Homogenous coordinates and their application to coordinate transformation and vertical section of point clouds

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1. Homogeneous coordinates

Homogeneous coordinates can describe the position of a point given in a space with n dimension with n+1 coordinates where the original coordinates are multiplied by a non-zero constant, and the n+1th coordinate is going to be the constant.

Let's have a point in the original space given with n coordinates: $(x_1, x_2, \dots, x_n)$. The same point expressed with n+1 homogenous coordinates is $(\lambda \cdot x_1, \lambda \cdot x_2, \dots, \lambda \cdot x_n, \lambda)$ where λ is the non-zero constant. Thus an individual point in the original space has an infinite number of equivalents with homogeneous coordinates. For instance, a point in the two-dimensional space corresponds to a line passing through the origin in the three-dimensional space with homogeneous coordinates. The most straightforward way of generating homogenous coordinates is to set the λ constant to 1; hence the original and homogeneous coordinates are the same.

2. Distance from a point to a line

A line can be defined in a two-dimensional space by the equation $a \cdot x + b \cdot y + c = 0$ in general. A point is on the line if its coordinates work out the equation. Using the homogeneous coordinates of the point ($\mathbf{p}$) and the vector ($\mathbf{e}$) generated from the coefficients of the equation of the line, you can get the left side of the equation as the scalar product of ($\mathbf{p}$) and ($\mathbf{e}$). The sign of the scalar product indicates which side of the line is the point, while its value is proportional to the distance from the point to the line. The scalar product must be normalized in order to get the distance itself.

Let's have two different points in the two-dimensional space defining a line. Their homogeneous coordinates are in the vectors $\mathbf{p_1}$and $\mathbf{p_2}$. The abovementioned vector $\mathbf{e}$ with the coefficients of the line defined by the two points can be computed as the vector product of $\mathbf{p_1}$and $\mathbf{p_2}$.

After a few simplifications:

$\mathbf{e} = \mathbf{p_1} \times \mathbf{p_2} = (y_1 - y_2 \quad x_2 - x_1 \quad x_1 \cdot y_2 - x_2 \cdot y_1)$

where x and y are the original coordinates.

You can check your computations by having the scalar products of $< \mathbf{p_1}\ \mathbf{e} >$ or $< \mathbf{p_2}\ \mathbf{e} >$ since they are definitely zero.

The formula can compute the distance from an arbitrary point to the line:

$$d = < \mathbf{p}\ \mathbf{e} > \cdot \frac{1}{\lambda \sqrt{a^2 + b^2}}$$

where d is the distance and $\mathbf{p}$ is the vector containing the homogeneous coordinates of the point.

As it has been stated above d might be plus or minus depending on geometry.

Check out your knowledge. Let's have the two points with coordinates (4,6) and (1,2) defining a line. Compute the distance from the origin to the line, then the distance from the point (1,5) to the line. Pay attention to the geometry. Which point is on the left and the right side of the line?

3. Coordinate transformation

3x3 matrixes can describe fundamental 2D coordinate transformations. The transformation can be applied by multiplying the vector containing the homogenous coordinates and the transformation matrix. The transformation matrix, in general, has four parts:

1.		3.
2.		4.

1. rotation and separated scale factor along x and y axises
2. shift
3. projective transformation
4. the same scale factor along both x and y axes

Let the shift value along the x and y axes be T_x and T_y, respectively. The transformation matrix containing both shifts is:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_x & 1 \end{bmatrix}$$

Let the rotation angle around the origin be α. The transformation matrix containing this rotation is:

$$\mathbf{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation with a positive sign means a clockwise rotation.

Let the same scale factor along both x and y axes be s. The transformation matrix containing this scale factor is:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

In case the scale factors are different along the x and y axes, the transformation matrix is:

$$\mathbf{R} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This way, you can reflect over x or y-axes. All you need to do is set the scale factors to +1 and -1. In case the reflection over the x-axis, for instance, $s_x = -1$ and $s_x = +1$.

One of the underlying advantages of using homogeneous coordinates is that a 3x3 matrix can describe any fundamental transformations. Consequently, more complex transformations can be derived by multiplying the different transformation matrixes. For instance, rotation around a given point has three steps, as well as three matrixes:

1. shift the origin of the original system to the given point
2. rotation around the origin
3. the inverse of the 1st step, shift back to the original origin.

Check out your knowledge. Let's have the two points with coordinates (4,6) and (1,2) defining a line. Transform the origin and the point (1,5) to a local system defined by the line. Compare the transformed coordinates to the distances computed in the previous exercise.