# The Equations of Motion
# &
# Overview of Numerical Solution Techniques

# THIS PRESENTATION

→ The Equations of Motion

(1) Perfectly **rigid** elements

(2) Elements being deformable because of an internal **FEM mesh**

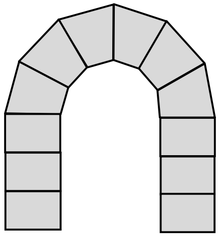
(3) Elements being deformable because of a **uniform strain field**

→ Overview of Numerical Solution Techniques

- The aim
- Initial remarks
- **Euler** method
- Method of **Central Differences**
- **Newmark's** $\beta$ – method

# REMEMBER:

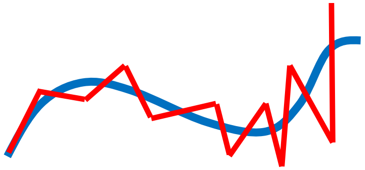
<u>Main steps of the analysis of an engineering problem:</u>



- the model: collection of separate elements ('**discrete elements**')
  {1 body ↔ 1 element} or {several bodies ↔ few elements}
  **<u>Step 1.:</u> define the initial geometry**

- rigid or deformable ***elements***; rigid or deformable ***contacts***
  **<u>Step 2.:</u> specify the material characteristics**

- the loading process:
  ( e.g. external forces acting on the elements; e.g. prescribed displacements)
- calculation of the state changing: ***series of small increments***
  **<u>Step 3.:</u> calculation of the actual displacement increments**



The main techniques:

→ ~~Quasi-static~~ „ ~~**f** = **K**·**u**~~ ”

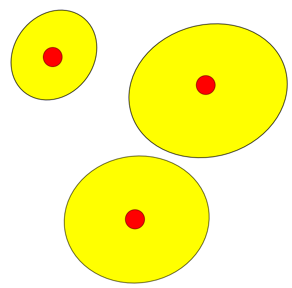
→ Timestepping „ **f** = m·**a** ”
  – explicit
  – implicit

# THE EQUATIONS OF MOTION

Three main types of the elements:

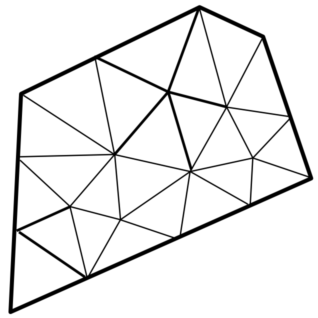
(1) perfectly **rigid** elements

$\rightarrow$ reference point



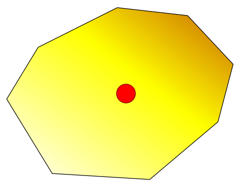
(2) elements being deformable because of an **internal FEM mesh**

$\rightarrow$ nodes



(3) elements being deformable because of a **uniform strain field**

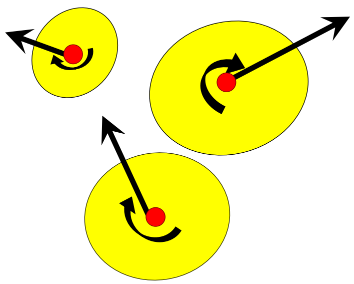
$\rightarrow$ a reference point + a constant strain function



# THE EQUATIONS OF MOTION

„f = ma”

a) Perfectly rigid elements



Reference point

to every element

the displacement vector of the p-th element:

$$\mathbf{u}^p(t) = \begin{bmatrix} u_x^p(t) \\ u_y^p(t) \\ u_z^p(t) \\ \varphi_x^p(t) \\ \varphi_y^p(t) \\ \varphi_z^p(t) \end{bmatrix}$$

total displacement vector:

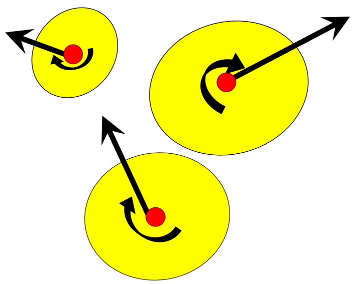
$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}^1(t) \\ \mathbf{u}^2(t) \\ \vdots \\ \mathbf{u}^N(t) \end{bmatrix}$$

summed up of small increments!

# THE EQUATIONS OF MOTION

„f = ma”

a) Perfectly rigid elements



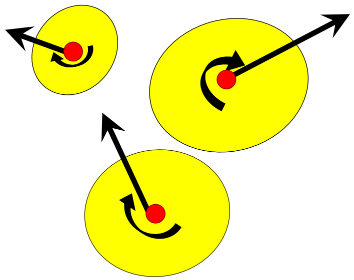
velocity vector: $\mathbf{v}(t) = \dfrac{d\mathbf{u}(t)}{dt}$

pl. $$\mathbf{v}^p(t) = \begin{bmatrix} v_x^p(t) \\ v_y^p(t) \\ v_z^p(t) \\ \omega_x^p(t) \\ \omega_y^p(t) \\ \omega_z^p(t) \end{bmatrix} = \begin{bmatrix} \dfrac{du_x^p(t)}{dt} \\ \dfrac{du_y^p(t)}{dt} \\ \dfrac{du_z^p(t)}{dt} \\ \dfrac{d\varphi_x^p(t)}{dt} \\ \dfrac{d\varphi_y^p(t)}{dt} \\ \dfrac{d\varphi_z^p(t)}{dt} \end{bmatrix}$$

