \end{cases}$$

Function representing a **discrete** material density, i.e. the minimization unknown

$$C_{ijhk}(\chi) = \psi(\chi) C^0_{ijhk}$$

$$E(\chi) = \psi(\chi) E^0$$

properties of the “full material”

Material model to interpolate the constitutive tensor (i.e. the Young modulus for isotropic material)

$$C^0_{ijhk} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{hk} + \frac{E}{2(1+\nu)}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh})$$

The elastic problem depends on $\psi(\chi)$: find $\underline{u} \in H^1$ such that $\underline{u}\,|_{\Gamma_u} = \underline{u}_0$ and

$$\int_\Omega \psi(\chi) C^0_{ijhk} \varepsilon_{hk}(\underline{u})\, \varepsilon_{ij}(\underline{v})\; dx = \int_{\Gamma_t} t_{0j}\, v_j\; ds, \qquad \forall \underline{v} \in H^1$$

# Structural topology optimization: mathematical formulation

- Objective function: "structural compliance", work of the external loads at equilibrium, twice the strain energy stored in the structure (Clapeyron th.), measure of the structural deformability

$$\mathcal{C} = W(\underline{u}) = \int_{\Gamma_t} t_{0j}\, u_j \; ds = \int_{\Omega} \psi(\chi) C^0_{ijhk} \varepsilon_{hk}(\underline{u}) \varepsilon_{ij}(\underline{u}) \; d\Omega$$

- Volume constraint: enforcement of the available "volume fraction" of material

$$\Omega_{mat}/\Omega \leq V_f \qquad \int_{\Omega} \chi d\Omega \Big/ \int_{\Omega} d\Omega \leq V_f \qquad V_f \leq 1$$

$$\left\{ \begin{array}{ll} \min\limits_{\psi \in \psi_{ad}} & \mathcal{C} = W(\underline{u}) \\ \text{s.t.} & \int_{\Omega} \psi(\chi) C^0_{ijhk} \varepsilon_{hk}(\underline{u})\, \varepsilon_{ij}(\underline{v}) \; dx = \int_{\Gamma_t} t_{0j}\, v_j \; ds, \;\; \forall \underline{v} \in H^1, \\ & \int_{\Omega} \chi d\Omega \Big/ \int_{\Omega} d\Omega \leq V_f, \end{array} \right.$$

Objective function

State equation, as a constraint

Volume constraint

# Structural topology optimization: NAND/SAND

- Two unknown fields arise in the formulation: density and displacement

In the conventional Nested Analysis and Design Approach (NAND) only the structural design variable is treated as the optimization variable. In the more demanding Symultaneous Analysis and Design Approach (SAND) both variables enter the optimization problem

$$\begin{cases} \min\limits_{\psi \in \psi_{ad}} \quad \mathcal{C} = W(\underline{u}) \\ \text{s.t.} \quad \int_\Omega \psi(\chi) C^0_{ijhk} \varepsilon_{hk}(\underline{u})\, \varepsilon_{ij}(\underline{v})\ dx = \int_{\Gamma_t} t_{0j}\, v_j\ ds, \ \forall \underline{v} \in H^1, \\ \int_\Omega \chi d\Omega \Big/ \int_\Omega d\Omega \le V_f, \end{cases}$$

Objective function

Nested (or treated as an equality constraint)

Volume constraint

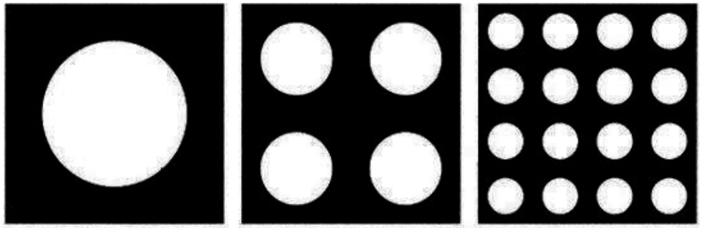
$$\begin{cases} \max\limits_{\psi \in \psi_{ad}} \ \min\limits_{\underline{v} \in \underline{v}_{ad}} \quad \Pi(\underline{u}) \\ \text{s.t.} \quad \int_\Omega \chi d\Omega \Big/ \int_\Omega d\Omega \le V_f \end{cases}$$

$$\Pi(\underline{u}) = U(\underline{u}) - W(\underline{u}) = W(\underline{u})/2 - W(\underline{u}) = -W(\underline{u})/2 = -\mathcal{C}(\underline{u})/2$$

*Max-min problem in terms of the total potential energy*

# Structural topology optimization: is the continuous problem well-posed?

- Minimization problem with discrete values (0-1): the formulation has no feasible solution in the case of isotropic material. The stiffest geometry calls for the largest number of holes, finally achieving “optimal microstructures”



- An enlargement of the design domain is needed

“Void” “Full material”

$$0 \leq \rho(x) \leq 1$$

Function representing a **continuous** material density, i.e. the minimization unknown

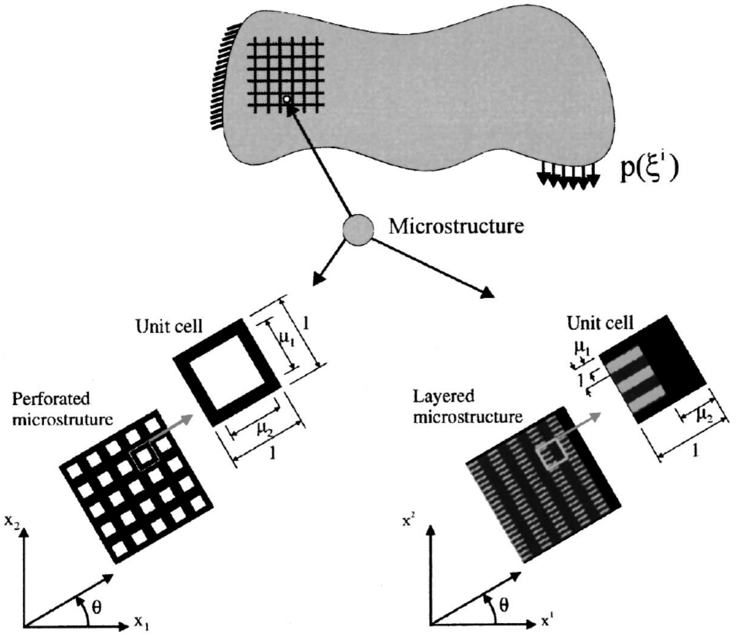
1. allowing for microstructures with intermediate densities between 0-1: optimization of composite materials

2. alternatively, introducing a penalization of the intermediate densities to achieve pure 0-1 design: optimization by distribution of isotropic material

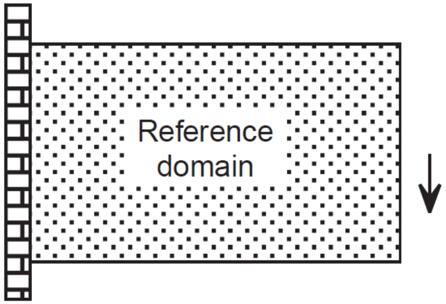
# Structural topology optimization:
# topology optimization of composite materials

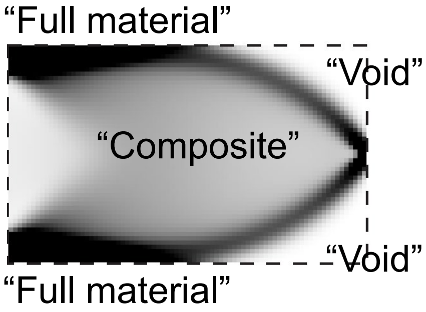
1. Material density represents the material as a microstructure (“gray” is allowed)

The microstructure is a composite material with an infinite number of infinitely small voids, leading to a porous composite with a density varying between 0 and 1. Macroscopic mechanical features can be derived through homogenization methods.



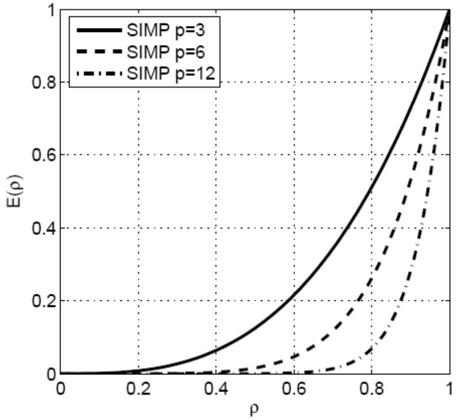
Since the macroscopic properties of common types of microstructure are not isotropic an orientation angle is also needed





# Structural topology optimization: topology optimization by distribution of isotropic material

2. Intermediate material density is penalized to achieve 0-1 design (no “gray”)

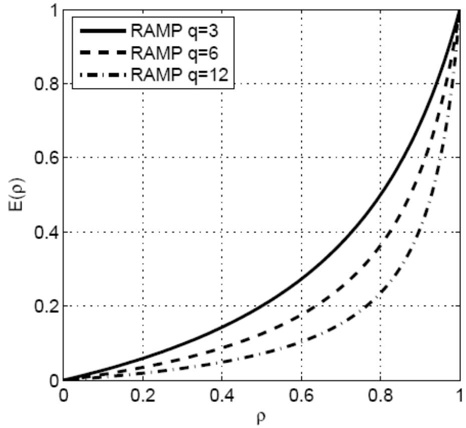


SIMP: Solid Isotropic Microstructure with Penalty

(Rozvany et al, 92; Bendsøe and Sigmund, 99; Berke 70)

$$E(\rho) = \rho^p E^0 \qquad \rho_{min} \leq \rho(\chi) \leq 1 \qquad \rho_{min} = 10^{-3}$$

- The power $p > 1$ penalizes intermediate densities to achieve pure 0-1 design (usually $p$=3). $\rho_{min}$ is needed against FEM singularities
- It makes even the compliance minimization a nonconvex problem



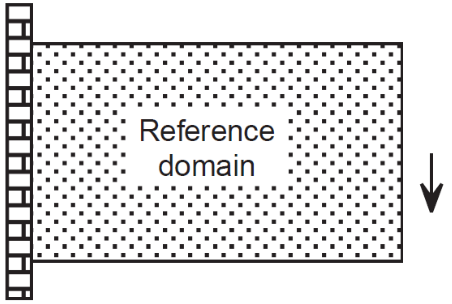
RAMP: Rationale Approximation of Material Properties

