Exercises in the field of error propagation

Exercise 1. We have to measure the elevation difference of two points with the mean error of 1,5 mm. We carry out repeated two-way levellings, where the elevation difference computed from a one-way levelling has the mean error of 6 mm. How many times do we have to repeat the two-way levellings to achieve the required accuracy?

The mean error of a one-way levelling:

$$m_{one-way} = \pm 6mm$$

The elevation difference computed from forward and backward levellings:

$$\Delta h_{two-way} = \frac{\Delta h_{forward} - \Delta h_{backward}}{2}$$

In order to compute the mean error of a two-way levelling, we apply the law of error propagation:

$$\frac{\partial \Delta h_{two-way}}{\partial \Delta h_{forward}} = \frac{1}{2}, \text{ and}$$
$$\frac{\partial \Delta h_{two-way}}{\partial \Delta h_{bacward}} = -\frac{1}{2}.$$

since the mean error of a one-way levelling is given, therefore:

$$m_{forward} = m_{backward} = m_{one-way}.$$

According to the law of error propagation:

$$m_{two-way} = \sqrt{\left(\frac{1}{2}\right)^2 m_{forward}^2 + \left(-\frac{1}{2}\right)^2 m_{backward}^2} = \frac{m_{one-way}}{\sqrt{2}} = \pm 4{,}2mm$$

How many times should a two-way levelling be repeated, in order to achieve the required mean error of ±1,5 mm?

The most likely value of the results of repeated observations is computed by the arithmetic mean (average value):

$$\overline{\Delta h} = \frac{\sum_{i=1}^{n} \Delta h_i^{two-way}}{n}.$$

When the law of error propagation is applied to the previous formula, the following result is obtained:

$$m_{\overline{\Delta h}} = \frac{m_{two-way}}{\sqrt{n}}.$$

Since the required mean error $m_{\overline{\Delta h}}$ is given, and the mean error of a two-way levelling has been already computed, the number of repetition (n) can be determined as well:

$$\pm 1{,}5mm \geq \frac{\pm 4{,}2mm}{\sqrt{n}},$$

therefore:

$$n \geq \left(\frac{\pm 4{,}2}{\pm 1{,}5}\right)^2 = 8.$$

The two-way levelling should be repeated at least 8 times in order to achieve the required mean error.

Exercise 2. The inner diameter and the height of two cylindrical water towers have been measured. The diameters were d = 32,12 m ± 5 cm and the heights were h = 24,45m ± 3 cm. Compute the cummulative volume of the towers and its mean error!

The volume of one cylinder:

$$V = \frac{d^2\pi}{4}h.$$

Since we have two cylinders, the total volume is:

$$V = \frac{d^2\pi}{2}h = 39623{,}22\ \text{m}^3$$

Applying the law of error propagation to the formula of the volume:

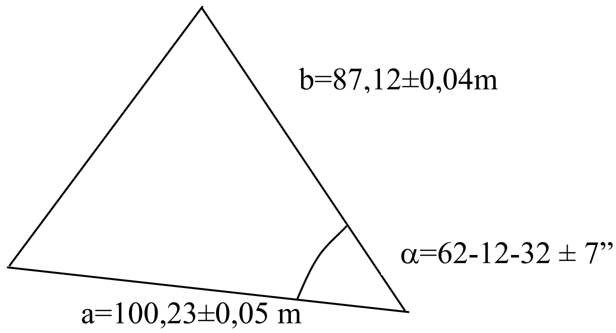
$$\frac{\partial V}{\partial d} = dh\pi = 2467{,}199\ \text{m}^2$$
$$\frac{\partial V}{\partial h} = \frac{d^2\pi}{2} = 1620{,}582\ \text{m}^2$$

The mean error of the volume is:

$$m_V = \sqrt{\left(\frac{\partial V}{\partial d}\right)^2 m_d^2 + \left(\frac{\partial V}{\partial h}\right)^2 m_h^2} = \pm 132{,}5\ \text{m}^3$$

Finally the volume of the two towers is: 39623,22 ± 132,5 m^3.

Exercise 3. In order to compute the perimeter of a triangle, we have measured the sides a,b and the angle α. How much is the area of the triangle, and its mean error?



The area of the triangle is:

$$A = \frac{ab}{2}\sin\alpha = 3862{,}4 \text{ m}^2$$

Let’s apply the law of error propagation:

$$\frac{\partial A}{\partial a} = \frac{b}{2}\sin\alpha = 38{,}54 \text{ m}$$

$$\frac{\partial A}{\partial b} = \frac{a}{2}\sin\alpha = 44{,}34 \text{ m}$$

$$\frac{\partial A}{\partial \alpha} = \frac{ab}{2}\cos\alpha = 2035{,}66 \text{ m}^2$$

The respective mean error values are:

$$m_a = \pm 0{,}05m$$
$$m_b = \pm 0{,}04m$$
$$m_\propto = \pm 7" = \pm\frac{7}{206264{,}8}[rad]$$

Please note that the mean error of angular variables must be converted into radian!

The law of error propagation:

$$m_A = \sqrt{\left(\frac{\partial A}{\partial a}\right)^2 m_a^2 + \left(\frac{\partial A}{\partial b}\right)^2 m_b^2 + \left(\frac{\partial A}{\partial \propto}\right)^2 m_\propto^2} =$$
$$\sqrt{38{,}54^2 \cdot 0{,}05^2 + 44{,}34^2 \cdot 0{,}04^2 + 2035{,}66^2 \cdot \left(\frac{7}{206264{,}8}\right)^2} = \pm 2{,}62 \text{ m}^2$$

So the area of the triangle is: 3862,4 ±2,62 m^2.