# THE EQUATIONS OF MOTION

„f = ma"

a) Perfectly rigid elements



velocity vector: $\mathbf{v}(t) = \dfrac{d\mathbf{u}(t)}{dt}$

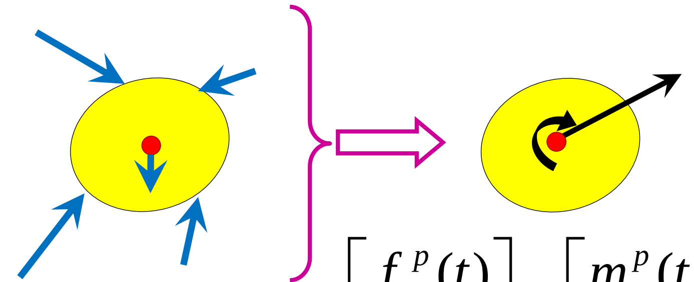
acceleration vector: $\mathbf{a}(t) = \dfrac{d^2\mathbf{u}(t)}{dt^2}$

$$\text{pl. } \mathbf{a}^p(t) = \begin{bmatrix} a_x^p(t) \\ a_y^p(t) \\ a_z^p(t) \\ \beta_x^p(t) \\ \beta_y^p(t) \\ \beta_z^p(t) \end{bmatrix} = \begin{bmatrix} \dfrac{d^2 u_x^p(t)}{dt^2} \\ \dfrac{d^2 u_y^p(t)}{dt^2} \\ \dfrac{d^2 u_z^p(t)}{dt^2} \\ \dfrac{d^2 \varphi_x^p(t)}{dt^2} \\ \dfrac{d^2 \varphi_y^p(t)}{dt^2} \\ \dfrac{d^2 \varphi_z^p(t)}{dt^2} \end{bmatrix}$$

# THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Equations of motion of the *p*-th element:



$$\begin{bmatrix} f_x^p(t) \\ f_y^p(t) \\ f_z^p(t) \end{bmatrix} ; \begin{bmatrix} m_x^p(t) \\ m_y^p(t) \\ m_z^p(t) \end{bmatrix}$$

$$m^p a_x^p = f_x^p$$
$$m^p a_y^p = f_y^p$$
$$m^p a_z^p = f_z^p$$

$$I_{xx}^p \beta_x - I_{xy}^p \beta_y - I_{xz}^p \beta_z + \omega_y^p \left( \omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) - \omega_z^p \left( \omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) = m_x^p$$

$$I_{yy}^p \beta_y - I_{yx}^p \beta_x - I_{yz}^p \beta_z - \omega_x^p \left( \omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) + \omega_z^p \left( \omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right) = m_y^p$$

$$I_{zz}^p \beta_z - I_{zx}^p \beta_x - I_{zy}^p \beta_y + \omega_x^p \left( \omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) - \omega_y^p \left( \omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right) = m_z^p$$

$$I_{xy}^p = \int_{V^p} (x - x^p) \cdot (y - y^p) \cdot \mu(x,y,z) \cdot dV \quad ;$$

$$I_{zy}^p = \int_{V^p} (z - z^p) \cdot (y - y^p) \cdot \mu(x,y,z) \cdot dV \quad \text{etc.}$$

# THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Special case: e.g. Spheres:

$$I^p_{xy} = 0; \quad I^p_{zy} = 0; \quad \text{etc.}; \quad I^p_{xx} = I^p_{yy} = I^p_{zz} := I^p$$

$$m^p a^p_x = f^p_x$$

$$m^p a^p_y = f^p_y$$

$$m^p a^p_z = f^p_z$$

$$I^p \beta_x = m^p_x$$

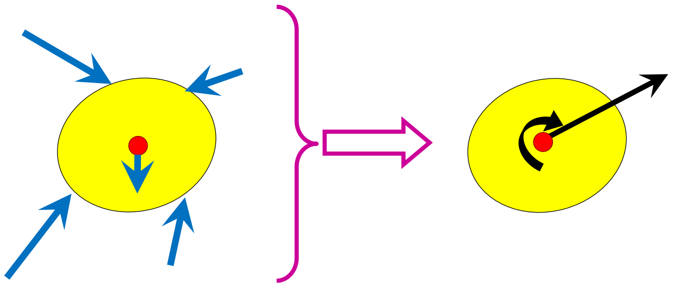
$$I^p \beta_y = m^p_y$$

$$I^p \beta_z = m^p_z$$

# THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Equations of motion of the *p*-th element:

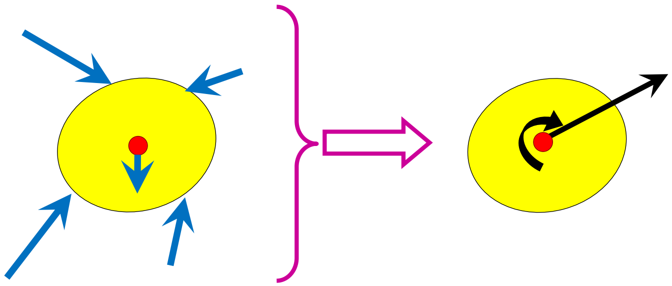


the load vector:  forces reduced to the reference point

$\rightarrow$ partly from the **external** forces
acting on the elements (e.g. weight)
*depend on position and velocity*

$\rightarrow$ partly from the **contact** forces
expressed by the neighbouring elements
*depend on position and velocity*

# THE EQUATIONS OF MOTION



a) Perfectly rigid elements

Equations of motion of the *p*-th element:

$$m^p a_x^p = f_x^p$$

$$m^p a_y^p = f_y^p$$

$$m^p a_z^p = f_z^p$$

$$I_{xx}^p \beta_x - I_{xy}^p \beta_y - I_{xz}^p \beta_z = m_x^p - \omega_y^p \left( \omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) + \omega_z^p \left( \omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right)$$

$$I_{yy}^p \beta_y - I_{yx}^p \beta_x - I_{yz}^p \beta_z = m_y^p + \omega_x^p \left( \omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) - \omega_z^p \left( \omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right)$$

$$I_{zz}^p \beta_z - I_{zx}^p \beta_x - I_{zy}^p \beta_y = m_z^p - \omega_x^p \left( \omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) + \omega_y^p \left( \omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right)$$

$$\mathbf{M}^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

# THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Equations of motion of the *p*-th element:

$$\mathbf{M}^p = \begin{bmatrix} m^p & & & & & \\ & m^p & & & & \\ & & m^p & & & \\ & & & I_{xx}^p & -I_{xy}^p & -I_{xz}^p \\ & & & -I_{yx}^p & I_{yy}^p & -I_{yz}^p \\ & & & -I_{zx}^p & -I_{zy}^p & I_{zz}^p \end{bmatrix}$$

$$m^p a_x^p = f_x^p$$

$$m^p a_y^p = f_y^p$$

$$m^p a_z^p = f_z^p$$

$$I_{xx}^p \beta_x - I_{xy}^p \beta_y - I_{xz}^p \beta_z = m_x^p - \omega_y^p \left( \omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) + \omega_z^p \left( \omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right)$$

$$I_{yy}^p \beta_y - I_{yx}^p \beta_x - I_{yz}^p \beta_z = m_y^p + \omega_x^p \left( \omega_z^p I_{zz}^p - \omega_x^p I_{zx}^p - \omega_y^p I_{zy}^p \right) - \omega_z^p \left( \omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right)$$