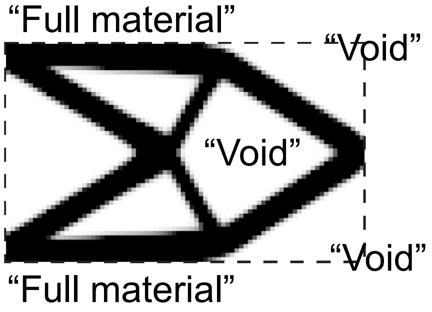
(Stolpe and Svanberg, 03)

$$E(\rho) = E_{min} + \frac{\rho}{1+q(1-\rho)}(E^0 - E_{min}) \qquad 0 \leq \rho(\chi) \leq 1 \qquad E_{min} = 10^{-9} E^0$$

- A convex interpolation model implies a convex objective function. Unfortunately, this result is achieved only if $q \geq E_0/E_{min}-1$
- It works fine for two-material problems with $E^0=E^{01}$ $E_{min}=E^{02}$

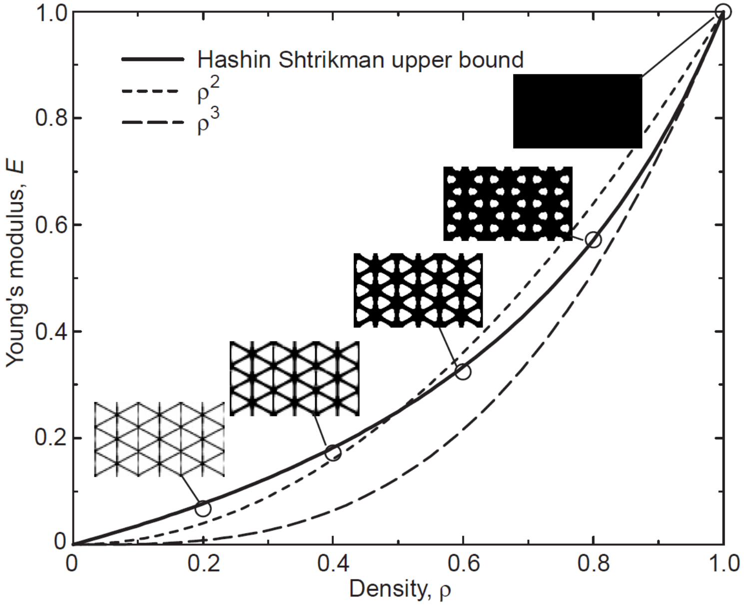
# Structural topology optimization: topology optimization by distribution of isotropic material





- SIMP/RAMP improves the numerical tractability of the continuous setting (topology → sizing), but does not completely solve the problem (mesh dependence of the discrete setting)

- For $p \geq 3$, SIMP can be seen as a «material model»



$C_{ijhk}(\rho)$ corresponds to a composite material constructed from void and the given material at a real density $\rho$, since the bulk modulus $k$ and the shear modulus μ satisfy the Hashin-Shtrikman bounds:

$$0 \leq \kappa \leq \frac{\rho\kappa^0\mu^0}{(1-\rho)\kappa^0+\mu^0}, \quad 0 \leq \mu \leq \frac{\rho\kappa^0\mu^0}{(1-\rho)(\kappa^0+2\mu^0)+\kappa^0}$$

$$0 \leq E \leq E^* = \frac{\rho E^0}{3-2\rho}$$

# Governing equations:
# SIMP-based elasticity problem

Given a domain with assigned loads and boundary conditions, find the distribution of linear elastic isotropic material that minimizes an assigned scalar function for a fixed set of constraints

"Void" "Full material"

$$\rho_{min} \leq \rho(\chi) \leq 1$$

Function representing the **continuous** material density, i.e. the minimization unknown

$$C_{ijhk}(\rho(\chi)) = \rho(\chi)^p C^0_{ijhk}$$

$$\downarrow$$

Elasticity tensor of the given isotropic material ("full material")

SIMP: Solid Isotropic Microstructure with Penalty i.e. the material interpolation scheme to represent the constitutive tensor depending on *ρ*

*p>1* to penalize intermediate densities and achieve a pure 0-1 design

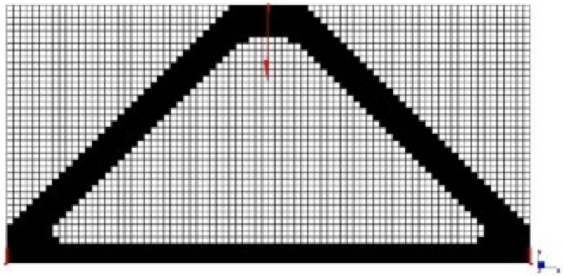
The elastic problem depends on $\rho(\chi)$ : find $\underline{u} \in H^1$ such that $\underline{u}\,|_{\Gamma_u} = \underline{u}_0$ and

$$\int_{\Omega} \rho^p C^0_{ijkl} \varepsilon_{ij}(\underline{u}) \varepsilon_{kl}(\underline{v})\ d\Omega = \int_{\Gamma_t} \underline{t}_0 \cdot \underline{v}\ d\Gamma \quad \forall \underline{v} \in H^1$$

# Governing equations:
# SIMP-based structural compliance

A classical scheme for the discretization of the above problem adopts the same mesh of four-node elements and a two-field interpolation with:

- A piecewise constant density discretization (with unknowns **x**)
- A bilinear displacement approximation (with unknowns **U**)



$$\mathbf{K}(\mathbf{x})\,\mathbf{U} \;=\sum_{e=1}^{N} x_e^p \mathbf{K}_e^0 \mathbf{U}_e \;=\; \mathbf{F}$$

*topology optimization → sizing optimization*

- Compliance: - work of the external loads at equilibrium (Clapeyron th.)
  - measure of the structural deformability

$$\mathcal{C} = \int_\Omega \rho_h^p C_{ijkl}^0 \varepsilon_{ij}(\underline{u}_h)\varepsilon_{kl}(\underline{u}_h)\, d\Omega \;=\; \mathbf{U}^T\mathbf{K}\mathbf{U} = \sum_{e=1}^{N} x_e^p \mathbf{U}_e^T \mathbf{K}_e^0 \mathbf{U}_e,$$

**U**$_e$: element displacements vector

**K**$_e{}^0$: element stiffness matrix for virgin material

# Problem formulation:
# classical formulation for maximum stiffness

**MCW** **M**inimum **C**ompliance with **W**eight (volume) constraint (Bendsøe and Kikuchi, 88)

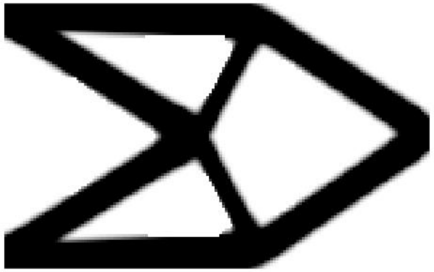
Given a domain with assigned loads and boundary conditions, find the distribution of an amount of linear elastic isotropic material that minimizes the compliance

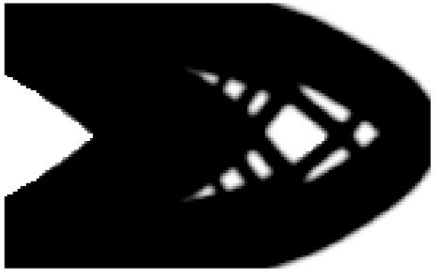
$$\left\{ \begin{array}{lll} \displaystyle\min_{x_{min} \le x_e \le 1} & \mathcal{C} = \mathbf{U}^T \mathbf{K} \mathbf{U} = \displaystyle\sum_{e=1}^{N} x_e^p \mathbf{U}_e^T \mathbf{K}_e^0 \mathbf{U}_e & \text{Structural compliance} \\ \text{bound/side constraints} & & \\ \text{s.t.} & \mathbf{K}(\mathbf{x})\ \mathbf{U} \ = \displaystyle\sum_{e=1}^{N} x_e^p \mathbf{K}_e^0 \mathbf{U}_e \ = \ \mathbf{F}, & \text{Governing eqns. elastic problem} \\ & \displaystyle\sum_{e=1}^{N} x_e A_e \Big/ \sum_{e=1}^{N} A_e \ \le \ V_f, & \text{Volume constraint} \end{array} \right.$$

- For low $V_f$, truss-like structures arise / for high $V_f$, beam with optimal openings are found

$V_f = 0.4$

$C/C_0 = 2.1$



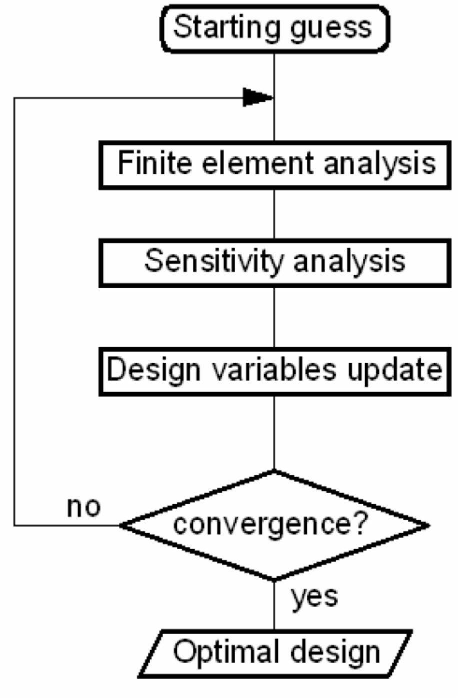


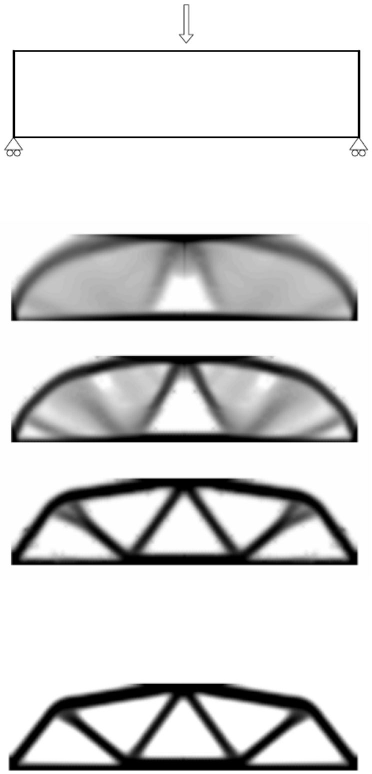
$V_f = 0.8$

$C/C_0 = 1.1$

# Solution of the minimization problem: gradient-based minimization

- Nested Analysis and Design Approach (NAND): only the structural design variable is treated as the optimization variable





Iterative algorithms:

macrostructural non-gradient approaches, in general heuristic methods: evolutionary approaches, fully stressed design method…

gradient-based approaches:
- optimality criteria
- mathematical programming
  (MMA, CONLIN…)

↓

sensitivity analysis

# Sensitivity analysis: fundamentals

- At each iteration, values and derivatives with respect to $x_k$ are computed for the objective function and the volume

$$\mathcal{C} = \mathbf{F}^T \mathbf{U}(\mathbf{x})$$ where **U** solves $\mathbf{K}\mathbf{U} = \mathbf{F}$

Assuming that **F** does not depend on $x_k$ one has: $\dfrac{\partial \mathcal{C}}{\partial x_k} = \mathbf{F}^T \dfrac{\partial \mathbf{U}}{\partial x_k}$

Using the chain rule on the equilibrium equation one has:

$$\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U} + \mathbf{K}\frac{\partial \mathbf{U}}{\partial x_k} = \mathbf{0} \qquad \rightarrow \qquad \mathbf{K}\frac{\partial \mathbf{U}}{\partial x_k} = -\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U}$$

$$\mathcal{V} = \sum_{e=1}^{N} x_e A_e \qquad \frac{\partial \mathcal{V}}{\partial x_k} = A_k$$

- 1+N systems of equations must be solved, per step

# Sensitivity analysis: the adjoint method

- Derivatives with respect to $x_k$ for the objective function can be computed more efficiently through the adjoint method

o.f. is re-written adding a «zero function», with $\tilde{\mathbf{U}}$ arbitrary but fixed real vector

$$\mathcal{C} = \mathbf{F}^T\mathbf{U} - \tilde{\mathbf{U}}^T(\mathbf{K}\mathbf{U} - \mathbf{F})$$

$$\frac{\partial \mathcal{C}}{\partial x_k} = \mathbf{F}^T\frac{\partial \mathbf{U}}{\partial x_k} - \tilde{\mathbf{U}}^T(\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U} + \mathbf{K}\frac{\partial \mathbf{U}}{\partial x_k})$$

$$\frac{\partial \mathcal{C}}{\partial x_k} = (\mathbf{F}^T - \tilde{\mathbf{U}}^T\mathbf{K})\frac{\partial \mathbf{U}}{\partial x_k} - \tilde{\mathbf{U}}^T\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U}$$

that can in turn be written as:

$$\frac{\partial \mathcal{C}}{\partial x_k} = -\tilde{\mathbf{U}}^T\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U}$$ where $\tilde{\mathbf{U}}$ satisfies the adjoint eqn. $\mathbf{F}^T - \tilde{\mathbf{U}}^T\mathbf{K} = \mathbf{0}$

# Sensitivity analysis: the adjoint method

The adjoint equation is in the form of an equilibrium equation (self-adjoint problem)

$$\mathbf{F}^T - \tilde{\mathbf{U}}^T\mathbf{K} = \mathbf{0} \qquad \rightarrow \qquad \mathbf{K}\tilde{\mathbf{U}} \;\; = \mathbf{F} \qquad \rightarrow \qquad \tilde{\mathbf{U}} = \mathbf{U}$$

$$\frac{\partial \mathcal{C}}{\partial x_k} = -\mathbf{U}^T \frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U} = -p x_k^{p-1} \mathbf{U}_k^T \mathbf{K}_k^0 \mathbf{U}_k$$

- only 1 system of equations must be solved, per step (very efficient)
- the sensitivity is “localized”: only information at the element level is involved (the effect of the other variables is “hidden” in $\mathbf{U}_k$)
- the sensitivity is negative for any element: according to physical intuition, additional material in any element decreases compliance, i.e. increases stiffness

# Update of the design variables: optimality criterion

- Heuristic method based on the Lagrangian function of the optimization problem

the o.j. is augmented by the constraints through a set of non-negative multipliers

$$\mathcal{L} = \underset{\substack{\downarrow \\ \text{o.f.}}}{\mathcal{C}} + \lambda\underset{\substack{\downarrow \\ \text{volume con.}}}{(V - V_f V_0)} + \boldsymbol{\lambda}_1^T \underset{\substack{\downarrow \\ \text{equilibrium con.}}}{(\mathbf{KU} - \mathbf{F})} + \sum_{e=1}^{N} \lambda_{2e} \underset{\substack{\downarrow \\ \text{lower bounds}}}{(x_{min} - x_e)} + \sum_{e=1}^{N} \lambda_{3e} \underset{\substack{\downarrow \\ \text{upper bounds}}}{(x_e - 1)}$$

optimality arises when all derivatives of the Lagrangian with respect to $x_k$ are zero

$$\frac{\partial \mathcal{L}}{\partial x_k} = \frac{\partial \mathcal{C}}{\partial x_k} + \lambda \frac{\partial V}{\partial x_k} + \boldsymbol{\lambda}_1^T \frac{\partial \mathbf{KU}}{\partial x_k} - \lambda_{2e} + \lambda_{3e} = 0 \qquad k = 1, 2, \ldots N$$

# Update of the design variables: optimality criterion

Assuming that side constraints are not active and **F** is design-independent:

$$\frac{\partial \mathcal{L}}{\partial x_k} = \frac{\partial \mathcal{C}}{\partial x_k} + \lambda \frac{\partial V}{\partial x_k} + \boldsymbol{\lambda}_1^T \cancel{\frac{\partial \mathbf{K}\mathbf{U}}{\partial x_k}} - \cancel{\lambda_{2e}} + \cancel{\lambda_{3e}} = -p x_k^{p-1} \mathbf{U}_k^T \mathbf{K}_k^0 \mathbf{U}_k + \lambda A_k = 0$$

$$k = 1, 2, \dots N$$

Hence, for intermediate densities, the **strain energy density-like term is constant all over the domain in the optimal solution**. Since areas with high energy are expected to be too low on stiffness, a fix-point type update scheme can be formulated. For intermediate density:

$$x_k^{j+1} = x_k^j \left( \frac{p x_k^{p-1} \mathbf{U}_k^T \mathbf{K}_k^0 \mathbf{U}_k}{\lambda A_k} \right)^{\eta} = x_k^j (B_k^j)^{\eta}$$

Add material where $B_k > 1$ otherwise remove material

↓ updated

↓ current

↓ current =1 at optimum

↓ η is a tuning parameter to stabilize the iteration (0.5 in general)

# Update of the design variables: optimality criterion

A general scheme accounting for side constraints reads:

$$x_k^{j+1} = \begin{cases} \max\{(1-\zeta)x_k^j, x_{min}\} & \text{if } x_k^j (B_k^j)^\eta \leq \max\{(1-\zeta)x_k^j, x_{min}\} \\ \min\{(1+\zeta)x_k^j, 1\} & \text{if } \min\{(1+\zeta)x_k^j, 1\} \leq x_k^j (B_k^j)^\eta \\ x_k^j (B_k^j)^\eta & \text{otherwise} \end{cases}$$

- Add material where $B_k$ >1, otherwise remove material
- This only takes place if the update does not violate the bounds on $x_k$
- A positive move limit ζ (0.2 in general) is introduced to ensure that no big change in relative density arises between two subsequent steps
- The lagrangian multiplier λ is updated in an inner iteration loop using bisection in order to satisfy the active volume constraint

# Update of the design variables: mathematical programming

***Sequential convex programming***: a sequence of explicit sub-problems is used that are convex approximations of the original one

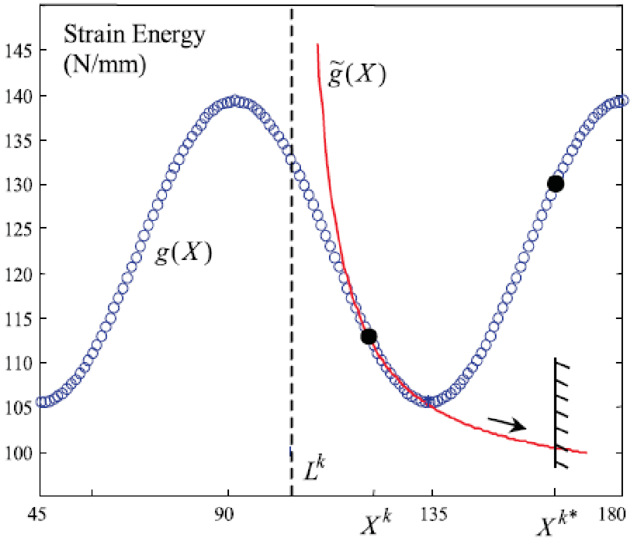
- Sequential linear and quadratic programming techniques attack the problem *without accounting for the specific characteristics of the involved functions*: they generate sub-problems by linearizing both objective function and constraints in the **direct variables**

- MMA (Svanberg, 1987) and CONLIN (Fleury, 1986) linearize objective function and constraints in the direct variables and in the **reciprocal variables**, depending on the sign of the gradient. Such an approximation perfectly suits a broad range of structural optimization problems

# Update of the design variables:
# Method of Moving Asymptotes (MMA)

- MMA (Svanberg, 1987) CONLIN (Fleury, 1986) use a sequence of simpler approximated sub-problems of given type. They are **separable**, **convex** and are constructed on the current sensitivity information as well as some history

$$\tilde{g}_j(\mathrm{x}) = g_j^0 + \sum_{i=1}^{n} \frac{p_{ij}}{U_i - x_i} + \sum_{i=1}^{n} \frac{q_{ij}}{x_i - L_i}$$

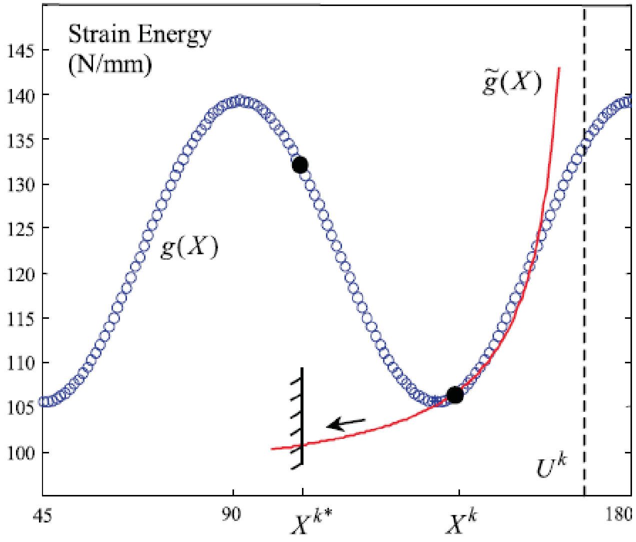


negative sensitivity

$$p_{ij} = \max\{0, (U_i - x_i^0)^2 \frac{\partial g_j}{\partial x_i}\}$$

$$q_{ij} = \max\{0, -(x_i^0 - L_i)^2 \frac{\partial g_j}{\partial x_i}\}$$

$U_i$ and $L_i$ control the range for which the approximation is reasonable, depending on the iteration history so far



positive sensitivity

# Update of the design variables: Method of Moving Asymptotes (MMA)

- separable subproblems → necessary conditions of optimality do not couple the primary variables (the design variables)
- convex approximations → efficient dual methods can be used