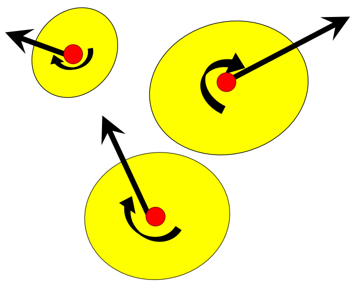
$$I_{zz}^p \beta_z - I_{zx}^p \beta_x - I_{zy}^p \beta_y = m_z^p - \omega_x^p \left( \omega_y^p I_{yy}^p - \omega_x^p I_{yx}^p - \omega_z^p I_{yz}^p \right) + \omega_y^p \left( \omega_x^p I_{xx}^p - \omega_y^p I_{xy}^p - \omega_z^p I_{xz}^p \right)$$

$$\mathbf{M}^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

# THE EQUATIONS OF MOTION

a) Perfectly rigid elements

Equations of motion of the *p*-th element: (6 scalar equations)



$$\mathbf{M}^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

for the complete system (*N* elements):

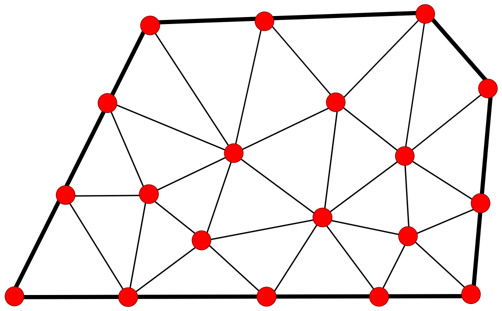
$$\boxed{\mathbf{M}(t)\mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N \end{bmatrix} \qquad \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t)) = \begin{bmatrix} \mathbf{f}^1(t, \mathbf{u}(t), \mathbf{v}(t)) \\ \mathbf{f}^2(t, \mathbf{u}(t), \mathbf{v}(t)) \\ \vdots \\ \mathbf{f}^N(t, \mathbf{u}(t), \mathbf{v}(t)) \end{bmatrix}$$

# THE EQUATIONS OF MOTION

„f = ma”

<u>b) Elements made deformable by being subdivided</u>



<u>most often:</u> SIMPLEX subdivision

displacement vector of the *p*-th node:

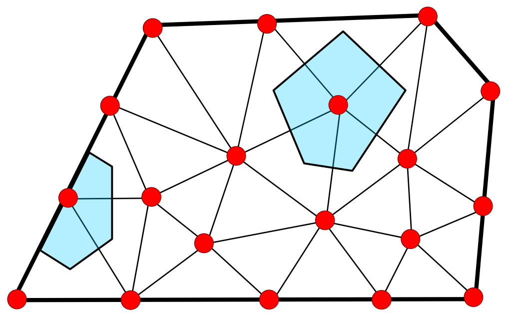
$$\mathbf{u}^{p}(t) = \begin{bmatrix} u_x^p(t) \\ u_y^p(t) \\ u_z^p(t) \end{bmatrix}$$

displacement vector of the whole system:

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}^{1}(t) \\ \mathbf{u}^{2}(t) \\ \vdots \\ \mathbf{u}^{N}(t) \end{bmatrix}$$

# THE EQUATIONS OF MOTION

b) Elements made deformable by being subdivided



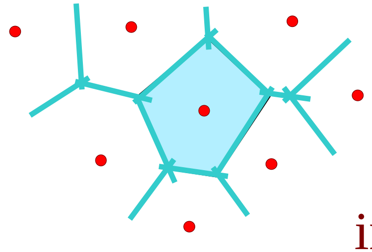
the equations of motion of the *p*-th node:

$$m^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

mass assigned to the *p*-th node: $m^p$

≡ the **Voronoi-cell** of the *p*-th node

Voronoi tessellation:



in 2D:

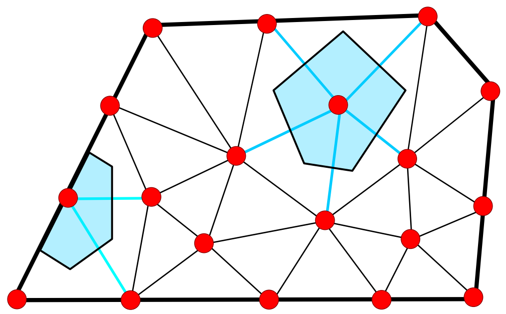
bisecting lines ⇒ 2D domains assigned to the nodes

in 3D:

bisecting planes ⇒ 3D domains assigned to the nodes

# THE EQUATIONS OF MOTION

b) Elements made deformable by being subdivided



the equations of motion of the *p*-th node:

$$m^p(t)\mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

mass assigned to the *p*-th node: $m^p$

the force acting on the *p*-th node: $\mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$ (3 components)

**ON THE NODE!**

- ← from the stresses inside the simplexes
- ← from the neighbouring element
- ← from external forces (e.g. self weight, drag force)

force from the stress within a simplex:

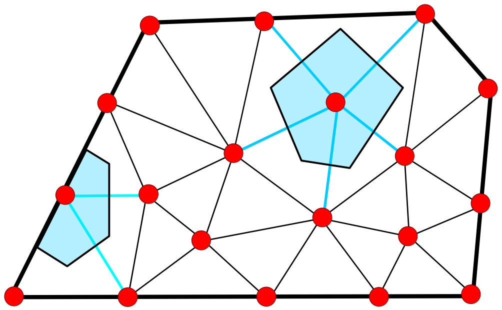
--- nodal translations ⇒ simplex strain ✓

--- from this and material characteristics ⇒ uniform stress in the simplex ✓

--- stress vector acting on the face of the cell: $\sigma_{ij} n_j = p_i$ ; resultant ✓

# THE EQUATIONS OF MOTION

b) Elements made deformable by being subdivided



the equations of motion of the whole system:

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$

($N\times 3$ scalar equations)

the complete inertial matrix consists of :

$$\mathbf{M}^p=\begin{bmatrix} m^p & & \\ & m^p & \\ & & m^p \end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix} \mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N \end{bmatrix}$$

the load vector: nodal forces

$$\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))=\begin{bmatrix} \mathbf{f}^1(t,\mathbf{u}(t),\mathbf{v}(t)) \\ \mathbf{f}^2(t,\mathbf{u}(t),\mathbf{v}(t)) \\ \vdots \\ \mathbf{f}^N(t,\mathbf{u}(t),\mathbf{v}(t)) \end{bmatrix}$$

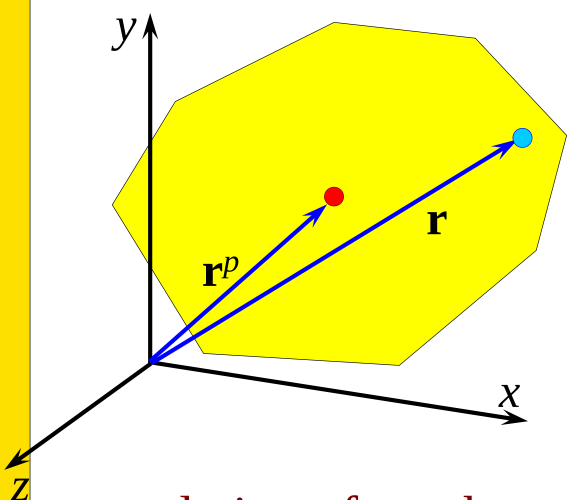
# THE EQUATIONS OF MOTION

„f = ma”

c) Uniform-strain deformable elements without subdivision

displacement vector of the *p*-th element:

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$



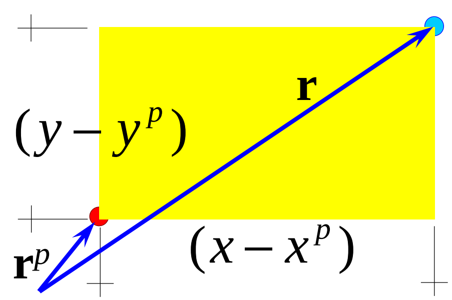

(reference point:
rigid-body translation and rotation;
the ***uniform*** strain of the element)

e.g. in 2D :

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \gamma_{xy}^p \end{bmatrix}$$

translation of another point in the element:

$$\mathbf{u} = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}$$



# THE EQUATIONS OF MOTION

„f = ma”

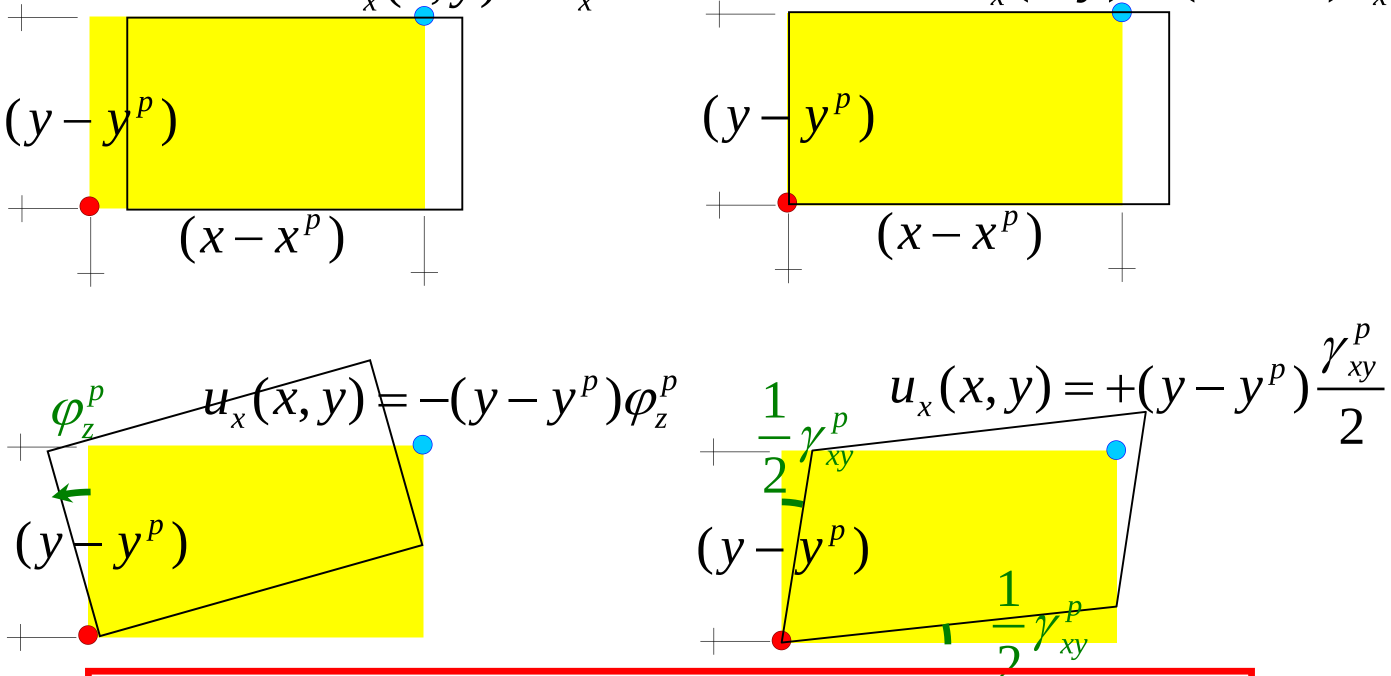
c) Uniform-strain deformable elements without subdivision HOME:

translation of another point in the element: with the help of superposition

$$u_x(x,y)=?$$

$$u_x(x,y)=u_x^p \qquad u_x(x,y)=(x-x^p)\varepsilon_x^p$$

$$u_x(x,y)=-(y-y^p)\varphi_z^p \qquad u_x(x,y)=+(y-y^p)\frac{\gamma_{xy}^p}{2}$$




$$\mathbf{u}^p=\begin{bmatrix}u_x^p\\u_y^p\\\varphi_z^p\\\varepsilon_x^p\\\varepsilon_y^p\\\gamma_{xy}^p\end{bmatrix}$$