For the volume-constrained minimum compliance problem, the sensitivity is negative and an MMA **convex** approximation of the o.f. after the j-th step reads:

$$\mathcal{C}(\mathbf{x}^j) - \sum_{e=1}^{N} \frac{(x_e^j - L_e)^2}{x_e - L_e} \frac{\partial \mathcal{C}}{\partial x_e}(\mathbf{x}^j)$$

A dual method can be used:

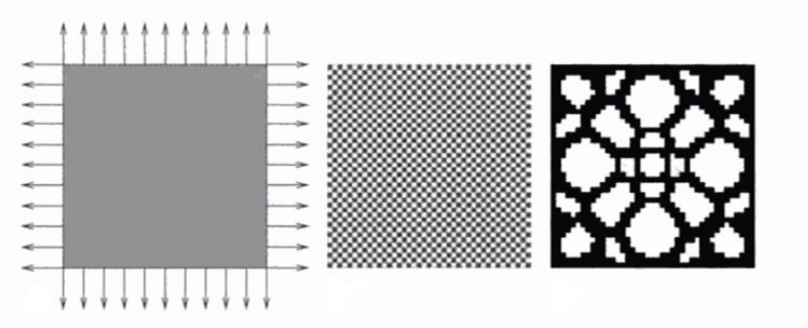
1) the Lagrange functional is minimized **element by element** with respect to all $x_e$
2) then, the resulting functional is maximized with respect to λ

$$\mathcal{L} = \mathcal{C}(\mathbf{x}^j) - \sum_{e=1}^{N} \frac{(x_e^j - L_e)^2}{x_e - L_e} \frac{\partial \mathcal{C}}{\partial x_e}(\mathbf{x}^j) + \lambda \left( \sum_{e=1}^{N} x_e A_e - V_f V_0 \right)$$

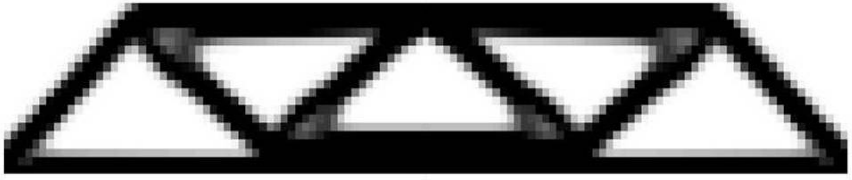
Flexibility + Excellent performance in case of a limited number of constraints

# Numerical issues

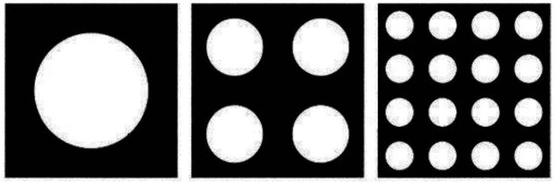
- Checkerboard patterns and mesh dependence (Sigmund and Petersson, 98)



Checkboarded layouts are optimal solutions from a mathematical point of view, but they are not feasible from a physical point of view (this depends on the adopted FEM and density discretization)



Mesh dependence: different solutions arise for different meshes (this is the discrete counterpart of not well-posedness)



# Numerical issues: filters

- Filtered sensitivities of the objective function can be successfully used to prevent numerical instabilities, i.e. mesh dependence and checkerboard patterns

$$\frac{\widetilde{\partial c}}{\partial x_e} = \frac{1}{\sum_N H_{el}} \sum_N H_{el} x_l \frac{\partial c}{\partial x_l}, \qquad H_{el} = \sum_N \max(0, r_{min} - \text{dist}(e,l))$$

*e*-th filtered sensitivity

*l*-th unfiltered sensitivity

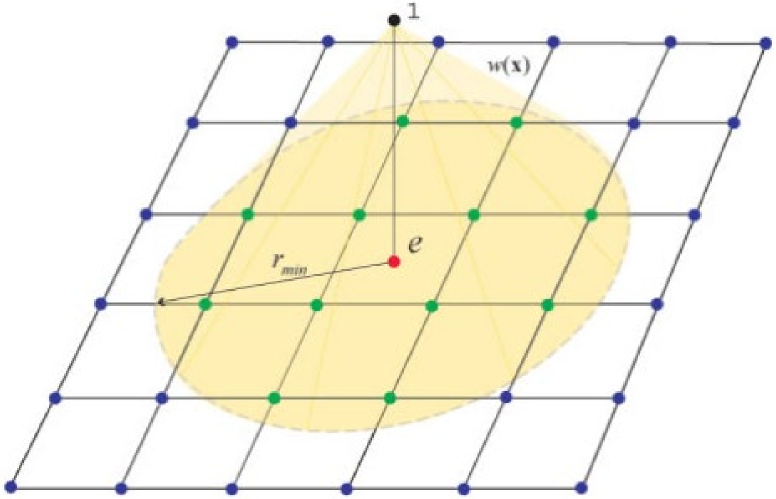
filter radius

distance between elements

Being $d_m$ the reference size of the mesh of finite elements:

$r_{min} = 1.5\ d_m$ is the minimum value to avoid the arising of undesired checkerboard patterns

$r_{min} > 1.5\ d_m$ provides control on the minimum thickness of any member of the design



# Numerical issues: filters

- Alternatively, filtered densities can be easily implemented for increased robustness

$$\underset{\substack{\downarrow \\ e\text{-th physical unknown}}}{\tilde{x}_e} = \frac{1}{\sum_N H_{el}} \sum_N H_{el} \underset{\substack{\downarrow \\ l\text{-th design variable}}}{x_l}, \qquad H_{el} = \sum_N \max(0, \underset{\substack{\downarrow \\ \text{filter radius}}}{r_{min}} - \underset{\substack{\downarrow \\ \text{distance between elements}}}{\text{dist}(e,l)})$$

The chain rule is needed to compute the derivatives of o.f. and constraints: $\frac{\partial \psi}{\partial x_k} = \sum_N \frac{\partial \psi}{\partial \tilde{x}_l} \frac{\partial \tilde{x}_l}{\partial x_k}$

- Heaviside projection filters can be implemented to reduce blurred edges in case of big $r_{min}$

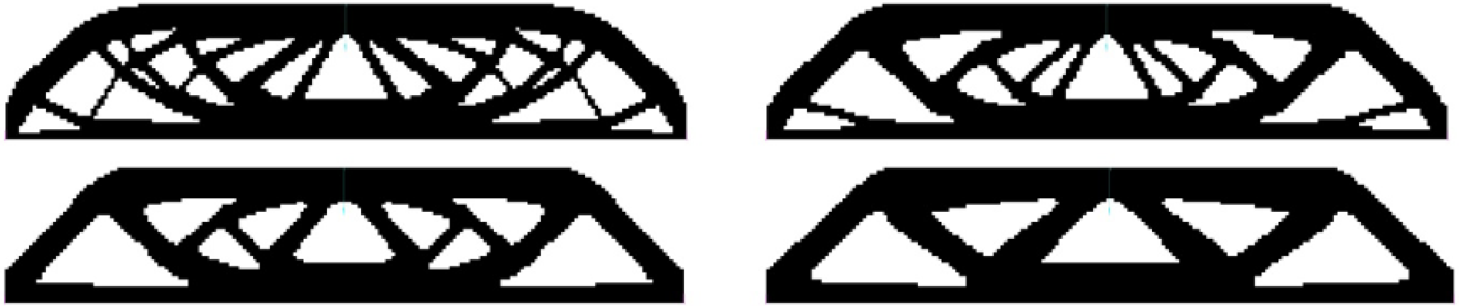
$$\underset{\substack{\downarrow \\ e\text{-th intermediate variable}}}{\tilde{x}_e} = \frac{1}{\sum_N H_{el}} \sum_N H_{el} \underset{\substack{\downarrow \\ l\text{-th design variable}}}{x_l}, \qquad \underset{\substack{\downarrow \\ e\text{-th physical unknown}}}{\overline{x}_e} = 1 - e^{-\beta \tilde{x}_e} + \underset{\substack{\downarrow \\ e\text{-th intermediate variable}}}{\tilde{x}_e} e^{-\beta}$$

β → ∞ Heaviside step function

A continuation approch is needed starting the first optimization from large values of β: high CPU cost

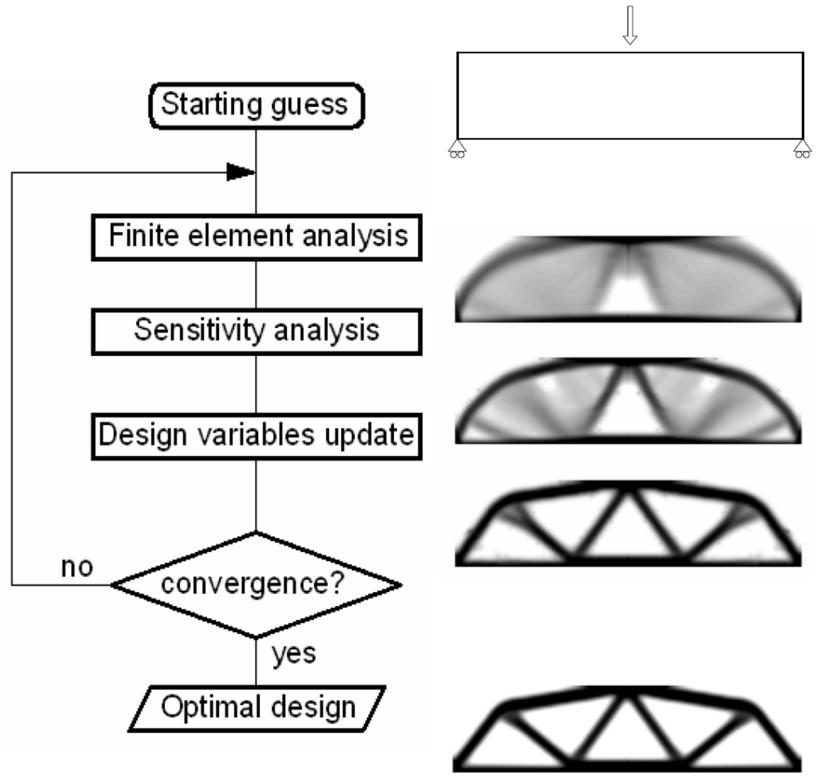
# Numerical issues: filters

Optimal layouts using the same filtering radius $r_{min}$ and finer meshes: density filter vs. HS projection filter



Optimal layouts using increasing values of the filtering radius $r_{min}$ *on the* same mesh: HS projection filter

# Other numerical issues



- What about uniqueness of the solution?

Most problems are non-convex → multiple solutions arise (one big bar/many small bars under uni-axial tension)

- What about local minima?

Most problems are non-convex → many local minima arise (multi-start procedures along with globally convergent algorithms can be used, but there is no guarantee of global optimality)

- How to choose the starting guess?

$x_e=V_f$ or $x_e=1$ are generally used to start the minimization all over the domain, no matter for their feasibility with respect to the volume constraint

- How to choose the convergence criterion?

In general, check on the maximum change in density / o.f. between two subsequent steps

# Efficient implementation in Matlab

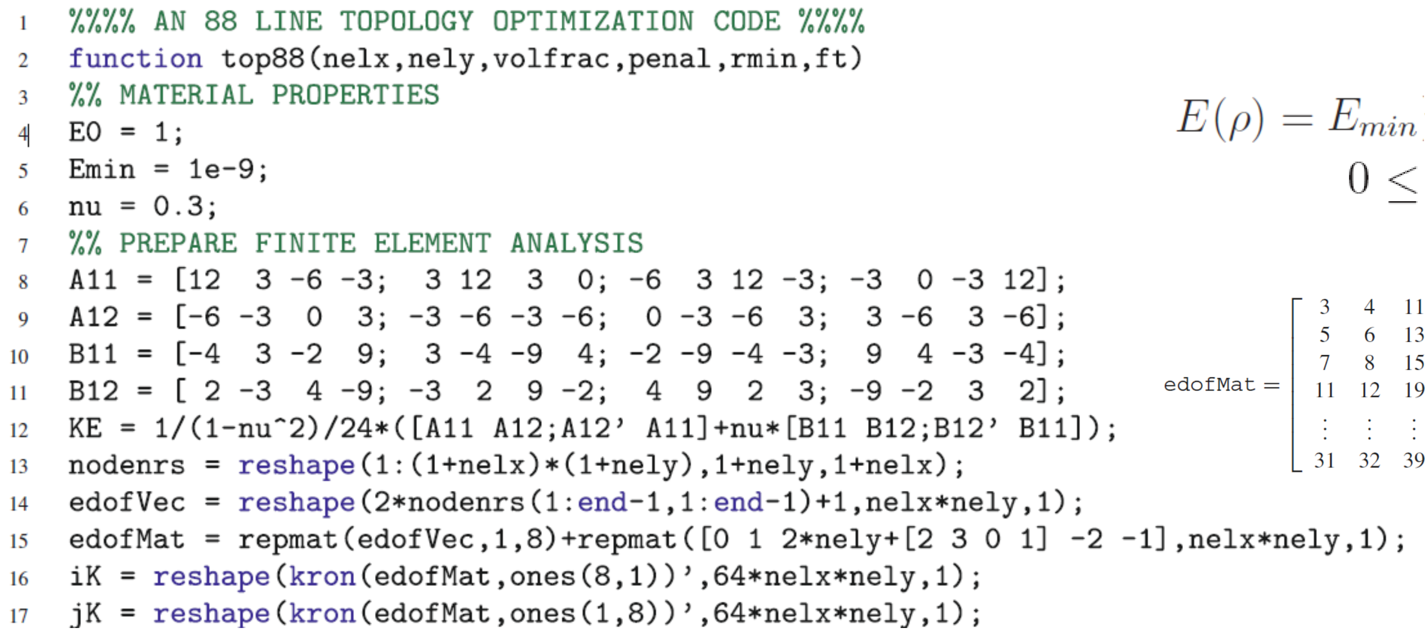
Efficient topology optimization in MATLAB using 88 lines of code (Andreassen et al, 2011)

It exploits a mesh of *nelx* x *nely* four-node finite elements with $d_m=1$. For each element, one density unknown.

It allows for two filtering techniques (ft=1/2 sensitivity/density filter)

$$E(\rho) = E_{min} + \rho^p (E^0 - E_{min})$$

$$0 \leq \rho(\chi) \leq 1$$

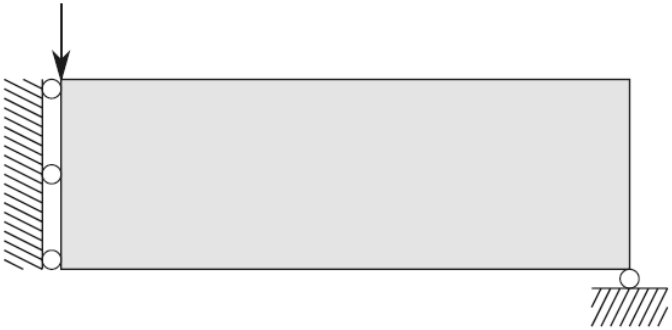


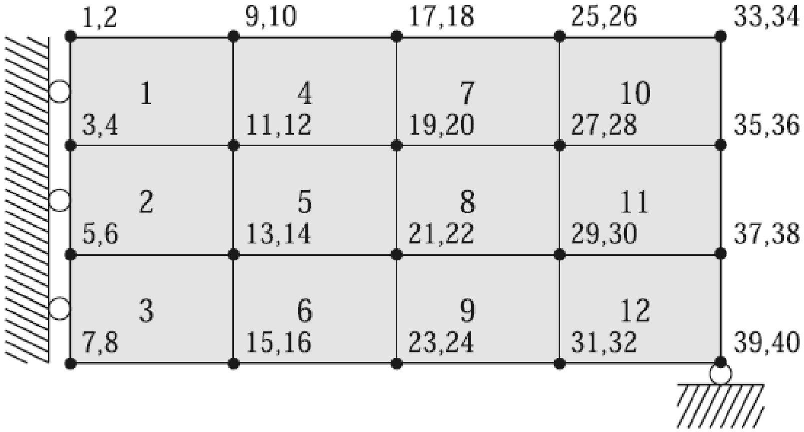
```
%%%% AN 88 LINE TOPOLOGY OPTIMIZATION CODE %%%%
function top88(nelx,nely,volfrac,penal,rmin,ft)
%% MATERIAL PROPERTIES
E0 = 1;
Emin = 1e-9;
nu = 0.3;
%% PREPARE FINITE ELEMENT ANALYSIS
A11 = [12  3 -6 -3;  3 12  3  0; -6  3 12 -3; -3  0 -3 12];
A12 = [-6 -3  0  3; -3 -6 -3 -6;  0 -3 -6  3;  3 -6  3 -6];
B11 = [-4  3 -2  9;  3 -4 -9  4; -2 -9 -4 -3;  9  4 -3 -4];
B12 = [ 2 -3  4 -9; -3  2  9 -2;  4  9  2  3; -9 -2  3  2];
KE = 1/(1-nu^2)/24*([A11 A12;A12' A11]+nu*[B11 B12;B12' B11]);
nodenrs = reshape(1:(1+nelx)*(1+nely),1+nely,1+nelx);
edofVec = reshape(2*nodenrs(1:end-1,1:end-1)+1,nelx*nely,1);
edofMat = repmat(edofVec,1,8)+repmat([0 1 2*nely+[2 3 0 1] -2 -1],nelx*nely,1);
iK = reshape(kron(edofMat,ones(8,1))',64*nelx*nely,1);
jK = reshape(kron(edofMat,ones(1,8))',64*nelx*nely,1);
```

$$\texttt{edofMat} = \left[\begin{array}{cccccccc} 3 & 4 & 11 & 12 & 9 & 10 & 1 & 2 \\ 5 & 6 & 13 & 14 & 11 & 12 & 3 & 4 \\ 7 & 8 & 15 & 16 & 13 & 14 & 5 & 6 \\ 11 & 12 & 19 & 20 & 17 & 18 & 9 & 10 \\ \vdots & \vdots & \vdots & \vdots & & & & \\ 31 & 32 & 39 & 40 & 37 & 38 & 29 & 30 \end{array}\right] \begin{array}{l} \leftarrow \text{Element 1} \\ \leftarrow \text{Element 2} \\ \leftarrow \text{Element 3} \\ \leftarrow \text{Element 4} \\ \\ \leftarrow \text{Element 12} \end{array}$$

# Efficient implementation in Matlab

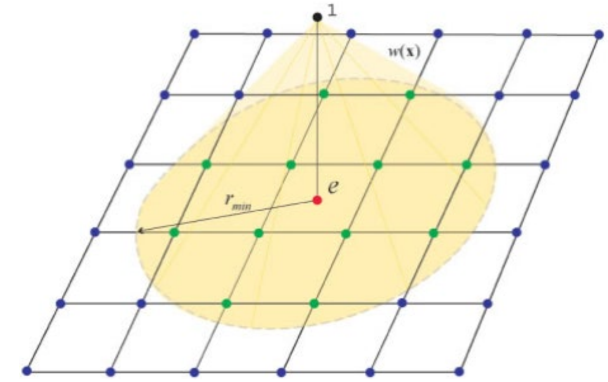
```
% DEFINE LOADS AND SUPPORTS (HALF MBB-BEAM)
F = sparse(2,1,-1,2*(nely+1)*(nelx+1),1);
U = zeros(2*(nely+1)*(nelx+1),1);
fixeddofs = union([1:2:2*(nely+1)],[2*(nelx+1)*(nely+1)]);
alldofs = [1:2*(nely+1)*(nelx+1)];
freedofs = setdiff(alldofs,fixeddofs);
```





# Efficient implementation in Matlab

```
%% PREPARE FILTER
iH = ones(nelx*nely*(2*(ceil(rmin)-1)+1)^2,1);
jH = ones(size(iH));
sH = zeros(size(iH));
k = 0;
for i1 = 1:nelx
  for j1 = 1:nely
    e1 = (i1-1)*nely+j1;
    for i2 = max(i1-(ceil(rmin)-1),1):min(i1+(ceil(rmin)-1),nelx)
      for j2 = max(j1-(ceil(rmin)-1),1):min(j1+(ceil(rmin)-1),nely)
        e2 = (i2-1)*nely+j2;
        k = k+1;
        iH(k) = e1;
        jH(k) = e2;
        sH(k) = max(0,rmin-sqrt((i1-i2)^2+(j1-j2)^2));
      end
    end
  end
end
H = sparse(iH,jH,sH);
Hs = sum(H,2);
%% INITIALIZE ITERATION
x = repmat(volfrac,nely,nelx);
xPhys = x;
loop = 0;
change = 1;
```