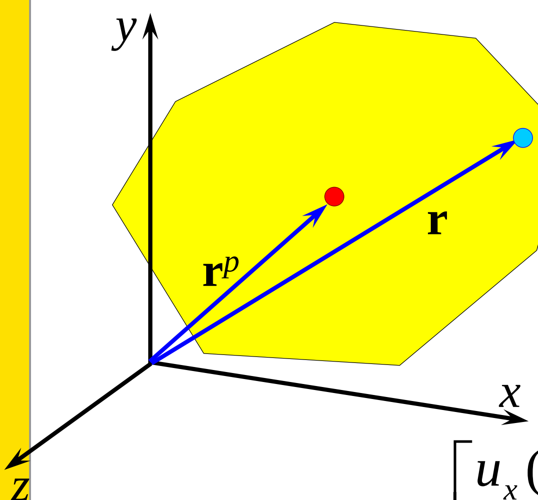
$$u_x(x,y)=u_x^p-(y-y^p)\varphi_z^p+(x-x^p)\varepsilon_x^p+(y-y^p)\frac{\gamma_{xy}^p}{2}$$

# THE EQUATIONS OF MOTION

„f = ma”

## c) Uniform-strain deformable elements without subdivision

translation of another point in the element:




$$u_x(x,y) = u_x^p - (y - y^p)\varphi_z^p + (x - x^p)\varepsilon_x^p + \frac{(y - y^p)}{2}\gamma_{xy}^p$$

$$u_y(x,y) = u_y^p + (x - x^p)\varphi_z^p + (y - y^p)\varepsilon_y^p + \frac{(x - x^p)}{2}\gamma_{xy}^p$$

$$\begin{bmatrix} u_x(x,y) \\ u_y(x,y \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(y - y^p) & (x - x^p) & 0 & \dfrac{(y - y^p)}{2} \\ 0 & 1 & (x - x^p) & 0 & (y - y^p) & \dfrac{(x - x^p)}{2} \end{bmatrix} \begin{bmatrix} u_x^p \\ u_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \gamma_{xy}^p \end{bmatrix}$$

similarly in 3D!

$\Rightarrow$ relative translations in the contacts:

can be expressed from $\mathbf{u}^p$

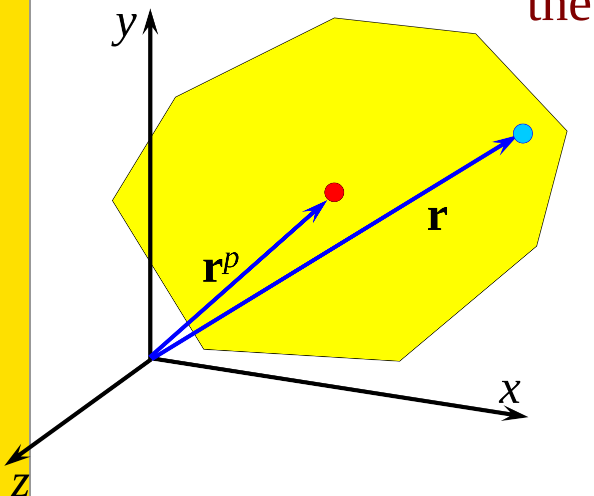
# THE EQUATIONS OF MOTION

„f = ma”

c) Uniform-strain deformable elements without subdivision

remember:

the displacement vector of the *p*-th element:




(reference point:

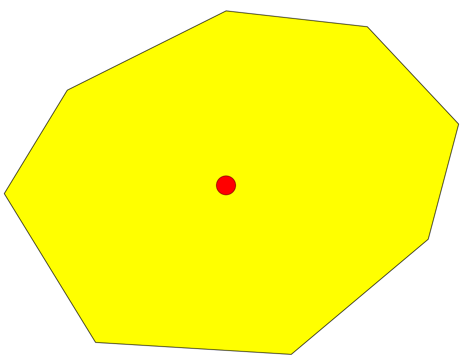
rigid-body translation and rotation;

the ***uniform*** strain of the element)

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

# THE EQUATIONS OF MOTION

c) Uniform-strain deformable elements without subdivision



load vector beloning to element *p*:

– from the contacts with neighbouring elements

– from the external forces directly acting on the element

$$\mathbf{f}^p = \begin{bmatrix} f_x^p \\ f_y^p \\ f_z^p \\ m_x^p \\ m_y^p \\ m_z^p \\ V^p \sigma_x^p \\ V^p \sigma_y^p \\ V^p \sigma_z^p \\ V^p \tau_{yz}^p \\ V^p \tau_{zx}^p \\ V^p \tau_{xy}^p \end{bmatrix} \qquad \mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

the equations of motion of the *p*-th element:

$$\mathbf{M}^p \cdot \mathbf{a}^p(t) = \mathbf{f}^p(t, \mathbf{u}(t), \mathbf{v}(t))$$

the equations of motion of the whole system:

$$\mathbf{M} \cdot \mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))$$

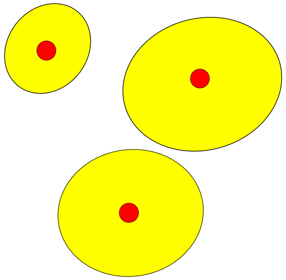
# THE EQUATIONS OF MOTION

$$\mathbf{M} \cdot \mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))$$

<u>Three main types of the elements:</u>

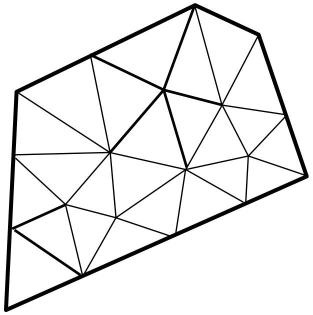
(1) perfectly **rigid** elements

$\rightarrow$ reference point



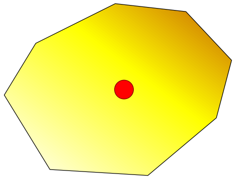
(2) elements being deformable because of an **internal FEM mesh**

$\rightarrow$ nodes



(3) elements being deformable because of a **uniform strain field**

$\rightarrow$ a reference point + a constant strain function



# THIS PRESENTATION

→ The Equations of Motion

(1) Perfectly **rigid** elements

(2) Elements being deformable because of an internal **FEM mesh**

(3) Elements being deformable because of a **uniform strain field**

→ Overview of Numerical Solution Techniques

- The aim
- Initial remarks
- **Euler** method
- Method of **Central Differences**
- **Newmark's $\beta$** – method