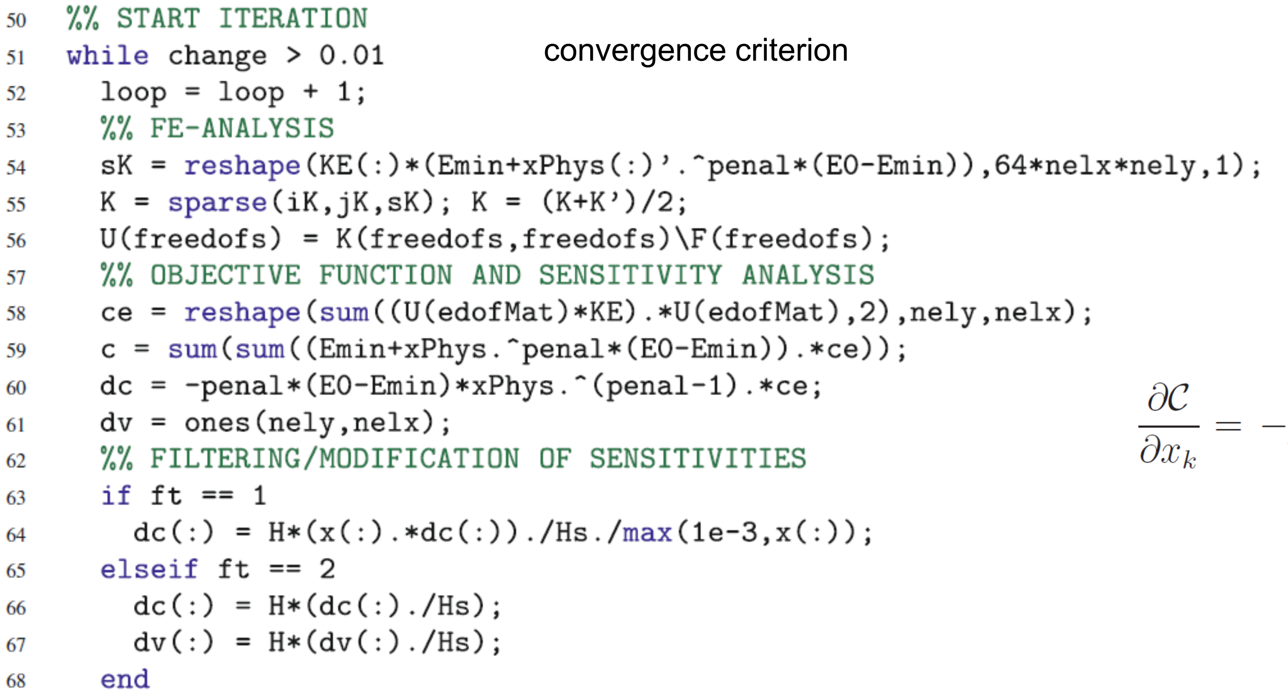


$$H_{el} = \sum_N \max(0, r_{min} - \mathrm{dist}(e,l))$$

x design unknown

xPhys physical unknown

# Efficient implementation in Matlab



```
%% START ITERATION
while change > 0.01
  loop = loop + 1;
  %% FE-ANALYSIS
  sK = reshape(KE(:)*(Emin+xPhys(:)'.^penal*(E0-Emin)),64*nelx*nely,1);
  K = sparse(iK,jK,sK); K = (K+K')/2;
  U(freedofs) = K(freedofs,freedofs)\F(freedofs);
  %% OBJECTIVE FUNCTION AND SENSITIVITY ANALYSIS
  ce = reshape(sum((U(edofMat)*KE).*U(edofMat),2),nely,nelx);
  c = sum(sum((Emin+xPhys.^penal*(E0-Emin)).*ce));
  dc = -penal*(E0-Emin)*xPhys.^(penal-1).*ce;
  dv = ones(nely,nelx);
  %% FILTERING/MODIFICATION OF SENSITIVITIES
  if ft == 1
    dc(:) = H*(x(:).*dc(:))./Hs./max(1e-3,x(:));
  elseif ft == 2
    dc(:) = H*(dc(:)./Hs);
    dv(:) = H*(dv(:)./Hs);
  end
```

convergence criterion

$$\mathbf{K}(\mathbf{x})\ \mathbf{U} \ = \sum_{e=1}^{N} x_e^p \mathbf{K}_e^0 \mathbf{U}_e \ = \ \mathbf{F}$$

$$\mathcal{C} = \mathbf{U}^T \mathbf{K} \mathbf{U} = \sum_{e=1}^{N} x_e^p \mathbf{U}_e^T \mathbf{K}_e^0 \mathbf{U}_e$$

$$\frac{\partial \mathcal{C}}{\partial x_k} = -p x_k^{p-1} \mathbf{U}_k^T \mathbf{K}_k^0 \mathbf{U}_k \qquad \frac{\partial \mathcal{V}}{\partial x_k} = A_k$$

# Efficient implementation in Matlab

```
  %% OPTIMALITY CRITERIA UPDATE OF DESIGN VARIABLES AND PHYSICAL DENSITIES
  l1 = 0; l2 = 1e9; move = 0.2;
  while (l2-l1)/(l1+l2) > 1e-3
    lmid = 0.5*(l2+l1);
    xnew = max(0,max(x-move,min(1,min(x+move,x.*sqrt(-dc./dv/lmid)))));
    if ft == 1
      xPhys = xnew;
    elseif ft == 2
      xPhys(:) = (H*xnew(:))./Hs;
    end
    if sum(xPhys(:)) > volfrac*nelx*nely, l1 = lmid; else l2 = lmid; end
  end
  change = max(abs(xnew(:)-x(:)));
  x = xnew;
  %% PRINT RESULTS
  fprintf(' It.:%5i Obj.:%11.4f Vol.:%7.3f ch.:%7.3f\n',loop,c, ...
    mean(xPhys(:)),change);
  %% PLOT DENSITIES
  colormap(gray); imagesc(1-xPhys); caxis([0 1]); axis equal; axis off; drawnow;
end
```

inner loop in the o.c. update to enforce the volume constraint through the multiplier λ

$$\tilde{x}_e = \frac{1}{\sum_N H_{el}} \sum_N H_{el} x_l,$$

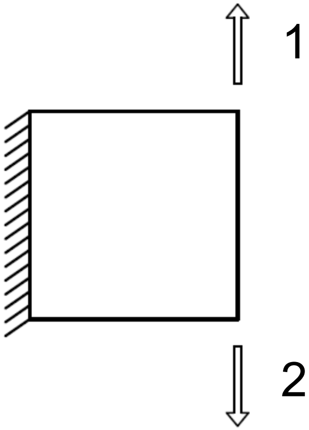
```
xPhys(passive==1) = 0;
xPhys(passive==2) = 1;
```

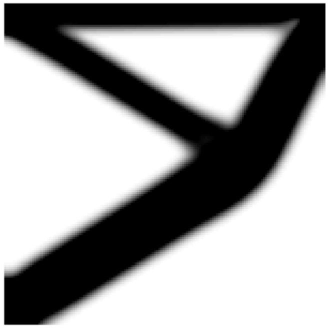
to enforce black/white regions

# A variation of the theme: multiple loads

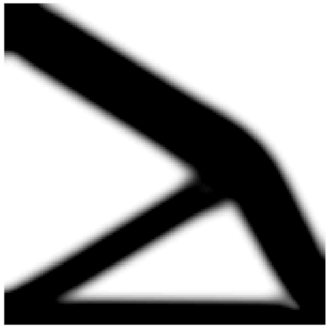
- Optimal design for more than one load case (M) can be achieved working with the sum of the relevant strain energies

$$\begin{cases} \min\limits_{x_{min} \le x_e \le 1} & \mathcal{C} = \sum\limits_{i=1}^{M} \mathbf{U}_i^T \mathbf{K} \mathbf{U}_i = \sum\limits_{i=1}^{M} \sum\limits_{e=1}^{N} x_e^p \mathbf{U}_{i,e}^T \mathbf{K}_e^0 \mathbf{U}_{i,e} \\ \text{s.t.} & \mathbf{K}(\mathbf{x})\ \mathbf{U}_i = \sum\limits_{e=1}^{N} x_e^p \mathbf{K}_e^0 \mathbf{U}_{i,e} = \mathbf{F}_i, \quad i = 1 \ldots M \\ & \sum\limits_{e=1}^{N} x_e A_e \Big/ \sum\limits_{e=1}^{N} A_e \le V_f, \end{cases}$$



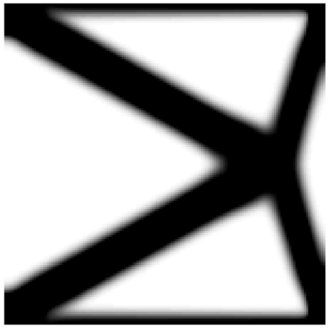


only 1



only 2

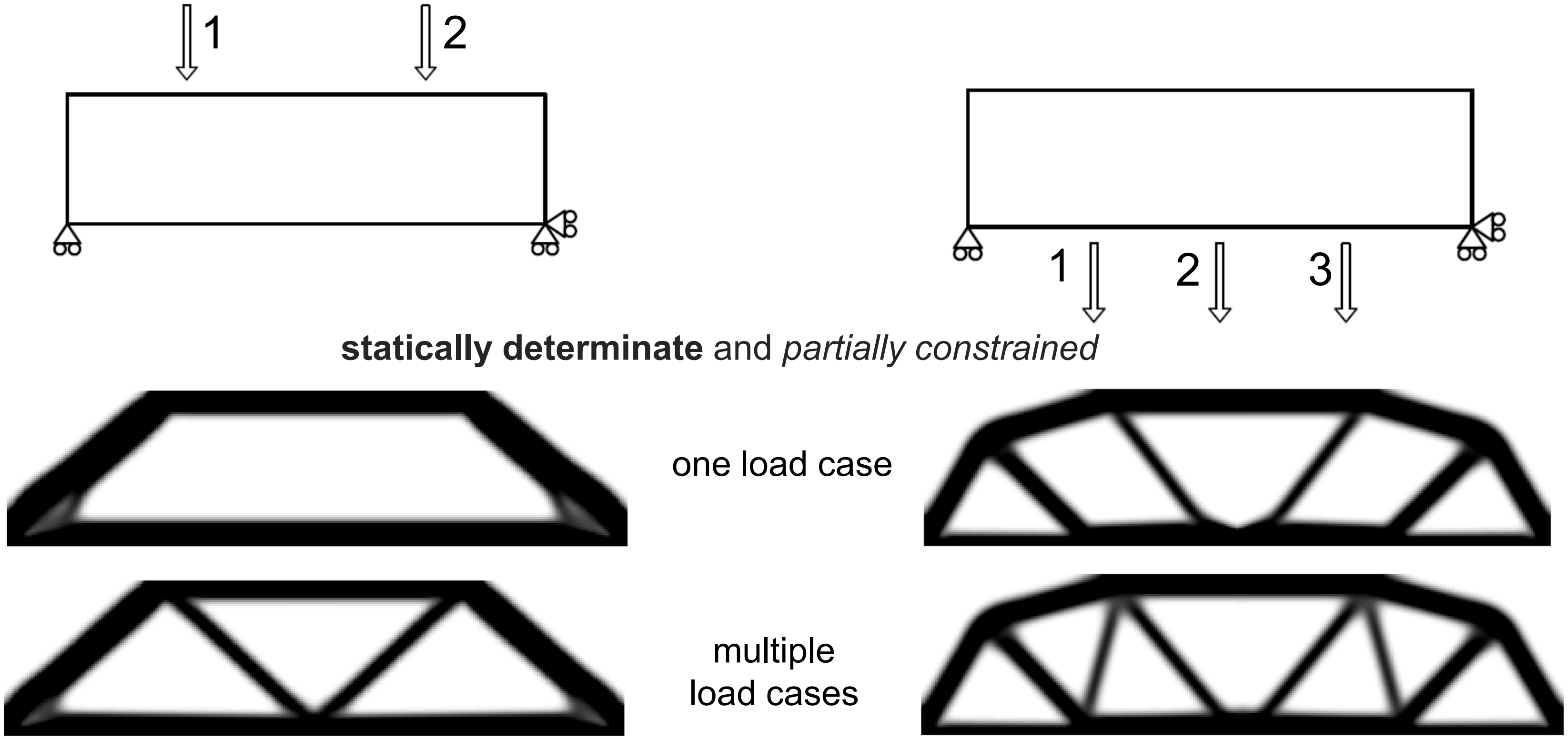
1 + 2 (1 load case)



1 and 2 (2 load cases)

# A variation of the theme: multiple loads

- Multiple load cases can be implemented to improve robustness of the layouts



**statically determinate** and *partially constrained*

one load case

multiple
load cases

**statically determinate** and *completely constrained*

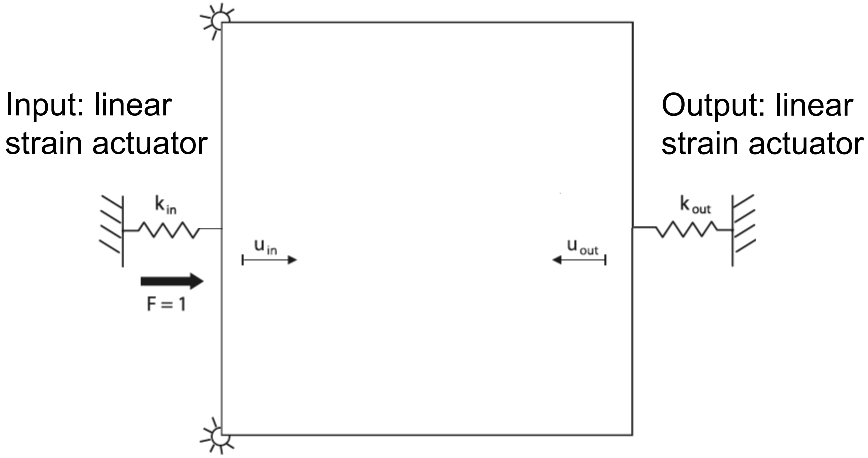
Load condition can be probabilistic to account for several uncertainties (Lógó 2013)

# A variation of the theme: compliant mechanisms

- Design of flexible (micro-)mechanisms that transfer an input force or displacement to another point through elastic body deformation

Given a domain with assigned loads and boundary conditions, find the distribution of a prescribed amount of linear elastic isotropic material that maximizes the output displacement:

$$\begin{cases} \max\limits_{x_{min} \leq x_e \leq 1} \quad u_{out} \\ \text{s.t.} \quad \underbrace{[\mathbf{K}(\mathbf{x})}_{\text{design domain}} + \underbrace{\mathbf{K}_s]}_{\text{springs}} \ \mathbf{U} \ = \ \mathbf{F}, \\ \qquad \sum_{e=1}^{N} x_e A_e \Big/ \sum_{e=1}^{N} A_e \ \leq \ V_f \end{cases}$$



Inverter

# Sensitivity analysis: the adjoint method

- Derivatives with respect to $x_k$ for the o.f. are computed via the adjoint method

o.f. is re-written adding a «zero function», with $\boldsymbol{\lambda}$ arbitrary but fixed real vector

$$u_{out} = \mathbf{L}^T\mathbf{U} - \boldsymbol{\lambda}^T\left([\mathbf{K}(\mathbf{x}) + \mathbf{K}_s]\ \mathbf{U} - \mathbf{F}\right)$$

$$\frac{\partial u_{out}}{\partial x_k} = \mathbf{L}^T\frac{\partial \mathbf{U}}{\partial x_k} - \ \boldsymbol{\lambda}^T(\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U} + [\mathbf{K}(\mathbf{x}) + \mathbf{K}_s]\frac{\partial \mathbf{U}}{\partial x_k})$$

$$\frac{\partial u_{out}}{\partial x_k} = (\mathbf{L}^T - \boldsymbol{\lambda}^T[\mathbf{K}(\mathbf{x}) + \mathbf{K}_s])\frac{\partial \mathbf{U}}{\partial x_k} - \boldsymbol{\lambda}^T\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U}$$

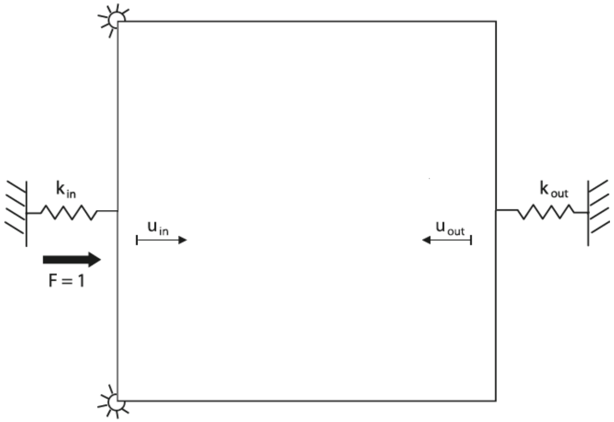
that can in turn be written as:

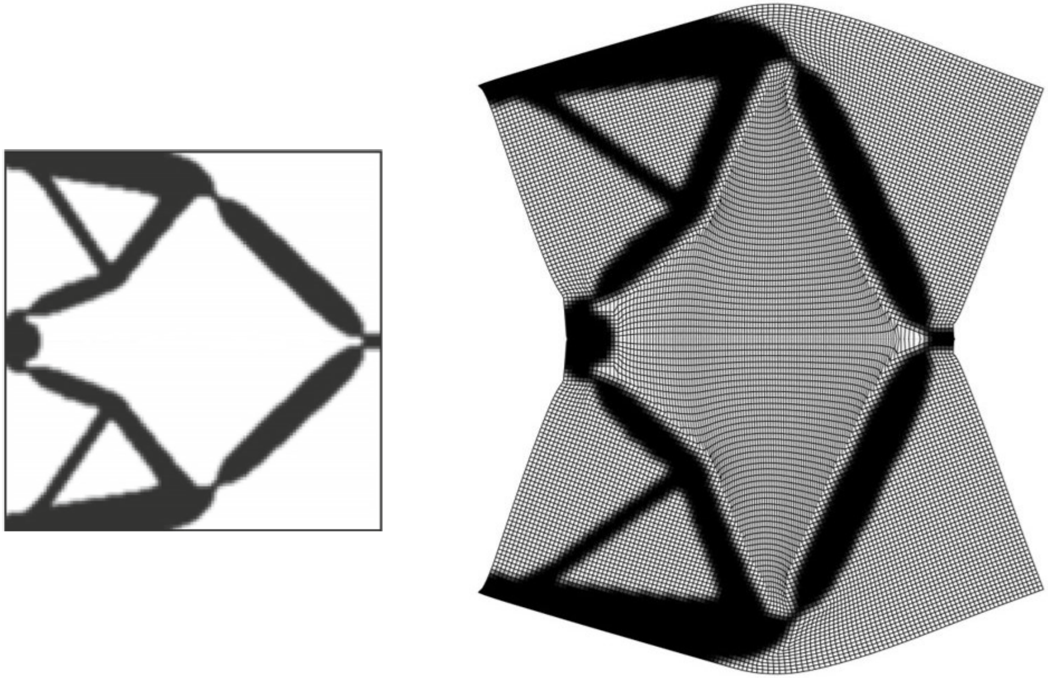
$$\frac{\partial u_{out}}{\partial x_k} = -\boldsymbol{\lambda}^T\frac{\partial \mathbf{K}}{\partial x_k}\mathbf{U} = -px_k^{p-1}\boldsymbol{\lambda}_k^T\mathbf{K}_k^0\mathbf{U}_k$$

(not a self-adjoint problem)

where $\boldsymbol{\lambda}$ satisfies the adjoint eqn. $[\mathbf{K}(\mathbf{x}) + \mathbf{K}_s]\boldsymbol{\lambda} \ \ = \mathbf{L}$

# A variation of the theme: compliant mechanisms

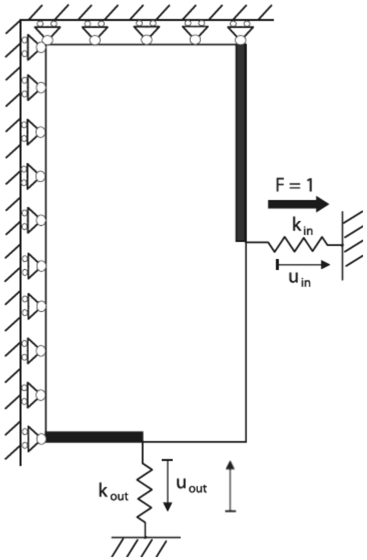


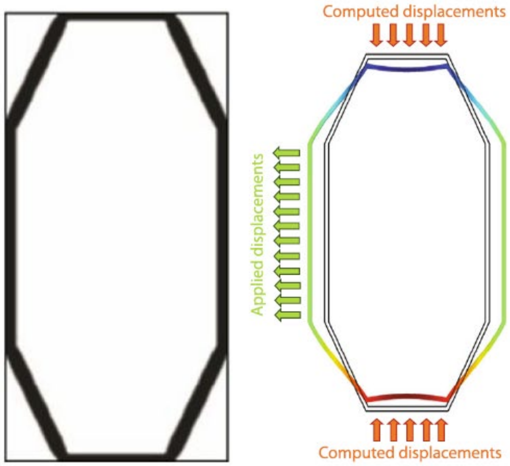


- Control of “hinge“ dimension in «lumped» mechanisms
- Need for geometric non-linear models