# SOLUTION OF THE EQUATIONS OF MOTION

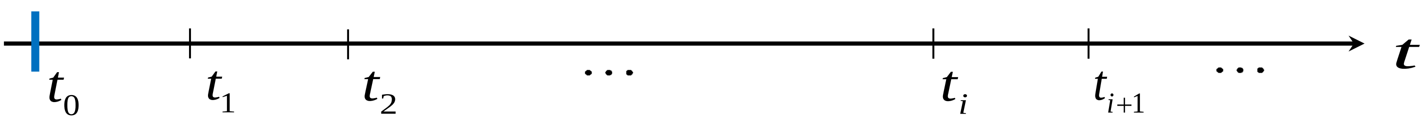
Numerical solutions only!

$$\mathbf{M} \cdot \mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))$$

The aim:

starting from a known $\mathbf{u}(t_0) = \mathbf{u}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$ state at a $t_0$ time instant,

the aim is to determine the approximative solutions $(\mathbf{u}_1, \mathbf{v}_1)$, $(\mathbf{u}_2, \mathbf{v}_2)$, …, $(\mathbf{u}_i, \mathbf{v}_i)$, $(\mathbf{u}_{i+1}, \mathbf{v}_{i+1})$, … belonging to the $t_1, t_2, \ldots, t_i, t_{i+1}, \ldots$ time instants.

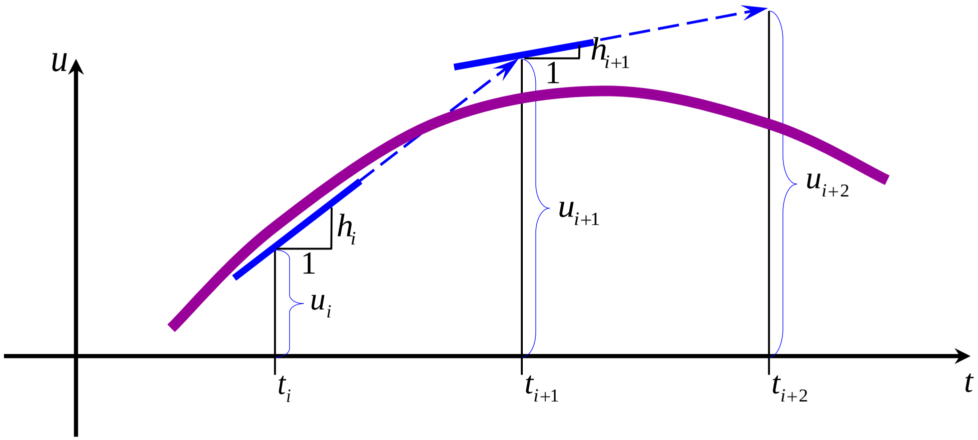



Initial remarks:

1. Explicit vs. implicit time integration methods
2. How to transform the equations of motion into first-order differential equations

# SOLUTION OF THE EQUATIONS OF MOTION

1. Explicit vs. implicit methods:



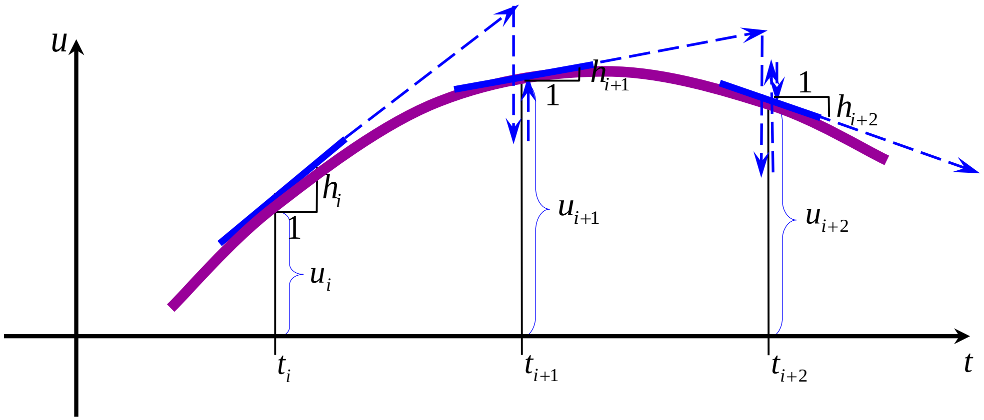

→ explicit methods:

in the state at $t_i$: ($\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{f}_i$) ⇒ equations of motion ⇒ approximate ($\mathbf{u}_{i+1}$, $\mathbf{v}_{i+1}$, $\mathbf{f}_{i+1}$) belonging to the state at $t_{i+1}$

NO checking of whether ($\mathbf{u}_{i+1}$, $\mathbf{v}_{i+1}$, $\mathbf{f}_{i+1}$) satisfy the eqs of motion, accept them and use them for the calculations of the next timestep ⇒ fast, but less reliable; numerical stability problems!

# SOLUTION OF THE EQUATIONS OF MOTION

## 1. Explicit vs. implicit methods:




→ implicit methods:

in the state at $t_i$: $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{f}_i) \Rightarrow$ equations of motion $\Rightarrow$

approximate $(\mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{f}_{i+1})$ belonging to the state at $t_{i+1}$;

then iterations, to improve this approximation belonging to $t_{i+1}$,

so that the eqs of motion be satisfied at $t_{i+1}$

$\Rightarrow$ slow, but longer timesteps,

more reliable, better numerical stability

# SOLUTION OF THE EQUATIONS OF MOTION

2. How to transform the equations of motion into <u>first-order DE</u>

The DE: $\mathbf{M}\cdot\frac{d^2\mathbf{u}(t)}{dt^2}=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$ where $\mathbf{v}(t)=\frac{d\mathbf{u}(t)}{dt}$