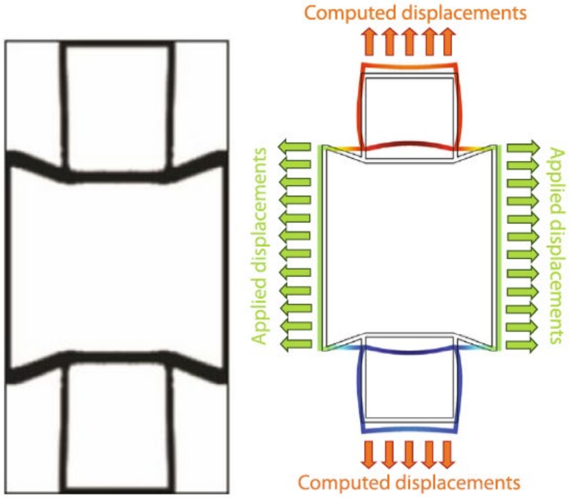
# A variation of the theme: compliant mechanisms

- Design of «distributed» compliant mechanism (structures with no internal hinge)



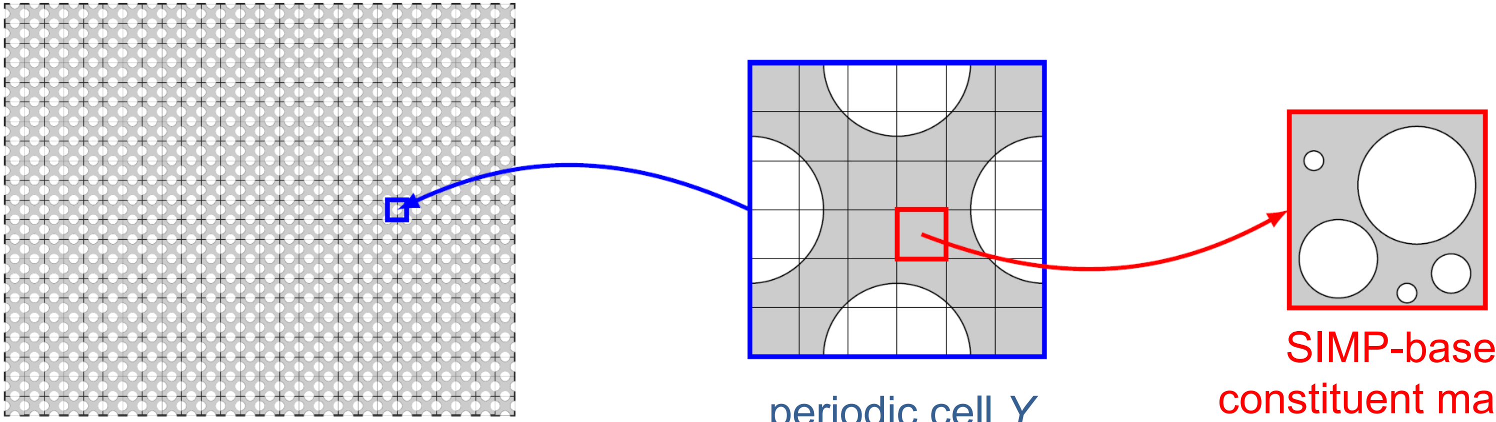


non-auxetic structure



auxetic structure

# A variation of the theme: periodic microstructures



**Macroscale:**
Structural level

**Microscale:**
RVE level

**Constituent material:**
Porous SIMP material

periodic cell $Y$

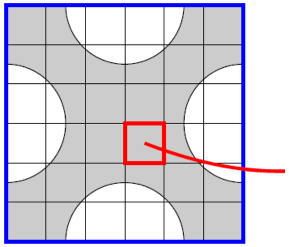
SIMP-based constituent material

$$\mathbf{E}^H = \begin{bmatrix} E_{11}^H & E_{12}^H & E_{13}^H \\ & E_{22}^H & E_{23}^H \\ \text{syms} & & E_{33}^H \end{bmatrix}$$

Define the homogenized costitutive tensor starting from the varying local constitutive tensor

# A variation of the theme: periodic microstructures

- Homogenization and periodic boundary conditions



periodic cell *Y*

SIMP-based constituent material

$$\mathbf{E}^H = \begin{bmatrix} E^H_{11} & E^H_{12} & E^H_{13} \\ & E^H_{22} & E^H_{23} \\ \text{syms} & & E^H_{33} \end{bmatrix}$$

Define the homogenized costitutive tensor starting from the varying local constitutive tensor

- Unit strains are enforced at the boundaries of the cell to compute the so-called SIMP-based mutual energies:

$$E^H_{ijkl} = \frac{1}{|Y|} \int_Y C_{pqrs} \varepsilon^{A(ij)}_{pq} \varepsilon^{A(kl)}_{rs} \, dY$$

$(11 \rightarrow 1), \boldsymbol{\varepsilon}^{0(1)}$

$(22 \rightarrow 2)\ \boldsymbol{\varepsilon}^{0(2)}$

$(12 = 21 \rightarrow 3)\ \boldsymbol{\varepsilon}^{0(3)}$

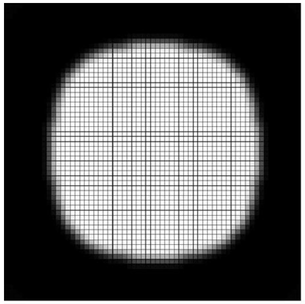
Unit strains in 2D

$$E^H_{ij} = \frac{1}{|Y|} \sum_e^N x^p_e (\mathbf{U}^{A(i)}_e)^T \mathbf{K}_{0e} \mathbf{U}^{A(j)}_e$$

# A variation of the theme: periodic microstructures

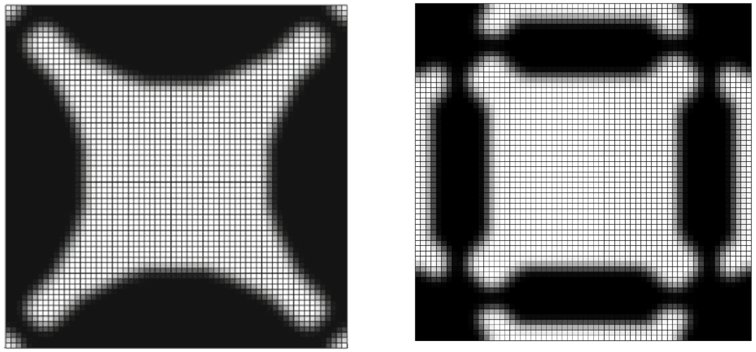
$$\max E_{11}^H + E_{22}^H \ \text{s.t.} \ \frac{1}{V}\sum_N x_e V_e \le V_f$$

Max bulk modulus



$$\begin{cases} \min\limits_{x_{min}<x_e<1} f\left(E_{ij}^H(\mathbf{U}^{A(i)},\mathbf{U}^{A(j)})\right) \\ \text{s.t.} \quad \left[\sum_e^N x_e^p \mathbf{K}_{0e}\right]\mathbf{U}^{A(i)} = \mathbf{F}^{A(i)} \quad i=1,2,3 \\ f(E_{ij}^H) \le \overline{f} \\ \frac{1}{V}\sum_N x_e V_e \le V_f \ . \end{cases}$$

$$\max E_{11}^H + E_{22}^H \ \text{s.t.} \ E_{33}^H \le \overline{f}$$



Max bulk modulus with limited shear modulus

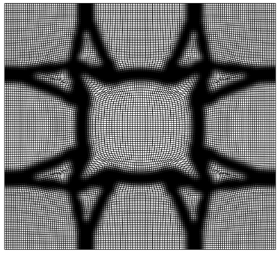
# A variation of the theme: periodic microstructures

- Auxetic microstructures → o.f. $-\varepsilon_2/\varepsilon_1$ , $-\varepsilon_1/\varepsilon_2$

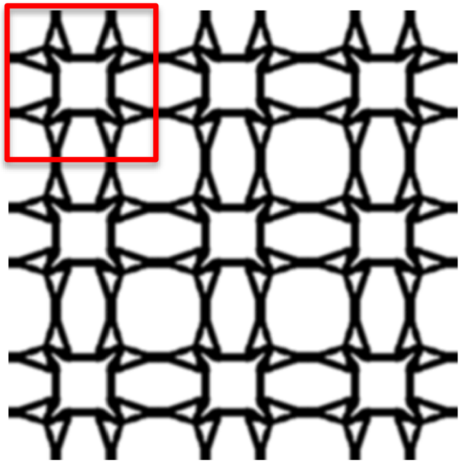
$$\begin{cases} \displaystyle\min_{x_{min}<x_e<1} E_{12}^H - \beta^{iter}(E_{11}^H + E_{22}^H) \\ \text{s.t.} \quad \displaystyle\frac{1}{V}\sum_N x_e V_e \leq V_f \; . \end{cases}$$

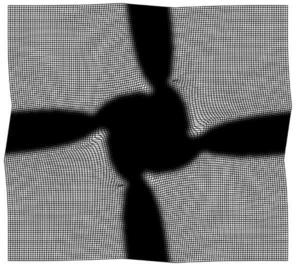
$0<\beta<1$

Highly nonconvex: many (local) solutions!

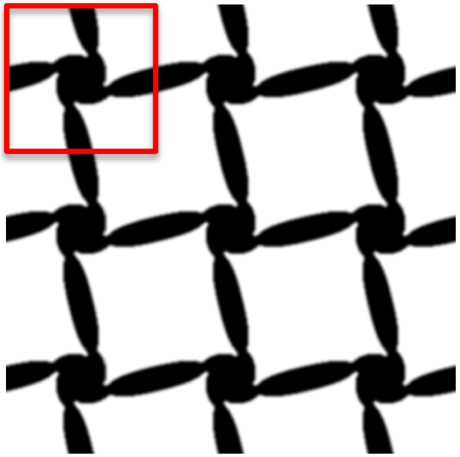


v*=-0.33





v*=-0.71



# Some remarks

- Topology optimization by distribution of isotropic material is a powerful numerical tool to perform **conceptual design** of structures, structural components and materials

- Key ingredients are: adoption of a “material model”, formulation of a constrained minimization problem, implementation of ad hoc/general algorithms for the solution, iterative computation of obj. fun., constraints and sensitivities

- Numerical issues: instabilities, mesh dependence, non uniqueness of the solution, local/global minima

- Very easy implementation for basic problems (design for stiff structures / compliant mechanisms / periodic microstructures)

- Many advanced issues can be dealt with…

# Some references

- Suggested readings

Bendsøe MP, Sigmund O (2003) Topology optimization - Theory, methods and applications. Springer, Berlin

Beghini LL, Beghini A, Katz N, Baker WF, Paulino GH (2014) Connecting architecture and engineering through structural topology optimization. Eng Struct;59:716-726

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Svanberg K (1987) Method of moving asymptotes-a new method for structural optimization. Int J Numer Methods Eng 24(2):359–373

Lógó J (2013) On the optimal layout of structures subjected to probabilistic or multiply loading. Struct Mutltidiscip Opt;48(6):1207-1212.

- Pictures and numerical examples from authored and co-authored papers

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