Notation:

new unknowns: $\mathbf{y}(t):=\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\end{bmatrix}$

new right-hand side:

$$\mathbf{a}(t,\mathbf{u}(t),\mathbf{v}(t)):=\mathbf{M}^{-1}\cdot\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t)) \quad \text{or:} \quad \mathbf{a}(t,\mathbf{y}(t)):=\mathbf{M}^{-1}\cdot\mathbf{f}(t,\mathbf{y}(t))$$

$$\hat{\mathbf{a}}(t,\mathbf{u}(t),\mathbf{v}(t)):=\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{a}(t,\mathbf{u}(t),\mathbf{v}(t))\end{bmatrix}$$

so the equations become:

$$\frac{d\mathbf{y}(t)}{dt}=\hat{\mathbf{a}}(t,\mathbf{y}(t))$$

$$\begin{bmatrix}\frac{d\mathbf{u}(t)}{dt}\\ \frac{d\mathbf{v}(t)}{dt}\end{bmatrix}=\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{M}^{-1}\cdot\mathbf{f}(t,\mathbf{y}(t))\end{bmatrix}$$

# THIS PRESENTATION

→ The Equations of Motion

(1) Perfectly **rigid** elements

(2) Elements being deformable because of an internal **FEM mesh**

(3) Elements being deformable because of a **uniform strain field**

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# EULER-METHOD

For the DEM eqs of motion:

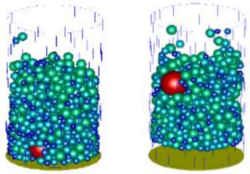
The problem:
$$\begin{bmatrix} \frac{d\mathbf{u}(t)}{dt} \\ \frac{d\mathbf{v}(t)}{dt} \end{bmatrix} = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{a}(t,\mathbf{u}(t),\mathbf{v}(t)) \end{bmatrix} \quad ; \quad \begin{bmatrix} \mathbf{u}(t_0) \\ \mathbf{v}(t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{bmatrix}$$

$$\mathbf{a}(t,\mathbf{u}(t),\mathbf{v}(t)) := \mathbf{M}^{-1} \cdot \mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))$$

at $t_i$: known $\mathbf{v}_i$ and $\mathbf{f}$ ;

From these, the new position and velocity:
$$\begin{bmatrix} \mathbf{u}_{i+1} \\ \mathbf{v}_{i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} + \Delta t \cdot \begin{bmatrix} \mathbf{v}_i \\ \mathbf{a}(t_i,\mathbf{u}_i,\mathbf{v}_i) \end{bmatrix}$$

meaning: the ***velocity*** and the ***acceleration*** are known at the beginning of $\Delta t$, and their values are ***kept constant*** along the time interval



or:
$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \cdot \mathbf{v}_i$$
$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta t \cdot \mathbf{a}(t_i,\mathbf{u}_i,\mathbf{v}_i)$$

DEM: ≈ contact dynamics methods (implicit vs)

disadvantage: oscillations

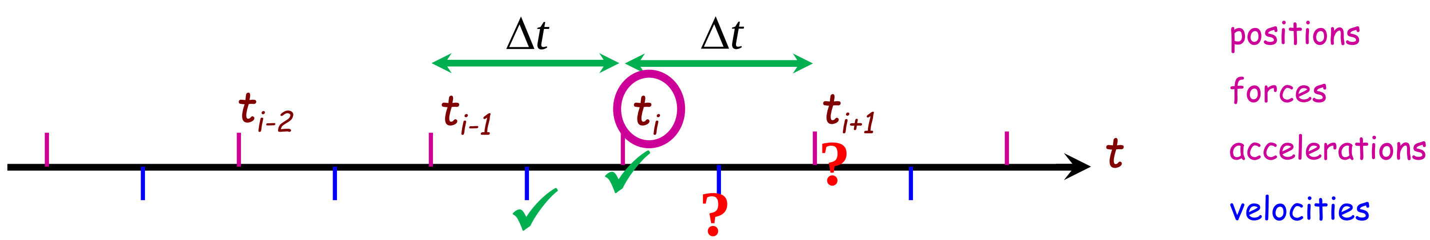
# METHOD OF CENTRAL DIFFERENCES

For the DEM eqs of motion:

The problem:

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{v}(t); \qquad \mathbf{u}(t_0) = \mathbf{u}_0;$$

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t, \mathbf{u}(t), \mathbf{v}(t)) \qquad \mathbf{v}(t_0) = \mathbf{v}_0$$




known: $\mathbf{v}_{i-1/2}$; $\mathbf{a}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$ (initially: e.g. $\mathbf{v}_{1-1/2} := \mathbf{v}_0$)

Let $\mathbf{v}_{i+1/2} := \mathbf{v}_{i-1/2} + \Delta t \cdot \mathbf{a}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$ ;

then from this: $\mathbf{u}_{i+1} := \mathbf{u}_i + \Delta t \cdot \mathbf{v}_{i+1/2}$

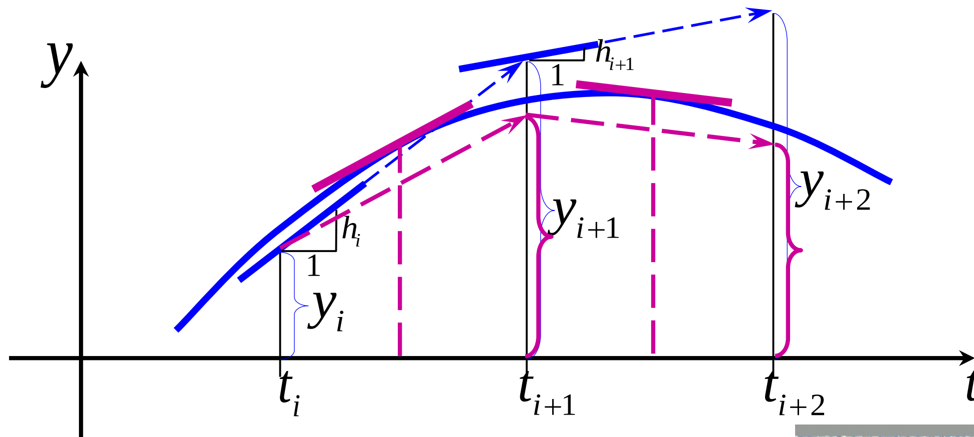
# METHOD OF CENTRAL DIFFERENCES

For the DEM eqs of motion:

The problem:

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{v}(t); \qquad \mathbf{u}(t_0) = \mathbf{u}_0;$$

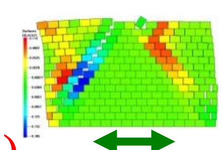
$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t, \mathbf{u}(t), \mathbf{v}(t)) \qquad \mathbf{v}(t_0) = \mathbf{v}_0$$

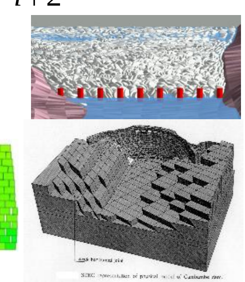



known: $\mathbf{v}_{i-1/2}$; $\mathbf{a}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$

Let $\mathbf{v}_{i+1/2} := \mathbf{v}_{i-1/2} + \Delta t \cdot \mathbf{a}(t_i, \mathbf{u}_i, \mathbf{v}_{i-1/2})$ ;

then from this: $\mathbf{u}_{i+1} := \mathbf{u}_i + \Delta t \cdot \mathbf{v}_{i+1/2}$





DEM: e.g. UDEC, PFC (most of the explicit timestepping methods)

# NEWMARK'S $\beta$-METHOD

For the DEM eqs of motion:

The problem: Find the $\mathbf{u}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ functions which satisfy the eqs.

$$\mathbf{M} \cdot \mathbf{a}(t) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t))$$

in which $\mathbf{v}(t) = \dfrac{d\mathbf{u}(t)}{dt}, \quad \mathbf{a}(t) = \dfrac{d^2\mathbf{u}(t)}{dt^2}$ .

Notation: „residual": $\mathbf{r}(t, \mathbf{u}(t), \mathbf{v}(t), \mathbf{a}(t)) = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t)) - \mathbf{M} \cdot \mathbf{a}(t)$

The $\mathbf{u}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ functions are the solutions of the differential eqs

if and only if: $\boxed{\mathbf{r}(t, \mathbf{u}(t), \mathbf{v}(t), \mathbf{a}(t)) = \mathbf{0}}$

→ Assume that the $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ numerical solutions belonging to $t_i$ satisfied this.

→ We would like to find $\mathbf{u}_{i+1}$, $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ belonging to $t_{i+1}$ so that:

$$\boxed{\mathbf{r}(t_{i+1}, \mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{a}_{i+1}) = 0}$$

# NEWMARK'S $\beta$-METHOD

For the DEM eqs of motion:

Approximation of the position and velocity at the end of the timestep:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}\left[(1-2\beta)\mathbf{a}_i + 2\beta \cdot \mathbf{a}_{i+1}\right]$$

$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

Expression for the unknown values $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ in terms of the unknown $\mathbf{u}_{i+1}$:

$$\mathbf{a}_{i+1} := \frac{1}{\beta \cdot \Delta t^2}\left[\mathbf{u}_{i+1} - \left(\mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}(1-2\beta)\mathbf{a}_i\right)\right]$$

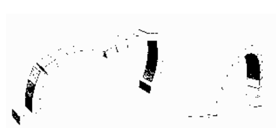
$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

here $\beta$ and $\gamma$ are constants controlling the behaviour of the method

The core of the method: Determine that $\mathbf{u}_{i+1}$, for which: $\mathbf{r}(t_{i+1}, \mathbf{u}_{i+1}, \mathbf{v}_{i+1}, \mathbf{a}_{i+1}) = 0$

$\rightarrow$ e.g. Newton-Raphson iteration to find $\mathbf{u}_{i+1}$, then express $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ ✓

DEM: e.g. DDA models



# NEWMARK'S $\beta$-METHOD

For the DEM eqs of motion:

Approximation of the position and velocity at the end of the timestep:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}\left[(1-2\beta)\mathbf{a}_i + 2\beta \cdot \mathbf{a}_{i+1}\right]$$

$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

Expression for the unknown values $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ in terms of the unknown $\mathbf{u}_{i+1}$:

$$\mathbf{a}_{i+1} := \frac{1}{\beta \cdot \Delta t^2}\left[\mathbf{u}_{i+1} - \left(\mathbf{u}_i + \Delta t \cdot \mathbf{v}_i + \frac{\Delta t^2}{2}(1-2\beta)\mathbf{a}_i\right)\right]$$

$$\mathbf{v}_{i+1} := \mathbf{v}_i + (1-\gamma)\cdot \Delta t \cdot \mathbf{a}_i + \gamma \cdot \Delta t \cdot \mathbf{a}_{i+1}$$

here $\beta$ and $\gamma$ are constants controlling the behaviour of the method

**≈ „no numerical blown-up for any time step length"**

specific $\beta$ and $\gamma$ values → several other methods

**UNCONDITIONALLY STABLE IF**: $2\beta \geq \gamma \geq ½$

e.g. $\gamma = ½$ , $\beta = 0$ : ***method of central differences***, which is

**≈ „time step length should be limited"**

**ONLY CONDITIONALLY STABLE**

# QUESTIONS

1. What are the kinematic degrees of freedom in case of ***perfectly rigid*** elements in 3D? For a model consisting of *n* perfectly rigid elements in 3D, what is the number of scalar equations of motion?

2. What are the kinematic degrees of freedom in case of an element being deformable because of being ***subdivided*** into uniform-strain simplexes in 3D? How could you determine the number of scalar equations of motion in 3D?

3. What are the kinematic degrees of freedom in case of ***uniform-strain deformable*** elements in 3D? For a model consisting of *n* uniform-strain deformable elements, what is the number of scalar equations of motion in 3D?

4. What is the ***difference*** between ***explicit*** and ***implicit*** time integration methods?

# QUESTIONS

5. You learnt about the ***Euler-method***, the ***central difference method*** and ***Newmark's β-method***. Which statements are true for which method(s)?

a) The velocity is constant along the timestep, and equal to its previously calculated value at the beginning of the timestep.
b) The velocity is constant along the timestep, and equal to its previously calculated value at the middle of the timestep.
c) The velocity is constant along the timestep, and equal to the weighted average of the values at the beginning and at the end of the timestep.
d) It is an explicit method.
e) It is an implicit method.
f) It contains an inner Newton-Raphson iteration.
g) This method is unconditionally stable.
h) This method is the special case of the Newmark method, with $\gamma = 1/2$ and $\beta = 0$.